INTEGRAL CRYSTALLINE TATE CLASSES OF CODIMENSION TWO AT SUPERSINGULAR POINTS OF MUMFORD'S FOURFOLDS

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Abstract. For a supersingular abelian variety X over $\bar{\mathbb{F}}_p$, let $T^4(X)$ be the lattice of integral crystalline Tate classes of codimension 2 and $S(X)$ the span of products of divisor classes. Let $p \geq 5$ and $\mathscr{G}_p$ be the p-divisible group of the supersingular points of Mumford's families of abelian fourfolds at primes inert in their cubic field, for the hyperspecial levels given by tensor-induced lattices. For every abelian fourfold X with $X[p^\infty] \cong \mathscr{G}_p$ we show that $T^4(X)/S(X) \cong (\mathbb{Z}/p)^4$, and that the integral crystalline Tate conjecture holds in codimension 2 on X if statements in a preprint of Markman hold. At these primes, Kisin's crystalline tensors make the Dieudonné module a tensor product of three rank-2 modules permuted cyclically by Frobenius, and at supersingular points Frobenius maps a tensor basis to multiples of basis vectors. For such monomial modules all lattices involved are spanned by multiples of basis vectors, with exponents given by min-plus formulas independent of p. Under the endomorphisms, T^4/S is a non-split extension of $(\mathbb{Z}/p)^2$ by $(\mathbb{Z}/p)^2$, generated by one Weil class of an inert imaginary quadratic action. Assuming these statements, an algebraic such class comes from specializing a Weil cycle on a fourfold with complex multiplication. A projector that is a polynomial in an endomorphism, and an intersection matrix with unit determinant, fix its p-adic normalization without a comparison theorem. At unramified non-inert primes with such levels, $T^4 = S$ at the supersingular points. Of the fourteen isomorphism classes of monomial supersingular Dieudonné modules of height 8, the conjecture holds unconditionally for nine, under the same assumption for that of $\mathscr{G}_p$, and is open for four. The indices are computed exactly with integers.

1. Introduction

Let X be a supersingular abelian variety of dimension g over $\bar{\mathbb{F}}_p$, let $W = W(\bar{\mathbb{F}}_p)$ with Frobenius σ, and let $M(X) = H^1_{\mathrm{cris}}(X/W)$, a free W-module of rank $2g$ with a σ-semilinear Frobenius F. The crystalline cohomology $H^n_{\mathrm{cris}}(X/W)$ is $\bigwedge^n M(X)$, and for even n the *integral crystalline Tate classes* are

$$T^n(X) = \{x \in H^n_{\mathrm{cris}}(X/W) \ : \ Fx = p^{n/2}x\} \ , \tag{1.1}$$

a free $\mathbb{Z}_p$-module of rank $\binom{2g}{n}$. We use this formulation only for supersingular X: for other X the F-fixed part can be larger than the span of algebraic classes, for instance for $X = E \times E'$ with E, E' non-isogenous ordinary elliptic curves, where $T^2(X)$ has rank 4 by (R2) in Section 2.1 and $\mathrm{NS}(X)$ has rank 2 (see e.g. [1, Chapter I, Proposition 14.2]). Let $\mathrm{Alg}^i(X) \subseteq T^{2i}(X)$ be the $\mathbb{Z}_p$-span of the crystalline classes of algebraic cycles of codimension i, and let $S(X) \subseteq T^4(X)$ be the $\mathbb{Z}_p$-span of products of two classes in $T^2(X)$. The *integral crystalline Tate conjecture in codimension i* for X is the statement $\mathrm{Alg}^i(X) = T^{2i}(X)$. For $i = 1$ it holds, because $\mathrm{NS}(X) \otimes \mathbb{Z}_p \cong T^2(X)$ (Section 2.1), and hence $S(X) \subseteq \mathrm{Alg}^2(X)$. The rational statement holds in every codimension. Indeed, $T^2(X) \otimes W$ has finite index in $\bigwedge^2 M(X)$ (all slopes of $p^{-1}\bigwedge^2 F$ are 0, and (R2)), and the even part of an exterior algebra is generated by its degree-2 part, so $T^{2i}(X) \otimes \mathbb{Q}$ is spanned by products of divisor classes. Hence the question in codimension 2 is whether the finite group $T^4(X)/\mathrm{Alg}^2(X)$, a quotient of $T^4(X)/S(X)$, vanishes. We answer it, under Hypothesis 7.1 below, for the abelian fourfolds with one particular supersingular p-divisible group, the one that occurs at the supersingular points of Mumford's families.

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Let $\mathscr{G}_p$ be the p-divisible group over $\bar{\mathbb{F}}_p$ whose Dieudonné module is

$$M = W^2 \otimes W^2 \otimes W^2 \ , \qquad F(x_1 \otimes x_2 \otimes x_3) = \sigma x_3 \otimes A\sigma x_1 \otimes \sigma x_2 \ , \qquad A = \begin{pmatrix} 0 & p \\ 1 & 0 \end{pmatrix} \ . \tag{1.2}$$

It exists by (R1) and depends only on p. It is isomorphic to $E_0[p^\infty] \times \mathscr{H}$ for a supersingular elliptic curve E_0 and a p-divisible group $\mathscr{H}$ of height 6 and dimension 3 (Remark 8.5).

Part (b) of the theorem below depends on Markman's preprint [2], which has not been refereed. Markman states that the Weil classes on polarized abelian sixfolds of Weil type with discriminant -1 are algebraic [2, Theorem 1.5.1], a statement with rational coefficients. We need an integral form of it with a multiplier prime to p: for every even $d \geq 3$, on every member of one connected family of abelian sixfolds of Weil type for $\mathbb{Q}(\sqrt{-d})$ with discriminant -1, the class $2(8d)^{8d}\kappa_3(\mathcal{E})$ is the class of an algebraic cycle with integer coefficients, where $\mathcal{E}$ is the reflexive sheaf of rank $8d$ of [2, Theorem 1.4.1] (Proposition 7.3). We prove it from three statements made in Markman's proof of [2, Theorem 1.5.1] and from other statements of [2], which together form Hypothesis 7.1 in Section 7; we have not verified Markman's proofs of them.

Theorem 1.3 (computer-assisted)**.** *Let $p \geq 5$ and let X be an abelian fourfold over $\bar{\mathbb{F}}_p$ with $X[p^\infty] \cong \mathscr{G}_p$.*

(a) $T^4(X)/S(X) \cong (\mathbb{Z}/p)^4$.

(b) Assume Hypothesis 7.1. Then $\mathrm{Alg}^2(X) = T^4(X)$: the integral crystalline Tate conjecture holds in codimension 2 on X. Moreover, the Künneth component in $H^2_{\mathrm{cris}}(X) \otimes H^2_{\mathrm{cris}}(X)$ of every class in $T^4(X \times X)$ is algebraic.

Besides Hypothesis 7.1, the proof uses the standard facts of Section 2.1, a closed-immersion criterion of SGA 3 in Lemma 3.1, the exact computations of Appendix A, and Lemma 7.2 on Chern classes in families, which rests on results of Atiyah and Hirzebruch, of Borel and Haefliger, and of Grothendieck.

Part (b) is proved first at the supersingular points of Mumford's families. Let k be a totally real cubic field and B a quaternion algebra over k that is split at exactly one real place and satisfies $\mathrm{Cor}_{k/\mathbb{Q}} B \cong \mathrm{Mat}_8(\mathbb{Q})$. Mumford [3, Section 4] constructed from (k, B) a family of Hodge type of polarized abelian fourfolds: the group B^1 of elements of B of reduced norm 1 acts on $\mathbb{V} = \mathbb{Q}^8$ through the norm map $B^\times \to (\mathrm{Cor}_{k/\mathbb{Q}} B)^\times$, its image is the Hodge group of a generic member, and the members with this Hodge group, which include those at very general points, have no non-trivial endomorphisms. The corresponding Shimura datum $(G, \mathbb{X})$ has $G \subset \mathrm{GL}(\mathbb{V})$ the image of $\mathbb{G}_m \times B^1$ (see e.g. [4, Section 2]), so that $G_{\bar{\mathbb{Q}}}$ is the image of $\mathbb{G}_m \times \mathrm{SL}_2^3$ acting on $\mathbb{U}_1 \otimes \mathbb{U}_2 \otimes \mathbb{U}_3$, $\dim \mathbb{U}_i = 2$. Fix a symplectic form ψ on $\mathbb{V}$ with $G \subset \mathrm{GSp}(\mathbb{V}, \psi)$, a lattice $\Lambda \subset \mathbb{V}$, a prime p, and a level $\mathsf{K} = \mathsf{K}_p\mathsf{K}^p$ with K_p the stabilizer of $\Lambda \otimes \mathbb{Z}_p$ in $G(\mathbb{Q}_p)$ and K^p neat. The reflex field is the image of k under the real place at which B splits, since the Hodge cocharacter is non-trivial only in the corresponding factor. Let $\mathscr{S}$ be Kisin's integral model [5] over the localization of its ring of integers at a prime above p, and $\mathcal{A} \to \mathscr{S}$ the abelian scheme. We assume

(h1) $p \geq 5$;
(h2) p is inert in k;
(h3) $B \otimes_k k_p \cong \mathrm{Mat}_2(k_p)$, where $k_p = k \otimes \mathbb{Q}_p$ and $O_{k_p} = O_k \otimes \mathbb{Z}_p$, so that $\mathbb{V} \otimes \mathbb{Q}_p = \mathrm{Cor}_{k_p/\mathbb{Q}_p} \mathbb{U}_p$ for the k_p-module $\mathbb{U}_p \cong k_p^2$ on which $B \otimes_k k_p$ acts, where Cor is tensor induction;
(h4) $\Lambda \otimes \mathbb{Z}_p = p^j \,\mathrm{Cor}(\Lambda_0)$ for an O_{k_p}-lattice $\Lambda_0 \subset \mathbb{U}_p$ and some $j \in \mathbb{Z}$.

By Lemma 3.1, (h4) implies that the closure of G in $\mathrm{GL}(\Lambda \otimes \mathbb{Z}_{(p)})$ is reductive, so that K_p is hyperspecial. For fixed (k, B, Λ), conditions (h3) and (h4) hold for all but finitely many p, and for every $p \geq 5$ there are data satisfying (h1)–(h4) (Lemma 8.2).

Under (h2)–(h4) every supersingular point of $\mathscr{S}(\bar{\mathbb{F}}_p)$ has p-divisible group isomorphic to $\mathscr{G}_p$ (Proposition 3.3), so Theorem 1.3 applies to these points. Theorem 5.2 describes T^4/S as a

module over the endomorphisms: it is a non-split extension of $(\mathbb{Z}/p)^2$ by $(\mathbb{Z}/p)^2$, whose quotient is carried by the Weil classes of an inert imaginary quadratic action. Theorem 8.3 proves part (b) at these points, with an algebraic class, specialized from a Weil cycle on a fourfold with complex multiplication, whose translates under the endomorphisms span T^4 together with S. At primes that are unramified and not inert in k, with B split above p and (h4), the supersingular points have $T^4 = S$ (Proposition 8.4). In Section 9 we compare $\mathscr{G}_p$ with the other monomial supersingular Dieudonné modules of height 8 and dimension 4: they form fourteen isomorphism classes, the integral crystalline Tate conjecture in codimension 2 holds unconditionally on every abelian fourfold whose Dieudonné module lies in one of nine of them, and four remain open (Proposition 9.3). For a very general complex fiber $X_{\mathbb{C}}$ of Mumford's family, $X_{\mathbb{C}} \times X_{\mathbb{C}}$ carries Hodge classes that are not in the span of products of divisor classes [6], and as far as we know their algebraicity is open; Remark 8.6 records the modest consequence of our results for them.

A preprint of Broe claims the Tate conjecture for abelian fourfolds over finite fields, through Markman's Weil classes, and deduces the crystalline Tate conjecture for them [7, Theorem 1.1 and Corollary 8.1(ii)]; it does not state an integral version. A preprint of Milne states that it deduces a similar statement for the integral Tate conjecture [8] from the failure of the integral Hodge conjecture for very general principally polarized abelian varieties of dimension at least 4, which is claimed in a preprint of Engel, de Gaay Fortman and Schreieder [9]. A preprint of Scavia and Suzuki claims applications of its results on direct summands of products of Jacobians to the integral Tate conjecture for divisors and 1-cycles on abelian varieties over finitely generated fields, and to a p-adic version of it over finite fields [10]. The same authors constructed non-algebraic classes over finite fields that vanish over the algebraic closure [11]. The classes of [11] have degree 4, and neither abstract concerns integral Tate classes over $\bar{\mathbb{F}}_p$. Cadoret and Pirutka give, assuming variational realization conjectures, uniform bounds for the obstructions to the integral Tate conjecture in one-dimensional families over infinite finitely generated fields [12]. Gregory proved that torsion codimension-2 cycles on supersingular abelian varieties are algebraically trivial [13], which concerns the kernel of the cycle class map rather than its image. For Mumford's families, a preprint of Chen claims that some of their fourfolds have infinitely many supersingular primes [14].

Use of large language models and status of the proofs. This note was produced by large language models, instances of the Claude models of Anthropic running as autonomous agents, in October 2026. They chose the question, found the proofs, wrote and ran the programs of Appendix A, and wrote the text. The proofs were checked only by other instances of the same models, which re-derived each result and repeated every computation with independently written programs. Such checks may share the blind spots of the models that wrote the proofs. The author has read the note and verified the bibliographic data of its references, but no human expert has checked the proofs. The least secure step is the dependence of part (b) on statements of [2] whose proofs we have not verified (Section 7).

Code availability. The programs for the computations of Appendix A are available under the MIT license at `https://github.com/ahippelainen/integral-tate-supersingular-fourfolds-public`.

Conventions. Throughout, $p \geq 5$, W and σ are as in the introduction, $\breve{\mathbb{Q}}_p = W[1/p]$, and for $r \geq 1$ we write $\mathbb{Z}_{p^r} = W(\mathbb{F}_{p^r})$ and $\mathbb{Q}_{p^r} = \mathbb{Z}_{p^r}[1/p]$. For every abelian variety X over $\bar{\mathbb{F}}_p$, the module $M(X)$, its Frobenius F and the identification $H^n_{\mathrm{cris}}(X/W) = \bigwedge^n M(X)$ are as in the introduction, $V = pF^{-1}$ is the Verschiebung, and for an endomorphism U of X, U^* acts on H^n_{cris} as $\bigwedge^n U^*$. All lattices are free modules of finite rank over $\mathbb{Z}_p$ or W, and $[\cdot : \cdot]$ is the index of a sublattice of full rank. A *monomial* lattice with respect to a basis (e_j) of a finite-dimensional $\breve{\mathbb{Q}}_p$-vector space is one of the form $\bigoplus_j p^{\nu_j} W e_j$. The letter k denotes the totally real cubic field of Mumford's families, K

an imaginary quadratic field and $L = kK$; $\iota, \bar{\iota}\colon \mathbb{Z}_{p^2} \to W$ are the two embeddings. We write E_0 for a supersingular elliptic curve.

The method. At an inert prime, the crystalline tensors of the integral model force $M(X)$ to be a tensor product of three Dieudonné modules of rank 2 permuted cyclically by Frobenius, with the Hodge filtration in one slot, and a lemma on rank-2 σ^3-crystals shows that at the supersingular points $M(X)$ is the model (1.2). In this model F maps the monomial basis e_s, $s \in \{0,1\}^3$, to multiples of monomials, so T^4, S and the span of endomorphism pull-backs are monomial lattices whose exponents are given by explicit min-plus formulas, independent of p. This gives $T^4/S \cong (\mathbb{Z}/p)^4$, carried by four monomials of $\bigwedge^4 M$; two of them are the top exterior powers of the eigenspaces of an inert imaginary quadratic action of signature $(2,2)$, and the endomorphisms move one Weil class onto all of T^4/S.

To produce one algebraic Weil class with a unit coefficient, we choose a fourfold X_c in Mumford's family with complex multiplication by $K \times L$, with K inert at p, whose reduction is supersingular and whose first homology is a unimodular Hermitian O_K-lattice at p. The algebraic classes on abelian sixfolds of Weil type that Proposition 7.3 derives from Hypothesis 7.1, carried to X_c by Schoen's degeneration argument through an isogeny of degree prime to p, give a cycle Z on X_c whose Weil component is $\lambda\omega + \bar{\lambda}\bar{\omega}$ with λ a p-adic unit. Let $\beta \in O_K$ generate $(O_K/p)^\times$. A projector P onto the Weil part, given by a polynomial with $\mathbb{Z}_{(p)}$-coefficients in $[\beta]^*$, produces the two cycles $Z_1 = PZ$ and $Z_2 = [\beta]^* Z_1$, whose intersection matrix has determinant

$$\det\big(\deg(Z_i \cdot Z_j)\big) = -u^2 \operatorname{Nm}(\lambda)^2 (\beta^4 - \bar{\beta}^4)^2 \ , \tag{1.4}$$

with u a p-adic unit. Since $\beta^4 \not\equiv \bar{\beta}^4 \bmod p$ for $p \geq 5$, this determinant is a p-adic unit. Specialization preserves intersection numbers, so the specialized classes span a sublattice of unit discriminant of the rank-2 lattice of integral crystalline Weil classes, hence all of it. All supersingular points have isomorphic p-divisible groups, so an isogeny of degree prime to p carries the result from the one specialization to every supersingular point, and to every abelian fourfold with p-divisible group $\mathscr{G}_p$.

Section 2 collects standard facts and the monomial computation, Section 3 derives (1.2) at the supersingular points of Mumford's families, Sections 4 and 5 prove part (a) and Theorem 5.2, Sections 6 and 7 construct and specialize the Weil cycle, Section 8 proves Theorems 1.3 and 8.3 and Proposition 8.4, and Section 9 treats the other monomial modules. Appendix A describes the computations. The proof of Theorem 1.3 does not use Proposition 8.4, Remark 8.6 or Section 9.

What is new. We searched arXiv, Hexagon and the web in October 2026 for integral and crystalline Tate classes of codimension 2 on supersingular abelian varieties, for integral Tate classes on abelian fourfolds over finite fields, and for Mumford's fourfolds at supersingular points. Within that search we found no source that decides the integral crystalline Tate conjecture in codimension 2 for a supersingular abelian variety that is not superspecial, none that computes T^4/S for supersingular Dieudonné modules, and none that uses intersection numbers to fix the p-adic normalization of a specialized Weil cycle. We expect that Proposition 8.4, the equality $T^4 = S$ for superspecial fourfolds, and the uniqueness of the supersingular p-divisible group in Proposition 3.3 are known to experts, but we did not find them stated. The new results are therefore Theorem 1.3, whose part (b) assumes Hypothesis 7.1, Theorem 5.2, the specialization argument of Proposition 7.6, and Proposition 9.3. Proposition 7.3 adapts Markman's argument to integer coefficients, under Hypothesis 7.1.

2. Preliminaries

2.1. **Standard facts.** We use the following facts.

(R1) For an abelian variety X of dimension g over $\bar{\mathbb{F}}_p$, $M(X)$ is free of rank $2g$ and is the contravariant Dieudonné module of $X[p^\infty]$, and $H^\bullet_{\mathrm{cris}}(X/W) = \bigwedge^\bullet M(X)$, both compatibly

with F (see e.g. [15, Examples 1.8(1) and Section 3.4]; for the contravariant Dieudonné module over finite fields, see e.g. [16, (14.3.1)]). Dieudonné theory is an anti-equivalence between p-divisible groups over $\bar{\mathbb{F}}_p$ and Dieudonné modules (see e.g. [17, Corollary 3.16, Definition 3.17 and Proposition 3.18]), and $VM/pM = \ker(F \bmod p)$ is identified with $H^0(X, \Omega^1)$, compatibly with endomorphisms [17, Corollary 5.11].

(R2) A σ^r-semilinear bijection φ of a free W-module R of finite rank has a basis of fixed vectors. Modulo p this is the existence of a basis of fixed vectors for a p^r-linear bijection of a finite-dimensional $\bar{\mathbb{F}}_p$-vector space (see e.g. [18]). If (e_i) is a basis with $\varphi e_i \equiv e_i \bmod p^n$, write $\varphi e_i = e_i + p^n \sum_j a_{ij} e_j$ and choose $b_{ij} \in W$ with $\sigma^r(b_{ij}) - b_{ij} \equiv -a_{ij} \bmod p$, which is possible because $\bar{\mathbb{F}}_p$ is algebraically closed; then $e_i + p^n \sum_j b_{ij} e_j$ is fixed modulo p^{n+1}, and the limit is a basis by Nakayama's lemma. If all Newton slopes of a σ^r-semilinear injective endomorphism F of R are positive, then $F^n R \subseteq pR$ for large n, so its reduction modulo p is nilpotent (see e.g. [19, Section 4.5], which treats σ-semilinear maps; for $r > 1$ apply it to the σ-crystal induced from (R, F), whose slopes are those of F divided by r).

(R3) For abelian varieties X, X' over $\bar{\mathbb{F}}_p$, $\mathrm{Hom}(X', X) \otimes \mathbb{Z}_p \to \mathrm{Hom}(M(X), M(X'))^F$ is injective with torsion-free cokernel (see e.g. [19, proof of Theorem 2.6]). A supersingular abelian variety of dimension g is isogenous to E_0^g [20, Theorem 4.2], so for supersingular X, X' both sides have rank $4g^2$ and the map is an isomorphism. Consequently $\mathrm{NS}(X) \otimes \mathbb{Z}_p \cong T^2(X)$ for every supersingular abelian variety X over $\bar{\mathbb{F}}_p$.

(R4) For smooth projective Y over $\bar{\mathbb{F}}_p$ there is a crystalline cycle class $\mathrm{cl}\colon \mathrm{CH}^i(Y) \to H^{2i}_{\mathrm{cris}}(Y/W) \otimes \mathbb{Q}$, compatible with products and pull-backs [21, I, Theorem 4.3.1] and with push-forwards along morphisms of smooth projective varieties [21, II, Theorem 1.1.3], with $F\,\mathrm{cl}(Z) = p^i\,\mathrm{cl}(Z)$ for Z of codimension i [21, Lemma B.2.2(a) and Remark B.2.3], [22, Proposition 7.13]. If Y is connected, the trace $\int\colon H^{2g}_{\mathrm{cris}}(Y/W) \to W$, $g = \dim Y$, is an isomorphism with $\int Fx = p^g \sigma(\int x)$ (see e.g. [15, Section 1.5(2) and Example 3.2], the latter applied to the pairing of 1 with x; at finite level, see e.g. [21, II, proof of Lemma 1.1.9]), and $\int \mathrm{cl}(Z) = \deg Z$ for 0-cycles Z. Classes of codimension 2 are integral and Tate: $\mathrm{cl}(Z) = -c_2(\mathcal{O}_Z)$ for an integral subscheme Z of codimension 2, with integral crystalline Chern classes [21, I, Theorem 4.3.1], so $\mathrm{Alg}^2(Y) \subseteq T^4(Y)$. See [22, Sections 5–7] for the corresponding statements in rigid cohomology and their comparison with crystalline Chern classes.

(R5) Let $\mathcal{Y}$ be smooth and projective over a complete discrete valuation ring O with fraction field of characteristic 0 and perfect residue field. Specialization of cycles, from the Chow ring with rational coefficients of the generic fiber of $\mathcal{Y}$ to that of its special fiber, maps the class of a subvariety to the class of the special fiber of its closure, and is a ring homomorphism compatible with push-forwards and pull-backs along morphisms of smooth projective O-schemes, in particular with degrees of 0-cycles and with pull-backs by endomorphisms of $\mathcal{Y}$ over O (see e.g. [21, Appendix B, B.3], and [23, Tag 0H4H] for specialization of cycles over a discrete valuation ring).

(R6) For $t \in \mathscr{S}(\bar{\mathbb{F}}_p)$, t lifts to a point of $\mathscr{S}$ over the ring of integers of a finite extension of $W[1/p]$, since $\mathscr{S}$ is smooth [5], and there is an isomorphism $M(\mathcal{A}_t) \cong \Lambda^\vee \otimes W$ that carries the crystalline tensors, which are fixed by F, to the tensors on $\Lambda^\vee$ defining G, extended to $\Lambda^\vee \otimes W$ [24, Section 4.2]. The crystalline tensors correspond to the étale tensors of a lift under the p-adic comparison isomorphism [24, 4.2.1], which is functorial (see e.g. [25, Remark 3.3.10]), so endomorphisms of a lift that fix the étale tensors fix the crystalline tensors. Points of $\mathrm{Sh}_{\mathsf{K}}(G, \mathbb{X})$ over a number field whose abelian variety has good reduction at a prime above p extend to $\mathscr{S}$ over the localization of the ring of integers at that prime. Indeed, $\mathscr{S}$ is the normalization of the closure of $\mathrm{Sh}_{\mathsf{K}}(G, \mathbb{X})$ in the moduli scheme of polarized abelian schemes with prime-to-p level structure [5, (2.3.3) and Theorem (2.3.8)], hence finite over it, as that closure is of finite type over a Dedekind domain of characteristic 0

(see e.g. [23, Tags 0335 and 035S]), and the point extends to that moduli scheme by the Néron property (see e.g. [1, Chapter I, Remark 17.2]). Abelian varieties with complex multiplication have potentially good reduction (see e.g. [26, Proposition 10.5], applied to the simple isogeny factors).

(R7) Skolem–Noether: two homomorphisms from a simple algebra over a field to a central simple algebra Δ over the same field are conjugate by an element of $\Delta^\times$ (see e.g. [27, Main Theorem 7.7.1]).

(R8) Unimodular Hermitian lattices of equal rank over $O_{K,p}$, for p inert in K, are isometric (see e.g. [28, Section 2], where this is attributed to [29]); a Hermitian space over K_p contains a unimodular lattice if and only if its determinant has even valuation, since the norms from $K_p^\times$ are the elements of $\mathbb{Q}_p^\times$ of even valuation (see e.g. [28, Section 2] and [27, 13.2.5]). Let $\mathrm{U}_\mathcal{V}$ be the unitary group of a Hermitian space $\mathcal{V}$ over K, $g_p \in \mathrm{U}_\mathcal{V}(\mathbb{Q}_p)$, and $\Upsilon \subseteq \mathrm{U}_\mathcal{V}(\mathbb{Q}_p)$ an open subgroup containing the scalars of norm 1 in $O_{K,p}^\times$. Then $g_p\Upsilon \cap \mathrm{U}_\mathcal{V}(\mathbb{Q}) \neq \emptyset$. Indeed, $g_p\Upsilon$ contains the products of g_p with the infinitely many scalars of norm 1 in $O_{K,p}^\times$, and only finitely many scalars make -1 an eigenvalue of their product with g_p, so we may assume that $\det(1+g_p) \neq 0$. The Cayley transform $g \mapsto (1-g)(1+g)^{-1}$ is a homeomorphism, inverse to itself, between the open sets where $\det(1+g) \neq 0$ of $\mathrm{U}_\mathcal{V}(\mathbb{Q}_p)$ and of the $\mathbb{Q}_p$-points of the $\mathbb{Q}$-vector space $\mathfrak{u}_\mathcal{V}$ of K-linear maps that are skew-Hermitian for the form, and it maps $\mathfrak{u}_\mathcal{V}(\mathbb{Q})$ into $\mathrm{U}_\mathcal{V}(\mathbb{Q})$; as $\mathfrak{u}_\mathcal{V}(\mathbb{Q})$ is dense in $\mathfrak{u}_\mathcal{V}(\mathbb{Q}_p)$, the open neighborhood $g_p\Upsilon$ of g_p contains a point of $\mathrm{U}_\mathcal{V}(\mathbb{Q})$.

2.2. **Monomial Dieudonné modules.** Let M be a free W-module with basis $(e_s)_{s\in\mathcal{C}}$ and F a σ-semilinear map with $Fe_s = p^{a(s)}e_{\pi(s)}$, where π is a permutation of $\mathcal{C}$ and $a(s) \in \{0,1\}$; assume that on every π-orbit the sum of the $a(s)$ is half the length of the orbit, so that $F^q = p^{q/2}\sigma^q$ in the basis (e_s) for q the order of π, which is even because every orbit has even length. For $J \subseteq \mathcal{C}$ with $|J| = n$ let $e_J \in \bigwedge^n M$ be the wedge of the e_s, $s \in J$, in a fixed order, and $a(J) = \sum_{s\in J} a(s)$. Then $\bigwedge^n F(e_J) = \pm p^{a(J)} e_{\pi(J)}$.

Lemma 2.1 (descent)**.** *Let n be even and $T^n = \{x \in \bigwedge^n M : \bigwedge^n F(x) = p^{n/2}x\}$. Then $T^n \otimes_{\mathbb{Z}_p} W = \bigoplus_{|J|=n} p^{\nu_T(J)} W e_J$ with*

$$\nu_T(J) = \max_{0\le j<q} \sum_{i=1}^{j} \Big(a\big(\pi^{-i}(J)\big) - \frac{n}{2} \Big) , \tag{2.2}$$

and T^n has rank $\binom{|\mathcal{C}|}{n}$. The lattice S^n spanned by products $x \wedge y$, $x \in T^{n_1}$, $y \in T^{n_2}$, $n_1 + n_2 = n$, satisfies $S^n \otimes W = \bigoplus p^{\nu_S(J)} W e_J$ with $\nu_S(J) = \min_{J=J_1\sqcup J_2}(\nu_T(J_1)+\nu_T(J_2))$ over the decompositions with $|J_i| = n_i$, and $[T^n : S^n] = p^{\sum_J(\nu_S(J)-\nu_T(J))}$.

Proof. Put $\varphi = p^{-n/2}\bigwedge^n F$ on $\bigwedge^n M \otimes \mathbb{Q}$; it is σ-semilinear, $\varphi(e_J) = \pm p^{a(J)-n/2} e_{\pi(J)}$, and $\varphi^q = \sigma^q$ because $F^q = p^{q/2}\sigma^q$ on M. Let $\Lambda_\varphi = \bigcap_{0\le j<q} \varphi^j(\bigwedge^n M)$. Since $\varphi^q(\bigwedge^n M) = \bigwedge^n M$, $\varphi(\Lambda_\varphi) = \Lambda_\varphi$, so φ is a σ-semilinear bijection of Λ_φ, and by (R2) $\Lambda_\varphi = \Lambda_\varphi^\varphi \otimes_{\mathbb{Z}_p} W$. A φ-fixed vector of $\bigwedge^n M$ lies in every $\varphi^j(\bigwedge^n M)$, so $\Lambda_\varphi^\varphi = T^n$ and $T^n \otimes W = \Lambda_\varphi$. Each $\varphi^j(\bigwedge^n M)$ is monomial, with exponent $\sum_{i=1}^j (a(\pi^{-i}J) - n/2)$ at e_J, and an intersection of monomial lattices is monomial with the maximal exponents, which is (2.2). The product of $p^{\nu_T(J_1)}e_{J_1}$ and $p^{\nu_T(J_2)}e_{J_2}$ is $\pm p^{\nu_T(J_1)+\nu_T(J_2)}e_{J_1\cup J_2}$ if $J_1 \cap J_2 = \emptyset$ and 0 otherwise; spans commute with $\otimes W$, which gives $\nu_S(J)$. Finally W is faithfully flat over $\mathbb{Z}_p$, so the length of T^n/S^n over $\mathbb{Z}_p$ equals the length of $(T^n/S^n)\otimes W$ over W. □

The same argument applies to Hom-lattices: $\mathrm{End}(M,F)\otimes W$ is the intersection of the lattices $F^j \mathrm{End}_W(M) F^{-j}$, $0 \le j < q$, and is monomial in the matrix units of the basis (e_s). If $U_1, \dots, U_4$ are monomial multiples of matrix units, then the polarized form $D(U_1,\dots,U_4)$ of $U \mapsto \bigwedge^4 U$ maps each

e_J to a signed monomial or to 0. Since 4! is a unit, the $\mathbb{Z}_p$-span of the $\bigwedge^4 U(x)$, $U \in \mathrm{End}(M,F)$, x in a monomial lattice, equals the $\mathbb{Z}_p$-span of the values of D, and after $\otimes W$ it is the monomial lattice whose exponents are the minima over all such monomial values. In particular every lattice in Sections 4–5 and 9 is monomial, and its exponents are min-plus expressions in the integers $a(s)$, independent of $p \geq 5$.

3. The Dieudonné module at a supersingular point

We show that all supersingular points of $\mathscr{S}(\bar{\mathbb{F}}_p)$ have the Dieudonné module (1.2), starting from the hyperspecial lattice.

Lemma 3.1. *Assume (h2)–(h4), let $\mathcal{G}$ be the closure of G in $\mathrm{GL}(\Lambda \otimes \mathbb{Z}_{(p)})$, write $G(W) = \mathcal{G}(W)$, and let $\Lambda_1, \Lambda_2, \Lambda_3$ be the base changes of Λ_0 to W along the three embeddings $O_{k_p} \to W$. Then $\mathcal{G}$ is reductive, $\mathcal{G}_W$ is $\mathbb{G}_m \cdot \mathrm{SL}_2^3$ acting on $\Lambda_1 \otimes \Lambda_2 \otimes \Lambda_3$, and $G(\breve{\mathbb{Q}}_p)$, $G(W)$ are the images of $\mathrm{GL}_2(\breve{\mathbb{Q}}_p)^3$, $\mathrm{GL}_2(W)^3$. The $G(W)$-stable lattices in $\Lambda \otimes \breve{\mathbb{Q}}_p$ are the $c(\Lambda \otimes W)$, $c \in \breve{\mathbb{Q}}_p^\times$. The same holds for $\Lambda^\vee$, with Λ_0 replaced by its dual.*

Proof. Since p is inert in k, $O_{k_p} \otimes_{\mathbb{Z}_p} W = W^3$ through the three embeddings, so $\Lambda \otimes W = p^j\Lambda_1 \otimes \Lambda_2 \otimes \Lambda_3$ and $G_{\breve{\mathbb{Q}}_p}$ is the image of GL_2^3. The kernel of $\mathrm{GL}_{2,W}^3 \to \mathrm{GL}(\Lambda_1 \otimes \Lambda_2 \otimes \Lambda_3)$ is the split torus $\mathcal{T}$ of triples of scalars with product 1, as one sees by comparing matrix entries of $\gamma_1 \otimes \gamma_2 \otimes \gamma_3$. The induced monomorphism of the reductive group $\mathrm{GL}_{2,W}^3/\mathcal{T}$ into $\mathrm{GL}(\Lambda \otimes W)$ is a closed immersion by [30, Exp. XVI, Corollaire 1.5(a)]. Its image is flat and closed with generic fiber $G_{\breve{\mathbb{Q}}_p}$, hence it is the closure of $G_{\breve{\mathbb{Q}}_p}$, which is $\mathcal{G}_W$ because forming the closure commutes with the flat base change $\mathbb{Z}_{(p)} \to W$ (see e.g. [23, Tag 081I]); so $\mathcal{G}_W$, and with it $\mathcal{G}$, is reductive. As $\mathcal{T}$ is a split torus and $\mathrm{Pic}(W) = 0$, Hilbert's Theorem 90 shows that $\mathrm{GL}_2(W)^3 \to \mathcal{G}(W)$ and $\mathrm{GL}_2(\breve{\mathbb{Q}}_p)^3 \to G(\breve{\mathbb{Q}}_p)$ are surjective. The reduction of $\Lambda_1 \otimes \Lambda_2 \otimes \Lambda_3$ modulo p is an external tensor product of standard representations of $\mathrm{SL}_2(\bar{\mathbb{F}}_p)^3$, hence irreducible, and $\mathrm{GL}_2(W)^3 \to \mathrm{GL}_2(\bar{\mathbb{F}}_p)^3$ is surjective. So a $G(W)$-stable lattice, scaled to lie in $\Lambda \otimes W$ but not in $p(\Lambda \otimes W)$, maps onto $\Lambda \otimes \bar{\mathbb{F}}_p$, and it is $\Lambda \otimes W$ by Nakayama's lemma. For $\Lambda^\vee$, note that $\Lambda^\vee \otimes W = p^{-j}\Lambda_1^\vee \otimes \Lambda_2^\vee \otimes \Lambda_3^\vee$. □

Lemma 3.2. *Let R be free of rank 2 over W and F_R a σ^r-semilinear map with $pR \subset F_R R \subset R$, $R/F_R R$ of length 1, and all Newton slopes positive. Then $(R, F_R) \cong (W^2, A\sigma^r)$ with A as in (1.2).*

Proof. Put $V = pF_R^{-1}$, so that $F_R R/pR$ is the image and VR/pR the kernel of F_R mod p. Both are lines in R/pR. If they differed, F_R mod p would map its image isomorphically onto itself and would not be nilpotent, contradicting (R2). Hence $F_R R = VR$ and $F_R^2 R = pR$, so $\Pi = p^{-1}F_R^2$ is a σ^{2r}-semilinear bijection of R, and by (R2) $R = R_0 \otimes_{\mathbb{Z}_{p^{2r}}} W$ with $R_0 = R^\Pi$, which F_R preserves. Choose $e_0 \in R_0 \setminus F_R R_0$ and put $e_1 = F_R e_0$. Then $F_R e_1 = pe_0$, and $e_1 \notin pR_0$ since otherwise $e_0 \in VR_0 = F_R R_0$. By Nakayama's lemma (e_0, e_1) is a basis. □

Proposition 3.3. *Assume (h2)–(h4) and let $t \in \mathscr{S}(\bar{\mathbb{F}}_p)$. Then $M = M(\mathcal{A}_t)$ is either μ-ordinary, with slopes $0, \frac{1}{3}, \frac{2}{3}, 1$ of multiplicities $1, 3, 3, 1$, or supersingular. In the supersingular case (M, F) is isomorphic to the model (1.2). In particular all supersingular $\mathcal{A}_t$, $t \in \mathscr{S}(\bar{\mathbb{F}}_p)$, have p-divisible group isomorphic to $\mathscr{G}_p$.*

For Mumford's families, that the Newton polygon jumps only at supersingular points is stated, under a mild assumption on p, in [4]. We include a proof because the model (1.2) is needed.

Proof. By (R6) and Lemma 3.1, $M \cong \Lambda^\vee \otimes W = p^{-j}\Lambda_1^\vee \otimes \Lambda_2^\vee \otimes \Lambda_3^\vee$, compatibly with the tensors. Since p is inert in k, σ permutes the three embeddings $O_{k_p} \to W$ cyclically, and after numbering them suitably $1 \otimes \sigma$ acts on $\Lambda^\vee \otimes W$ as $\sigma_\Lambda(x_1 \otimes x_2 \otimes x_3) = \sigma x_3 \otimes \sigma x_1 \otimes \sigma x_2$ in bases of the $\Lambda_i^\vee$ related by σ. Both F and σ_Λ fix the tensors, so $F = b\sigma_\Lambda$ with $b \in G(\breve{\mathbb{Q}}_p)$, and by Lemma 3.1 $b = b_1 \otimes b_2 \otimes b_3$ with $b_i \in \mathrm{GL}_2(\breve{\mathbb{Q}}_p)$. Since $\dim \mathcal{A}_t = 4$, F has elementary divisors $(1^4, p^4)$ on M.

If b_i has Cartan exponents (ν_i, ν_i'), the exponents of the elementary divisors of F are the eight sums of one of ν_i, ν_i' for each i, so $\sum_i |\nu_i - \nu_i'| = 1$: exactly one b_i is not a scalar times an element of $\mathrm{GL}_2(W)$. Numbering the slots cyclically we may assume it is b_2, and after moving scalars, $b_2 \in \mathrm{GL}_2(W)\mathrm{diag}(1,p)\,\mathrm{GL}_2(W)$ and $b_1, b_3 \in \mathrm{GL}_2(W)$.

Conjugation by $\gamma = \gamma_1 \otimes \gamma_2 \otimes \gamma_3 \in G(W)$ replaces b_i by $\gamma_i b_i \sigma(\gamma_{i-1})^{-1}$, indices modulo 3. Taking $\gamma_2 = 1$, $\gamma_3 = \sigma(\gamma_2) b_3^{-1}$ and $\gamma_1 = \sigma(\gamma_3) b_1^{-1}$ makes $b_1 = b_3 = 1$, and then F^3 preserves $\Lambda_2^\vee$, on which it is $F_2 = b_2\sigma^3$, and $F^3 = \sigma^2 F_2 \otimes F_2 \otimes \sigma F_2$ in the evident sense. The rank-2 σ^3-crystal $(\Lambda_2^\vee, F_2)$ has Hodge slopes $0, 1$, so its Newton slopes are $\theta, 1-\theta$ with $\theta \in \{0, \frac{1}{2}\}$, and the slopes of F are $\frac{1}{3}\{3\theta, 1+\theta, 2-\theta, 3-3\theta\}$ with multiplicities $1, 3, 3, 1$. So $\theta = 0$ is the μ-ordinary case and $\theta = \frac{1}{2}$ the supersingular one. In the latter, Lemma 3.2 with $r = 3$ gives $\gamma_2 \in \mathrm{GL}_2(W)$ with $\gamma_2 b_2 \sigma^3(\gamma_2)^{-1} = A$, and conjugation by $\sigma^2(\gamma_2) \otimes \gamma_2 \otimes \sigma(\gamma_2)$ keeps $b_1 = b_3 = 1$ and makes $b_2 = A$. Multiplication by p^j, which commutes with F, then identifies (M, F) with (1.2). The last assertion follows from Dieudonné theory (R1). □

4. Tate lattices of the model

In the model (1.2) put $e_s = e_{s_1} \otimes e_{s_2} \otimes e_{s_3}$ for $s \in \mathcal{C} = \{0,1\}^3$, with $Ae_0 = e_1$ and $Ae_1 = pe_0$. Then

$$Fe_s = p^{s_1} e_{\pi(s)}\ , \qquad \pi(s_1, s_2, s_3) = (s_3, 1 - s_1, s_2)\ , \tag{4.1}$$

and π has the orbits $(000, 010, 011, 111, 101, 100)$ and $(001, 110)$, on which the exponents $a(s) = s_1$ read 000111 and 01. So Section 2.2 applies with $q = 6$, and $\ker(F \bmod p)$ is spanned by the e_s with $s_1 = 1$. Consider the four subsets of $\mathcal{C}$

$$\begin{aligned} \mathsf{T}_+ &= \{000, 011, 101, 110\}\ , \qquad \mathsf{T}_- = \{001, 010, 100, 111\}\ , \\ \mathsf{M}_+ &= \{000, 001, 011, 101\}\ , \qquad \mathsf{M}_- = \{010, 100, 110, 111\}\ . \end{aligned} \tag{4.2}$$

Here T_+ and T_- are the two regular tetrahedra inscribed in the cube, and M_+, M_- are the sets on which the majority of the coordinates of $(-1)^s$ is $+1$, respectively -1, after the change of coordinates $s \mapsto (s_1, s_2, 1 - s_3)$.

Proposition 4.3 (computer-assisted)**.** *For the model* (1.2)*, $\nu_S(J) = \nu_T(J)$ for every 4-set $J \subseteq \mathcal{C}$ except the four sets* (4.2)*, where $(\nu_T(J), \nu_S(J)) = (0,1)$ for $J = \mathsf{T}_+, \mathsf{T}_-, \mathsf{M}_-$ and $(\nu_T(J), \nu_S(J)) = (1,2)$ for $J = \mathsf{M}_+$. Hence $T^4/S \cong (\mathbb{Z}/p)^4$, spanned by the images of $p^{\nu_T(J)} e_J$, $J \in \{\mathsf{T}_+, \mathsf{T}_-, \mathsf{M}_+, \mathsf{M}_-\}$, in $(T^4/S) \otimes W$. Moreover T^6 is spanned by products of three classes in T^2.*

Proof. Lemma 2.1 reduces the statement to the values of (2.2) and of $\nu_S(J)$ for the $\binom{8}{2} = 28$ two-sets and $\binom{8}{4} = 70$ four-sets, an exact integer computation (Appendix A). The quotient T^4/S has length 4, and it is killed by p because every defect $\nu_S(J) - \nu_T(J)$ is 1. □

Lemma 4.4. *Let K be imaginary quadratic with p inert in K, and suppose that $O_{K,p} = \mathbb{Z}_{p^2}$ acts on (M, F) in the model by endomorphisms that are diagonal in the basis (e_s), with characters $\iota_s \in \{\iota, \bar\iota\}$. Let $J_\iota = \{s : \iota_s = \iota\}$.*

- *(i) J_ι is one of the four sets* (4.2)*, and $\pi(J_\iota)$ is its complement.*
- *(ii) If the action is the reduction of an action of O_K on an abelian scheme over a discrete valuation ring of characteristic 0 whose generic fiber has signature $(2,2)$, then $J_\iota \in \{\mathsf{T}_+, \mathsf{T}_-\}$.*
- *(iii) The Weil part $\bigwedge^4_W M_\iota \oplus \bigwedge^4_W M_{\bar\iota}$ of $\bigwedge^4 M$, where M_ι and $M_{\bar\iota}$ are the eigenspaces of the characters ι and $\bar\iota$, is $We_{\mathsf{T}_+} \oplus We_{\mathsf{T}_-}$ in case (ii). The lattice $\mathcal{L} = T^4 \cap (We_{\mathsf{T}_+} \oplus We_{\mathsf{T}_-})$ has rank 2, $\mathcal{L} \otimes W = We_{\mathsf{T}_+} \oplus We_{\mathsf{T}_-}$, and every $x \in \mathcal{L} \setminus p\mathcal{L}$ is $\lambda e_{\mathsf{T}_+} \pm \sigma(\lambda) e_{\mathsf{T}_-}$ with $\lambda \in W^\times$.*

Proof. For $\alpha \in \mathbb{Z}_{p^2}$, $F(\iota_s(\alpha) e_s) = \sigma(\iota_s(\alpha)) p^{s_1} e_{\pi(s)}$ must equal $\iota_{\pi(s)}(\alpha) p^{s_1} e_{\pi(s)}$, so $\iota_{\pi(s)} = \sigma \circ \iota_s = \bar\iota_s$ because p is inert in K. So membership in J_ι alternates along both π-orbits: J_ι contains $\{000, 011, 101\}$ or $\{010, 111, 100\}$, and 001 or 110. The four resulting sets are (4.2), which proves (i).

For (ii), $\ker(F \bmod p) = VM/pM \cong H^0(\Omega^1)$ by (R1), and the eigen-ranks of O_K on the cotangent space of an abelian scheme with O_K-action are locally constant, p being unramified in K. So $|J_\iota \cap \{s_1 = 1\}| = 2$, while this number is $2, 2, 1, 3$ for $\mathsf{T}_+, \mathsf{T}_-, \mathsf{M}_+, \mathsf{M}_-$. For (iii), M_ι is spanned by the e_s, $s \in J_\iota$, so $\bigwedge^4 M_\iota = We_{J_\iota}$. The W-module $We_{\mathsf{T}_+} \oplus We_{\mathsf{T}_-}$ is stable under $\varphi = p^{-2}\bigwedge^4 F$, which exchanges e_{T_+} and e_{T_-} up to sign since $a(\mathsf{T}_+) = a(\mathsf{T}_-) = 2$; so the proof of Lemma 2.1 applies to it, with $\nu_T(\mathsf{T}_+) = \nu_T(\mathsf{T}_-) = 0$. A φ-fixed vector $\lambda e_{\mathsf{T}_+} + \lambda' e_{\mathsf{T}_-}$ satisfies $\lambda' = \pm\sigma(\lambda)$, and it lies outside $p(\mathcal{L} \otimes W)$ if and only if $\lambda \in W^\times$. □

Let $\mathcal{N} = T^4 \cap (We_{\mathsf{M}_+} \oplus We_{\mathsf{M}_-})$, a lattice of rank 2 with $\mathcal{N} \otimes W = pWe_{\mathsf{M}_+} \oplus We_{\mathsf{M}_-}$ by the same argument, since $\pi(\mathsf{M}_+) = \mathsf{M}_-$.

5. Endomorphisms and the extension

By (R3), $\mathrm{End}(X) \otimes \mathbb{Z}_p = \mathrm{End}(M, F)$ for every supersingular X with $M(X) \cong M$, and the $\mathbb{Z}_p$-span of $\{U^*x\}$ over $U \in \mathrm{End}(X)$ equals the span over $U \in \mathrm{End}(X) \otimes \mathbb{Z}_p$, because $U \mapsto \bigwedge^4 U(x)$ is polynomial and spans of finitely generated $\mathbb{Z}_p$-modules are closed. So the following proposition is about $\mathrm{End}(M, F)$.

Proposition 5.1 (computer-assisted)**.** *In the model* (1.2)*, with* $\mathrm{End} = \mathrm{End}(M, F)$ *acting on* $\bigwedge^4 M$ *by* $U \mapsto \bigwedge^4 U$*:*

(i) $\mathrm{End}^* S \subseteq S$*;*
(ii) $[T^4 : S + \mathcal{L}] = p^2$ *and* $S + \mathrm{End}^* \mathcal{L} = T^4$*;*
(iii) $\mathrm{End}^* \mathcal{N} \subseteq S + \mathcal{N}$ *and* $[T^4 : S + \mathcal{N}] = p^2$*;*
(iv) let T_{22} *be the image of* $T^4(X \times X)$ *under the Künneth projection to* $\bigwedge^2 M \otimes \bigwedge^2 M$*. Then* T_{22} *is spanned by the Künneth* $(2,2)$*-components of* $S(X \times X)$ *and of the classes* $(U_1, U_2)^* x$*,* $x \in \mathcal{L}$*, for the homomorphisms* $(U_1, U_2)\colon X \times X \to X$*,* $(y_1, y_2) \mapsto U_1 y_1 + U_2 y_2$*,* $U_i \in \mathrm{End}$*.*

Proof. By Section 2.2 every lattice here is monomial with exponents given by min-plus expressions in the $a(s)$; their values are an exact integer computation (Appendix A). □

Theorem 5.2. *Let* X *be an abelian fourfold over* $\bar{\mathbb{F}}_p$ *with* $M(X) \cong M$ *as in* (1.2)*. Then* $T^4/S \cong (\mathbb{Z}/p)^4$*,* $(S + \mathcal{N})/S \cong (\mathbb{Z}/p)^2$ *is stable under* $\mathrm{End}(X)$*,* $T^4/(S + \mathcal{N})$ *is the image of* $\mathcal{L}$*, and for every* $x \in T^4 \setminus (S + \mathcal{N})$ *the classes* S *and* U^*x*,* $U \in \mathrm{End}(X)$*, span* T^4*. Hence* T^4/S *is a non-split extension of* $T^4/(S + \mathcal{N}) \cong (\mathbb{Z}/p)^2$ *by* $(S + \mathcal{N})/S$ *as a module over* $\mathrm{End}(X)$*. Here* $\mathcal{L}$ *is the lattice of integral Weil classes of the diagonal* $\mathbb{Z}_{p^2}$*-action with eigenset* T_+ *(Lemma 4.4(iii)), and* $((S + \mathcal{N})/S) \otimes W$ *is spanned by the images of* pe_{M_+} *and* e_{M_-}*, where* M_+ *and* M_- *are the eigensets of Lemma 4.4(i) that meet* $\{s : s_1 = 1\}$ *in* 1 *and* 3 *elements (proof of Lemma 4.4(ii)).*

Proof. The first statement is Proposition 4.3, the stability is Proposition 5.1(i), (iii), and $(S+\mathcal{N})/S \cong (\mathbb{Z}/p)^2$ because, by Proposition 4.3 and the exponents of $\mathcal{N}$, the monomial lattices $(S + \mathcal{N}) \otimes W$ and $S \otimes W$ differ only at e_{M_+} and e_{M_-}, each by one power of p. Since $S \otimes W$ has exponent 1 at e_{T_+} and e_{T_-}, $(S + \mathcal{N}) \cap \mathcal{L} = p\mathcal{L}$ and $S + \mathcal{N} + \mathcal{L} = T^4$. Let $x \in T^4 \setminus (S + \mathcal{N})$; then $x \equiv y \bmod S + \mathcal{N}$ with $y \in \mathcal{L} \setminus p\mathcal{L}$, so $y = \lambda e_{\mathsf{T}_+} \pm \sigma(\lambda) e_{\mathsf{T}_-}$ with $\lambda \in W^\times$ by Lemma 4.4(iii). The diagonal action of $\mathbb{Z}_{p^2}$ with characters ι on T_+ and $\bar{\iota}$ on $\mathsf{T}_- = \pi(\mathsf{T}_+)$ commutes with F, as in the proof of Lemma 4.4, so it lies in $\mathrm{End}(M, F)$, and its idempotent $P_\iota \in \mathrm{End}(M, F) \otimes W$ is the projection onto the span of the e_s, $s \in \mathsf{T}_+$. Then $\bigwedge^4 P_\iota$ is the coordinate projection onto We_{T_+}; it preserves $S \otimes W$ and kills e_{M_+}, e_{M_-}, so $\bigwedge^4 P_\iota(x) \equiv \lambda e_{\mathsf{T}_+} \bmod S \otimes W$. By polarization, $\bigwedge^4 P_\iota(x)$ lies in the W-span of the $\bigwedge^4 U(x)$, $U \in \mathrm{End}(M, F)$. With the conjugate idempotent the same holds for e_{T_-}, so $(S + \mathrm{End}^* x) \otimes W \supseteq \mathcal{L} \otimes W$, hence $S + \mathrm{End}^* x \supseteq \mathcal{L}$ by faithful flatness, and Proposition 5.1(ii) gives $S + \mathrm{End}^* x = T^4$. If the extension split, a complement of $(S + \mathcal{N})/S$ would be an $\mathrm{End}(X)$-stable subgroup of order p^2 containing the image of some $x \notin S + \mathcal{N}$, whereas the $\mathrm{End}(X)$-translates of x span all of T^4/S. □

6. A fourfold with complex multiplication

Assume (h1)–(h4). The proof of Theorem 8.3 needs one supersingular point with an integral action of an inert imaginary quadratic field of signature $(2,2)$ that lifts to characteristic 0 together with a unimodular Hermitian structure.

Proposition 6.1. *There are an imaginary quadratic field K in which p is inert, a point $c \in \mathrm{Sh}_{\mathsf{K}}(G,\mathbb{X})(\bar{\mathbb{Q}})$ with abelian variety X_c, and an embedding of $K \times L$ in $\mathrm{End}(X_c)\otimes\mathbb{Q}$, $L = kK$, such that:*

(a) $(O_K \times O_L)\otimes\mathbb{Z}_p$ preserves $H_1(X_c,\mathbb{Z}_p)$;
(b) an embedding $K \hookrightarrow K\times L$ makes X_c of Weil type for K with signature $(2,2)$;
(c) c reduces to a point $t_0 \in \mathscr{S}(\bar{\mathbb{F}}_p)$ with $X_{t_0} = \mathcal{A}_{t_0}$ supersingular, and $K\times L$ acts on X_{t_0};
(d) after an automorphism of $(M(X_{t_0}),F)$ and the identification of Proposition 3.3, $(O_K \times O_L)\otimes\mathbb{Z}_p$ acts diagonally on the basis (e_s), and the ι-eigenset of $O_{K,p}$ is T_+ or T_-.

Proof. Choice of K. Let $\mathrm{Ram}(B)$ be the set of finite places of k at which B ramifies; p lies below none of them by (h3). For each prime ℓ below a place in $\mathrm{Ram}(B)$, choose a quadratic extension K_ℓ of $\mathbb{Q}_\ell$ contained in no $k_{\mathfrak{q}}$, $\mathfrak{q}\mid\ell$, $\mathfrak{q}\in\mathrm{Ram}(B)$; this is possible because at most one such $k_{\mathfrak{q}}$ is quadratic over $\mathbb{Q}_\ell$ and $\mathbb{Q}_\ell$ has at least three quadratic extensions. Choose K imaginary quadratic, inert at p and with completion K_ℓ at each such ℓ (finitely many local conditions). Then $L\otimes_k k_{\mathfrak{q}}$ is a field at every place $\mathfrak{q}$ where B ramifies, including the two ramified real places, where it is $\mathbb{C}$; so L embeds in B (see e.g. [27, Proposition 14.6.7]). Let L^1 be the kernel of the norm $L^\times\to k^\times$ and $\mathbb{T} = \mathbb{G}_m\cdot\mathrm{Res}_{k/\mathbb{Q}}L^1\subset G$, a maximal torus with $\mathbb{T}_{\mathbb{R}}$ compact modulo the center. Some $h\in\mathbb{X}$ factors through $\mathbb{T}_{\mathbb{R}}$. Indeed, G is the image of $\mathbb{G}_m\times B^1$, and $\mathbb{X}$ is the $G(\mathbb{R})$-conjugacy class of the morphism h_0 induced by the map $\mathbb{R}^\times\times S^1\to\mathbb{R}^\times\times B^1(\mathbb{R})$ that is the identity on $\mathbb{R}^\times$ and maps S^1 into $B^1(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})\times\mathrm{SU}(2)^2$, trivially at the two ramified real places and, at the real place ∞_1 where B splits, by the restriction to S^1 of the embedding $\mathbb{C}\to\mathrm{Mat}_2(\mathbb{R})$, $x+iy\mapsto\left(\begin{smallmatrix}x & y\\ -y & x\end{smallmatrix}\right)$ (see e.g. [4]). By (R7), the embedding of $L\otimes_{k,\infty_1}\mathbb{R}\cong\mathbb{C}$ in $B\otimes_{k,\infty_1}\mathbb{R}\cong\mathrm{Mat}_2(\mathbb{R})$ is conjugate to this one by some $\gamma_\infty\in\mathrm{GL}_2(\mathbb{R})$; since $\mathrm{diag}(1,-1)$ normalizes the image of the latter, γ_∞ can be taken in $\mathrm{SL}_2(\mathbb{R})$. Conjugating h_0 by the image in $G(\mathbb{R})$ of $(1,\gamma_\infty,1,1)$ gives $h\in\mathbb{X}$ with $h(S^1)$ in the image of $(L\otimes_{k,\infty_1}\mathbb{R})^1\times\{1\}\times\{1\}$, so h factors through $\mathbb{T}_{\mathbb{R}}$.

The algebra $K\times L$. Over $\bar{\mathbb{Q}}$, $\mathbb{T}$ acts on $\mathbb{U}_1\otimes\mathbb{U}_2\otimes\mathbb{U}_3$ through eight characters indexed by sign vectors $\epsilon\in\{\pm\}^3$, the sign in slot i recording which eigenline of L in $\mathbb{U}_i$ is used. The absolute Galois group permutes the slots through its action on the embeddings of k and changes all signs at once exactly when it acts non-trivially on K. So the orbits are $\{+++,---\}$, with stabilizer $\mathrm{Gal}(\bar{\mathbb{Q}}/K)$, and the remaining six vectors, with stabilizers the conjugates of $\mathrm{Gal}(\bar{\mathbb{Q}}/L)$; hence $\mathbb{Q}[\mathbb{T}(\mathbb{Q})]\cong K\times L$, and we identify the two. An embedding $K\hookrightarrow K\times L$ acts on the eigenline of ϵ through ι or $\bar\iota$, by a rule invariant under permutations of the slots and reversed by changing all signs. On the six-element orbit the rule is therefore given by the majority sign, and the two possibilities on $\{+++,---\}$ give the two K-structures, whose ι-eigensets are, up to complement, $\{\epsilon:\epsilon_1\epsilon_2\epsilon_3=+\}$ and $\{\epsilon:\text{majority}=+\}$. The Hodge cocharacter is non-trivial only in the slot of the non-compact real place, so $H^{1,0}$ is spanned by the lines with a fixed sign in that slot, and the two structures have signatures $(2,2)$ and $(1,3)$. We use the first, which gives (b).

Integrality. Since p is inert in k and in K, $L_p = L\otimes\mathbb{Q}_p\cong\mathbb{Q}_{p^6}$, which embeds in $B\otimes_k k_p = \mathrm{End}_{k_p}(\mathbb{U}_p)$ as a maximal subfield, so $\mathbb{U}_p$ is a one-dimensional L_p-space. Let $\Lambda_0'\subset\mathbb{U}_p$ be an O_{L_p}-lattice and $\Lambda' = \mathrm{Cor}(\Lambda_0')$. Since L_p/k_p is unramified, the base change of Λ_0' to W along each embedding $O_{k_p}\to W$ is $W\times W$, compatibly with the two eigenlines, so $\Lambda'\otimes W$ is the direct sum of the integral parts of the eight weight lines, on each of which $(O_K\times O_L)\otimes\mathbb{Z}_p$ acts through an integral character; hence $(O_K\times O_L)\otimes\mathbb{Z}_p$ preserves Λ'. By (h4), $\Lambda\otimes\mathbb{Z}_p = p^j\,\mathrm{Cor}(\Lambda_0)$, and every lattice of $\mathbb{U}_p$ is $g\Lambda_0$ for some $g\in\mathrm{GL}_2(k_p) = \mathrm{GL}(\mathbb{U}_p)$, whose tensor induction lies in $G(\mathbb{Q}_p)$: G is also the image of $\mathrm{Res}_{k/\mathbb{Q}}B^\times$, which is generated by B^1 and its center $\mathrm{Res}_{k/\mathbb{Q}}\mathbb{G}_m$, and the center

acts on $\mathbb{V}$ through the norm, which maps it onto $\mathbb{G}_m$. So $\Lambda' = \gamma_p(\Lambda \otimes \mathbb{Z}_p)$ for some $\gamma_p \in G(\mathbb{Q}_p)$, the scalar p^{-j} being absorbed into γ_p, as G contains the scalars. The point $c = [h, (\gamma_p, 1)]$ has $H_1(X_c, \mathbb{Z}_p) = \Lambda'$, which gives (a).

Reduction. Since X_c has complex multiplication by $K \times L$, by (R6) c extends, after a finite extension of the base, to a point of $\mathscr{S}$ with special point t_0; the endomorphisms extend by the Néron property. If t_0 were μ-ordinary, the slope-0 part of $M(X_{t_0}) \otimes \mathbb{Q}$ would be an F-isocrystal of rank 1 preserved by $K \times L$, whose endomorphism algebra is the field $\mathbb{Q}_p$ of elements of $\breve{\mathbb{Q}}_p$ fixed by σ; but $(K \times L) \otimes \mathbb{Q}_p \cong \mathbb{Q}_{p^2} \times \mathbb{Q}_{p^6}$ has no ring homomorphism to $\mathbb{Q}_p$. By Proposition 3.3, t_0 is supersingular, which gives (c).

Diagonal action. The elements of $\mathbb{T}(\mathbb{Q})$ fix the étale tensors, so by (R6) they act on $M(X_{t_0}) \otimes \mathbb{Q}$ through the group $G_F = \{\gamma \in G(\breve{\mathbb{Q}}_p) : \gamma F = F\gamma\}$. In the model, writing $\gamma = \gamma_1 \otimes \gamma_2 \otimes \gamma_3$ and comparing $\gamma F \gamma^{-1}$ with F shows that, up to a scalar, $\gamma = \sigma^2(\gamma') \otimes \gamma' \otimes \sigma(\gamma')$ with γ' in the centralizer Δ of $A\sigma^3$ in $\mathrm{Mat}_2(\breve{\mathbb{Q}}_p)$. Comparing entries in $\gamma' A = A\sigma^3(\gamma')$ shows that Δ consists of the matrices $\left(\begin{smallmatrix} y & py' \\ \sigma^3(y') & \sigma^3(y) \end{smallmatrix}\right)$ with $y, y' \in \mathbb{Q}_{p^6}$, so it is the quaternion division algebra over $\mathbb{Q}_{p^3}$ (see e.g. [27, Corollary 13.3.14]). Let $\Delta_{\mathbb{T}} \subseteq \Delta$ be the $\mathbb{Q}_{p^3}$-span of these γ', for all $\gamma \in \mathbb{T}(\mathbb{Q})$. It is commutative. Indeed, the elements of $\mathbb{T}(\mathbb{Q})$ commute, so for two of them, γ and $\tilde{\gamma}$, with slot-2 components γ' and $\tilde{\gamma}'$ as above, $\gamma'\tilde{\gamma}'$ and $\tilde{\gamma}'\gamma'$ differ by a scalar factor, which is ± 1 by comparing determinants. The corresponding factors in slots 1 and 3 are its images under σ^2 and σ, that is, itself, and the kernel of $\mathrm{GL}_2^3 \to \mathrm{GL}(M \otimes \breve{\mathbb{Q}}_p)$ consists of the triples of scalars with product 1 (proof of Lemma 3.1), so its cube is 1 and it is 1. As Δ is a division algebra, $\Delta_{\mathbb{T}}$ is a field, and its degree over $\mathbb{Q}_{p^3}$ divides $\dim_{\mathbb{Q}_{p^3}} \Delta = 4$ and is not 4. The $\breve{\mathbb{Q}}_p$-span of the action of $\mathbb{T}(\mathbb{Q})$ on $M \otimes \breve{\mathbb{Q}}_p$ is the image of $(K \times L) \otimes \breve{\mathbb{Q}}_p \cong \breve{\mathbb{Q}}_p^8$, which acts faithfully, as $K \times L$ does on X_{t_0}, by (R3) and Section 2.2. It lies in the tensor product of the $\breve{\mathbb{Q}}_p$-spans of $\sigma^2(\Delta_{\mathbb{T}})$, $\Delta_{\mathbb{T}}$ and $\sigma(\Delta_{\mathbb{T}})$ in $\mathrm{Mat}_2(\breve{\mathbb{Q}}_p)$, which are commutative and so of dimension at most 2. Hence these spans have dimension 2, their tensor product is that image, and the $\breve{\mathbb{Q}}_p$-span of $\Delta_{\mathbb{T}}$, being a subalgebra of an algebra isomorphic to $\breve{\mathbb{Q}}_p^8$, is isomorphic to $\breve{\mathbb{Q}}_p \times \breve{\mathbb{Q}}_p$. So $\Delta_{\mathbb{T}}$ has degree 2 over $\mathbb{Q}_{p^3}$ and $\Delta_{\mathbb{T}} \otimes_{\mathbb{Q}_{p^3}} \breve{\mathbb{Q}}_p$ is not a field, that is, $\Delta_{\mathbb{T}}$ is a maximal subfield of Δ isomorphic to $\mathbb{Q}_{p^6}$. Another one is $\{\mathrm{diag}(y, \sigma^3 y) : y \in \mathbb{Q}_{p^6}\}$, which commutes with $A\sigma^3$. By (R7) they are conjugate by an element of $\Delta^\times = A^{\mathbb{Z}} O_\Delta^\times$, and A normalizes the diagonal subfield, so they are conjugate by some $\gamma_0' \in O_\Delta^\times$. The automorphism $\sigma^2(\gamma_0') \otimes \gamma_0' \otimes \sigma(\gamma_0')$ of (M, F) is integral, and after applying it the slot components of the elements of $\mathbb{T}(\mathbb{Q})$ are diagonal. So $\mathbb{T}(\mathbb{Q})$, and with it its $\mathbb{Q}$-span $K \times L$, $(K \times L) \otimes \mathbb{Q}_p$ and $(O_K \times O_L) \otimes \mathbb{Z}_p$, act diagonally on the e_s. Lemma 4.4(ii) applied to the K-structure of (b) gives (d). □

Lemma 6.2. *In Proposition 6.1, $H_1(X_c, \mathbb{Z}_{(p)})$ is an $O_{K,(p)}$-module, and for a suitable polarization of X_c it is a unimodular Hermitian $O_{K,(p)}$-lattice.*

Proof. By Proposition 6.1(a), $\mathrm{End}(X_c) \cap K$ is an order of conductor prime to p, so $O_{K,(p)}$ acts. Let ψ_c be the Riemann form of a polarization whose class is a multiple of the polarization of the Shimura datum. Since $G(W)$ acts on $\Lambda \otimes W$ by similitudes with unit multiplier and $\Lambda \otimes W$ is irreducible modulo p (Lemma 3.1), the ψ_c-dual of $\Lambda \otimes \mathbb{Z}_p$ is $p^{j'}\Lambda \otimes \mathbb{Z}_p$. As γ_p is a similitude, the ψ_c-dual of $\Lambda' = H_1(X_c, \mathbb{Z}_p)$ is $p^{j''}\Lambda'$, with $j'' \le 0$ because ψ_c is integral. So $\psi_c(\Lambda', \Lambda') \subseteq p^{|j''|}\mathbb{Z}_p$, and $p^{j''}\psi_c = p^{-|j''|}\psi_c$, which is still integral on $H_1(X_c, \mathbb{Z})$ because it is integral at p, as just shown, and differs from ψ_c by a unit at every prime $\ell \neq p$, is the Riemann form of a polarization that is perfect on Λ'. We replace ψ_c by it. The Rosati involution preserves $K \times L = \mathbb{Q}[\mathbb{T}(\mathbb{Q})]$, since it maps each $\gamma \in \mathbb{T}(\mathbb{Q})$ to a scalar multiple of γ^{-1}, and being positive it restricts to complex conjugation on K. So ψ_c is the composition of $\mathrm{Tr}_{K/\mathbb{Q}}$ with a unique skew-Hermitian form, and since p is unramified in K the inverse different of $O_{K,p}$ is trivial, so the ψ_c-dual of Λ' and its dual for this form coincide, and the form is unimodular at p. □

7. A p-integral Weil cycle and its specialization

7.1. **Markman's classes.** Let $d \geq 3$ be even, $K_d = \mathbb{Q}(\sqrt{-d})$, C a generic non-hyperelliptic curve of genus 3, $X_3 = \mathrm{Pic}^2(C)$ with its principal theta divisor Θ, $X_3^\vee$ its dual and $X_6 = X_3 \times X_3^\vee$. Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be the ideal sheaves of the unions of $d+1$ generic translates of the Abel–Jacobi curve $\mathrm{AJ}(C)$ and of $-\mathrm{AJ}(C)$, respectively, twisted by Θ. Markman states that the dual of $\Phi(\mathcal{I}_2 \boxtimes \mathcal{I}_1)$, with $\Phi\colon D^b(X_3 \times X_3) \to D^b(X_6)$ Orlov's equivalence [2, Section 1.3], is up to shift a simple reflexive sheaf $\mathcal{E}$ of rank $8d$ on X_6 [2, Theorem 1.4.1(2) and Proposition 9.2.2]. Write $\kappa(\mathcal{E}) = \exp(-c_1(\mathcal{E})/8d)\mathrm{ch}(\mathcal{E})$ and $\kappa_3(\mathcal{E}) \in H^6(X_6, \mathbb{Q})$ for its degree-6 part.

Let $\Gamma_\mathbb{Z} = H^1(X_6, \mathbb{Z}) = H^1(X_3, \mathbb{Z}) \oplus H^1(X_3^\vee, \mathbb{Z})$, let $(\cdot,\cdot)$ be the symmetric pairing on $\Gamma_\mathbb{Z}$ for which both summands are isotropic and which pairs them by duality [2, (1.2.2)], and let $\Theta^\sharp\colon H^1(X_3^\vee, \mathbb{Z}) \to H^1(X_3, \mathbb{Z})$ be the contraction with Θ [2, (2.4.3)], an isomorphism because Θ is principal. Markman defines an endomorphism η of $\Gamma_\mathbb{Q}$ with $\eta^2 = -d$ (his $f = \eta_{\sqrt{-d}}$, [2, (2.4.1)]), which, with the polarization $\psi_6(x,y) = (\eta x, y)$ [2, (2.4.2)], makes X_6 a polarized abelian sixfold of Weil type for K_d [2, Section 1.2 and Proposition 2.4.4] whose Hermitian form $\mathcal{H}$ has signature $(3,3)$ and discriminant -1 [2, Lemmas 3.1.2 and 3.1.3]. On $\Gamma_K = \Gamma_\mathbb{Z} \otimes K_d$ it acts by $\sqrt{-d}$ on $\Gamma_{1,K} = \{(-\sqrt{-d}\Theta^\sharp y, y)\}$ and by $-\sqrt{-d}$ on $\Gamma_{2,K} = \{(\sqrt{-d}\Theta^\sharp y, y)\}$, $y \in H^1(X_3^\vee, K_d)$ [2, Section 1.2 and (2.4.6)]. Decomposing $(w,0)$ and $(0,y)$ into these eigenspaces gives $\eta(w,0) = (0, -(\Theta^\sharp)^{-1}w)$ and $\eta(0,y) = (d\Theta^\sharp y, 0)$, so η preserves $\Gamma_\mathbb{Z}$, and ψ_6 has determinant d^6 on $\Gamma_\mathbb{Z}$. A class in $\bigwedge^{i+j}\Gamma_K$ has K_d-type (i,j) if it lies in $\bigwedge^i\Gamma_{1,K} \wedge \bigwedge^j\Gamma_{2,K}$. The complex structures on $\Gamma_\mathbb{R}$ that commute with η and for which ψ_6 is a polarization correspond to the 3-dimensional subspaces of $\Gamma_{1,K} \otimes_{K_d} \mathbb{C}$ on which $\mathcal{H}$ is definite of a fixed sign, so they form the Hermitian symmetric domain $\mathcal{D}$ of the special unitary group $\mathrm{SU}_\mathcal{H}$ of $(\Gamma_\mathbb{Q}, \mathcal{H})$, with $\mathrm{SU}_\mathcal{H}(\mathbb{R}) \cong \mathrm{SU}(3,3)$. We call the polarized abelian sixfolds $X_{6,t}$, $t \in \mathcal{D}$, with $H^1(X_{6,t}, \mathbb{Z}) = \Gamma_\mathbb{Z}$, the family of $(\Gamma_\mathbb{Z}, \eta, \psi_6)$, and identify the cohomology of each member with $\bigwedge^\bullet\Gamma_\mathbb{Z}$.

Let $\Gamma_1 = \Gamma_{1,K} \cap (\Gamma_\mathbb{Z} \otimes O_{K_d})$ and $\Gamma_2 = \bar\Gamma_1$. For a $\mathbb{Z}$-basis (y_i) of $H^1(X_3^\vee, \mathbb{Z})$, the vectors $(-\sqrt{-d}\Theta^\sharp y_i, y_i)$ form an O_{K_d}-basis of Γ_1, since an element of $\Gamma_{1,K}$ is integral if and only if its second component is. Let $\omega_{X_6} \in H^6(X_6, O_{K_d})$ be their wedge product and $\bar\omega_{X_6}$ its conjugate. For $p \nmid 2d$, $\Gamma_\mathbb{Z} \otimes O_{K_d,(p)} = (\Gamma_1 \oplus \Gamma_2) \otimes_{O_{K_d}} O_{K_d,(p)}$, and the integral Weil classes at p are $\{\lambda\omega_{X_6} + \bar\lambda\bar\omega_{X_6} : \lambda \in O_{K_d,(p)}\}$.

Hypothesis 7.1 (M)**.** For every even $d \geq 3$, [2, Lemmas 9.3.6 and 9.3.11] and [2, Conjecture 7.3.9], as proved for families of abelian varieties in [2, Section 7.4.2], hold, and so do the other statements of [2] cited in this section.

In these statements the translates in $\mathcal{I}_1$ and $\mathcal{I}_2$ are orbits of two cyclic subgroups of X_3 of order $d+1$ that meet trivially, which exist for generic C [2, Section 1.5 and Lemma 9.3.1]; this changes neither $\mathrm{ch}(\mathcal{I}_i)$ nor $\kappa(\mathcal{E})$. Let $\varpi\colon X_6 \to Y_6$ be the quotient by the induced group of translations, of order $(d+1)^2$ [2, Section 1.5 and Lemma 9.3.3]. Lemma 9.3.6 of [2] states that Y_6 carries a reflexive sheaf $\mathcal{B}$ of rank $8d$, twisted by a Čech 2-cocycle with values in μ_{8d} and with trivial determinant, and Lemma 9.3.11 that $\mathcal{B}$ is semiregular. Conjecture 7.3.9 states, for sheaves of rank $8d$, that if $\mathcal{Y}$ is a deformation of a smooth complex projective variety $\mathcal{Y}_0$ over a smooth germ $(\Sigma, 0)$, and $\mathcal{B}$ is a semiregular coherent sheaf of rank $8d$ on $\mathcal{Y}_0$, twisted by a cocycle with values in μ_{8d}, whose Chern character extends to a horizontal section of Hodge type, then $\mathcal{B}$ extends to a twisted coherent sheaf over the part of $\mathcal{Y}$ above an open neighborhood $\mathcal{U}$ of 0. For families of abelian varieties and $\det\mathcal{B}$ trivial, the extension constructed in [2, Section 7.4.2] is flat over $\mathcal{U}$ and twisted by a cocycle with values in μ_{8d} that restricts to the cocycle of $\mathcal{B}$ after a change of the gluing of $\mathcal{B}$ by a cochain with values in μ_{8d} [2, Remark 7.3.2]. This change does not affect $\mathcal{B}^{\otimes 8d}$. These are the statements specific to Markman's proof of [2, Theorem 1.5.1]; Hypothesis 7.1 also covers the other statements of [2] cited in this section, and part (b) of Theorem 1.3 is conditional on all of them.

The proof of Proposition 7.3 uses Chern classes of complexes of analytic sheaves, defined through the real-analytic resolutions of Atiyah and Hirzebruch. Let $\mathcal{R}$ be a complex manifold with countable topology and $\mathscr{C}^\omega$ its sheaf of complex-valued real-analytic functions. For a coherent $\mathcal{O}_\mathcal{R}$-module $\mathcal{J}$, $\mathcal{J}\otimes\mathscr{C}^\omega$ is a coherent $\mathscr{C}^\omega$-module [31, Proposition 2.9], and every compact subset of $\mathcal{R}$ has an open neighborhood on which $\mathcal{J}\otimes\mathscr{C}^\omega$ has a finite resolution by locally free $\mathscr{C}^\omega$-modules [31, Proposition 2.6 and Section 4]. We write $[\mathcal{J}]_{\mathrm{top}}$ for the alternating sum of the classes of the corresponding real-analytic vector bundles in topological K-theory, and, for a bounded complex $\mathcal{F}$ of $\mathcal{O}_\mathcal{R}$-modules with coherent cohomology, $[\mathcal{F}]_{\mathrm{top}}=\sum_j(-1)^j[H^j(\mathcal{F})]_{\mathrm{top}}$. If $\mathcal{R}$ is compact, the K-theory and cohomology of [31] are the usual ones [31, Section 1 and remark after Proposition 6.2], $[\mathcal{J}]_{\mathrm{top}}$ does not depend on the resolution [31, Section 4], and we write $c(\mathcal{F})$ and $\mathrm{ch}(\mathcal{F})$ for the total Chern class and the Chern character of $[\mathcal{F}]_{\mathrm{top}}$. For a nonsingular quasi-projective complex variety Y, $\mathrm{cl}\colon \mathrm{CH}^\bullet(Y)\to H^{2\bullet}(Y,\mathbb{Z})$ is the cycle class map, a ring homomorphism compatible with pull-backs [32, 4.15]. For algebraic coherent sheaves and complexes on a nonsingular projective Y, c and ch denote the classes of their analytifications.

Lemma 7.2 (Chern classes in families)**.** *(a) Let $\mathcal{R}$ be compact. Then $[\cdot]_{\mathrm{top}}$ is additive on distinguished triangles of bounded complexes of $\mathcal{O}_\mathcal{R}$-modules with coherent cohomology.*

(b) Let $\mathcal{R}\to\Sigma$ be a proper holomorphic submersion with fibers $\mathcal{R}_t$, $t\in\Sigma$, let Li_t^* be the derived restriction to $\mathcal{R}_t$, and let $\mathcal{F}$ be a bounded complex of $\mathcal{O}_\mathcal{R}$-modules with coherent cohomology. Then $t\mapsto c(Li_t^*\mathcal{F})$ and $t\mapsto\mathrm{ch}(Li_t^*\mathcal{F})$ are flat sections over Σ of $t\mapsto\bigoplus_i H^{2i}(\mathcal{R}_t,\mathbb{Z})$ and of $t\mapsto\bigoplus_i H^{2i}(\mathcal{R}_t,\mathbb{Q})$.

(c) Let Y be a nonsingular projective complex variety and $\mathcal{F}$ a bounded complex of $\mathcal{O}_{Y^{\mathrm{an}}}$-modules with coherent cohomology. Then $c(\mathcal{F})$ is cl *of an element of $\mathrm{CH}^\bullet(Y)$, so the Chern classes of $\mathcal{F}$ are classes of algebraic cycles with integer coefficients.*

(d) Let Y be as in (c) and $Z\subseteq Y$ a closed subscheme of dimension at most $\dim Y-i$. Then $\mathrm{ch}_i(\mathcal{O}_Z)=\mathrm{cl}([Z]_{\dim Y-i})$, where $[Z]_{\dim Y-i}$ is the $(\dim Y-i)$-cycle of Z [23, Tag 02QU].

Proof. (a) At a point of $\mathcal{R}$, the stalk of $\mathscr{C}^\omega$, the ring of convergent power series in local holomorphic coordinates and their complex conjugates, is flat over the stalk of $\mathcal{O}_\mathcal{R}$, the ring of convergent power series in the coordinates, since its completion is faithfully flat over it and flat over the latter (see e.g. [23, Tags 00MB and 00MC]). Locally free $\mathscr{C}^\omega$-modules are projective among coherent ones [31, Lemma 2.7], so by the horseshoe lemma $[\cdot]_{\mathrm{top}}$ is additive on short exact sequences of coherent $\mathscr{C}^\omega$-modules. Hence the alternating sum of the classes of the terms of a bounded complex of locally free $\mathscr{C}^\omega$-modules with coherent homology is the alternating sum of the $[\cdot]_{\mathrm{top}}$ of its homology modules, and, by the long exact cohomology sequence and the flatness of $\mathscr{C}^\omega$, $[\cdot]_{\mathrm{top}}$ is additive on distinguished triangles of bounded complexes of $\mathcal{O}_\mathcal{R}$-modules with coherent cohomology.

(b) As $\mathcal{R}\to\Sigma$ is proper, every point of Σ has an open neighborhood Σ' such that the $H^j(\mathcal{F})\otimes\mathscr{C}^\omega$ have finite locally free resolutions on the inverse image of Σ', and the restrictions to the fibers over Σ' of the Chern classes of the resulting class $\sum_j(-1)^j[H^j(\mathcal{F})]_{\mathrm{top}}$ form a flat section over Σ'. Near a point of $\mathcal{R}_t$, $t\in\Sigma'$, choose holomorphic coordinates on $\mathcal{R}$ among which are the pull-backs of coordinates on Σ centered at t. Then the sheaf of complex-valued real-analytic functions on $\mathcal{R}_t$ is the quotient of $\mathscr{C}^\omega$ by these pull-backs and their complex conjugates, and, by the flatness of $\mathscr{C}^\omega$ over $\mathcal{O}$ on $\mathcal{R}$ and on $\mathcal{R}_t$, a locally free resolution of $\mathcal{J}\otimes\mathscr{C}^\omega$, for a coherent $\mathcal{O}_\mathcal{R}$-module $\mathcal{J}$, restricted to $\mathcal{R}_t$, has homology $\mathrm{Tor}_i^{\mathcal{O}_\mathcal{R}}(\mathcal{J},\mathcal{O}_{\mathcal{R}_t})\otimes\mathscr{C}^\omega$. So the restriction of $[\mathcal{J}]_{\mathrm{top}}$ to $\mathcal{R}_t$ is $[Li_t^*\mathcal{J}]_{\mathrm{top}}$, and, by (a) applied to the triangles obtained by applying Li_t^* to the truncations $\tau_{\le j-1}\mathcal{F}\to\tau_{\le j}\mathcal{F}\to H^j(\mathcal{F})[-j]$, the restriction of $\sum_j(-1)^j[H^j(\mathcal{F})]_{\mathrm{top}}$ is $[Li_t^*\mathcal{F}]_{\mathrm{top}}$.

(c) For an algebraic vector bundle on Y, the topological Chern class of its analytification is cl of its Chern class in $\mathrm{CH}^\bullet(Y)$. Indeed, both are functorial and additive, by [33, Théorème 1] for the latter. For a line bundle with a rational section, the former is 1 plus cl of the divisor of the section by [32, 4.13] and linearity, and the latter is 1 plus the class of that divisor by [33, Section 2,

example 1°, and Théorème 1]. So they agree by the uniqueness in [33, Théorème 1], applied to example 3° of [33, Section 2] on nonsingular quasi-projective varieties. The cohomology sheaves of $\mathcal{F}$ are analytifications of coherent sheaves [34, Exposé XII, Théorème 4.4], each of which is quasi-isomorphic to a bounded complex of algebraic vector bundles (see e.g. [23, Tags 0FDC, 0F87 and 0F8E]), and analytification is exact [34, Exposé XII, Proposition 1.3.1]. So by (a), $[\mathcal{F}]_{\mathrm{top}}$ is an alternating sum of classes of analytified algebraic vector bundles, and $c(\mathcal{F})$ is cl of an element of $\mathrm{CH}^\bullet(Y)$.

(d) For a closed subvariety $Y' \subseteq Y$ of codimension i', Y'^{an} is reduced and irreducible of codimension i' [34, Exposé XII, Propositions 2.1 and 2.4], and $\mathrm{ch}(\mathcal{O}_{Y'})$ is the class of Y'^{an} defined in [31, Proposition 5.4] plus terms of degree greater than $2i'$ [31, Propositions 3.4 and 6.2 and Lemma 5.6]. That class is $\mathrm{cl}(Y')$, as both restrict to the class of the submanifold $Y' \smallsetminus Y'_{\mathrm{sing}}$ on the complement of the singular locus Y'_{sing} of Y' [32, proof of 4.5], and this restriction is injective in degree $2i'$ [31, Lemma 5.3]. Now let $[Z]_{\dim Y - i} = \sum_{Y'} m_{Y'} Y'$. By [23, Tags 02MX and 02S9] and induction on the dimension, $[\mathcal{O}_Z] - \sum_{Y'} m_{Y'} [\mathcal{O}_{Y'}]$ is, in the Grothendieck group of coherent sheaves on Y, an integral combination of classes $[\mathcal{O}_{Y''}]$ of subvarieties Y'' of dimension less than $\dim Y - i$, whose Chern characters vanish in degrees up to $2i$. As $[\cdot]_{\mathrm{top}}$ is additive on coherent sheaves on Y by (a), $\mathrm{ch}_i(\mathcal{O}_Z) = \mathrm{cl}([Z]_{\dim Y - i})$. □

Proposition 7.3. *Assume Hypothesis 7.1 and let $d \geq 3$ be even. On every member of the family of $(\Gamma_{\mathbb{Z}}, \eta, \psi_6)$, the class $2(8d)^{8d}\kappa_3(\mathcal{E}) \in \bigwedge^6 \Gamma_{\mathbb{Q}}$ is the class of an algebraic cycle with integer coefficients.*

Proof. Markman deduces the algebraicity with rational coefficients [2, proof of Theorem 1.5.1]; the integral argument below is ours.

Near X_6. Markman states that $\mathrm{ch}(\mathcal{B}) = \kappa(\mathcal{B})$ because $\det \mathcal{B}$ is trivial [2, Section 7.3, after Lemma 7.3.7] and that $\varpi^*\kappa(\mathcal{B}) = \kappa(\mathcal{E})$ [2, Section 9.3, after Remark 9.3.7]; his Chern character is defined through the Atiyah class, and for coherent sheaves such as $\mathcal{E}$ it is the usual one (see e.g. [35, Section 1]). He also states that the derived tensor power $\mathcal{B}^{\otimes 8d}$ is untwisted with $\mathrm{ch}(\mathcal{B}^{\otimes 8d}) = \mathrm{ch}(\mathcal{B})^{8d}$ [2, (7.3.4) and Section 7.3, after Definition 7.3.6], and we use, as he does there, that this is the rational Chern character of $\mathcal{B}^{\otimes 8d}$, the one of Lemma 7.2. As $\kappa_0(\mathcal{B}) = 8d$ and $\kappa_1(\mathcal{B}) = 0$, we get $c_1(\mathcal{B}^{\otimes 8d}) = \mathrm{ch}_1(\mathcal{B}^{\otimes 8d}) = 0$ and $\mathrm{ch}_3(\mathcal{B}^{\otimes 8d}) = (8d)^{8d}\kappa_3(\mathcal{B})$, and $\mathrm{ch}_3 = (c_1^3 - 3c_1c_2 + 3c_3)/6$ gives $c_3(\mathcal{B}^{\otimes 8d}) = 2(8d)^{8d}\kappa_3(\mathcal{B})$ in $H^6(Y_6, \mathbb{Z})$, which is torsion-free. Hence $\varpi^* c_3(\mathcal{B}^{\otimes 8d}) = 2(8d)^{8d}\kappa_3(\mathcal{E})$.

Deformation. The group of translations defining ϖ consists of torsion points, which are determined by $\Gamma_{\mathbb{Z}}$, so it acts on every $X_{6,t}$, $t \in \mathcal{D}$, and gives isogenies $\varpi_t \colon X_{6,t} \to Y_{6,t}$. The class $\kappa(\mathcal{E})$ is of Hodge type on every $X_{6,t}$ [2, Theorem 1.4.1(3) and Corollary 1.3.2], and ϖ_t^* is an isomorphism of rational Hodge structures, so the flat continuation of $\mathrm{ch}(\mathcal{B}) = \kappa(\mathcal{B})$ is of Hodge type on every $Y_{6,t}$. By Hypothesis 7.1, $\mathcal{B}$ extends to a twisted coherent sheaf $\mathcal{B}_{\mathcal{U}}$ on the family of the $Y_{6,t}$ over a contractible open neighborhood $\mathcal{U} \subseteq \mathcal{D}$ of the point of X_6, flat over $\mathcal{U}$ and twisted by a cocycle with values in μ_{8d}. Its derived tensor power $\mathcal{F} = \mathcal{B}_{\mathcal{U}}^{\otimes 8d}$ is untwisted, a bounded complex with coherent cohomology on the total space of the family, and by flatness its derived restriction to $Y_{6,t}$ is $\mathcal{B}_t^{\otimes 8d}$, where $\mathcal{B}_t$ is the restriction of $\mathcal{B}_{\mathcal{U}}$. By Lemma 7.2(b), the Chern classes of $\mathcal{B}_t^{\otimes 8d}$ form a flat section over $\mathcal{U}$, so they are the flat continuations of those of $\mathcal{B}^{\otimes 8d}$, as $\mathcal{U}$ is contractible. As $Y_{6,t}$ is projective, they are classes of algebraic cycles with integer coefficients by Lemma 7.2(c). Pulling back along ϖ_t [32, 4.15] shows that $2(8d)^{8d}\kappa_3(\mathcal{E})$ is the class of an algebraic cycle with integer coefficients on $X_{6,t}$ for every $t \in \mathcal{U}$.

Spreading out. Let A_0 be the group of elements of $\mathrm{GL}(\Gamma_{\mathbb{Z}})$ that lie in the group of [2, (3.1.1)], the elements of $\mathrm{SO}^+(\Gamma_{\mathbb{Q}}, (\cdot,\cdot))$ that commute with η and have determinant 1 on $\Gamma_{1,K}$ and on $\Gamma_{2,K}$. Here SO^+ is the image of $\mathrm{Spin}(\Gamma_{\mathbb{Q}}, (\cdot,\cdot))$ in $\mathrm{SO}(\Gamma_{\mathbb{Q}}, (\cdot,\cdot))$. Its elements fix $\kappa(\mathcal{E})$ [2, Corollary 1.3.2 and Lemma 3.1.1]. We show that A_0 is an arithmetic subgroup of $\mathrm{SU}_{\mathcal{H}}$. Since $(\cdot,\cdot) = \frac{1}{2d}\mathrm{Tr}_{K_d/\mathbb{Q}}\mathcal{H}$ [2, (3.1.2)], an element of $\mathrm{GL}(\Gamma_{\mathbb{Q}})$ that commutes with η preserves $\mathcal{H}$ if and only if it preserves $(\cdot,\cdot)$, and its K_d-linear determinant on $\Gamma_{\mathbb{Q}}$ is its determinant on $\Gamma_{1,K}$. So $\mathsf{A}_0 \subseteq \mathsf{A}_1 = \mathrm{SU}_{\mathcal{H}}(\mathbb{Q}) \cap \mathrm{GL}(\Gamma_{\mathbb{Z}})$, and

A_0 contains the intersection of A_1 with the image of $\mathrm{Spin}(\Gamma_{\mathbb{Q}},(\cdot,\cdot))$. The restriction of the covering map $\mathrm{Spin}\to\mathrm{SO}$ to the inverse image $\tilde{G}$ of $\mathrm{SU}_{\mathcal{H}}$ in $\mathrm{Spin}(\Gamma_{\mathbb{Q}},(\cdot,\cdot))$ is a surjective homomorphism $\tilde{G}\to\mathrm{SU}_{\mathcal{H}}$, so it maps an arithmetic subgroup of $\tilde{G}(\mathbb{Q})$ to an arithmetic subgroup A_2 of $\mathrm{SU}_{\mathcal{H}}(\mathbb{Q})$ (see e.g. [26, Proposition 3.2]). Then $\mathsf{A}_2\cap\mathsf{A}_1$ has finite index in A_1 and lies in A_0, so A_0 has finite index in A_1 and is arithmetic.

Let $\mathsf{A}\subseteq\mathsf{A}_0$ be a neat subgroup of finite index (see e.g. [26, Proposition 3.5]). Then $\mathcal{M}=\mathsf{A}\backslash\mathcal{D}$ is a nonsingular quasi-projective variety (see e.g. [26, Proposition 3.1 and Theorem 3.12]), and the polarized variation of Hodge structure $\Gamma_{\mathbb{Z}}$ descends to $\mathcal{M}$. Its dual, of type $(-1,0),(0,-1)$, comes from a polarized, hence projective, family of abelian varieties $\mathfrak{X}\to\mathcal{M}$ with fibers $\mathfrak{X}_t$, $t\in\mathcal{M}(\mathbb{C})$, whose first cohomology groups form the local system $\Gamma_{\mathbb{Z}}$ (see e.g. [26, Theorem 14.8], and for projectivity [36, Section 2.1] and [23, Tag 0B45]). The class $2(8d)^{8d}\kappa_3(\mathcal{E})$ is a flat section over $\mathcal{M}$ of $t\mapsto H^6(\mathfrak{X}_t,\mathbb{Q})$; let $\mathcal{M}^{\mathrm{alg}}\subseteq\mathcal{M}(\mathbb{C})$ be the set of points at which it is the class of an algebraic cycle with integer coefficients.

For finitely many irreducible components $\mathfrak{H}_1,\dots,\mathfrak{H}_m$ of the relative Hilbert scheme of $\mathfrak{X}/\mathcal{M}$ whose points are subschemes of dimension at most 3, each projective over $\mathcal{M}$ (see e.g. [37, Section 1 and Theorem 5.1]), and integers $n_1,\dots,n_m$, let $\mathcal{Q}$ be a connected component of $\mathfrak{H}_1\times_{\mathcal{M}}\cdots\times_{\mathcal{M}}\mathfrak{H}_m$, and let $Z_{i,\mathfrak{s}}$ be the fiber at $\mathfrak{s}\in\mathcal{Q}$ of the i-th universal subscheme $\mathcal{Z}_i$. By Lemma 7.2(d), $\mathsf{z}(\mathfrak{s})=\sum_i n_i\mathrm{ch}_3(\mathcal{O}_{Z_{i,\mathfrak{s}}})$ is the class of the algebraic cycle $\sum_i n_i[Z_{i,\mathfrak{s}}]_3$ with integer coefficients, and every algebraic cycle of codimension 3 with integer coefficients on a fiber $\mathfrak{X}_t$ is of this form, with integral $Z_{i,\mathfrak{s}}$. For $\mathfrak{s},\mathfrak{s}'\in\mathcal{Q}$, $\mathsf{z}(\mathfrak{s}')$ is the parallel transport of $\mathsf{z}(\mathfrak{s})$ along a path in $\mathcal{Q}$. Indeed, $\mathcal{Q}$ is connected and has finitely many irreducible components, so $\mathfrak{s}$ and $\mathfrak{s}'$ are joined by a chain of irreducible components, consecutive ones meeting, and two points of an irreducible component lie on an irreducible curve in it (see e.g. [1, Chapter I, Lemma 5.20], applied to the closure of the component in a projective space). Let $\tilde{\Sigma}$ be the normalization of such a curve, a connected nonsingular curve (see e.g. [23, Tag 0B8Y]). The pull-back of $\mathfrak{X}$ to $\tilde{\Sigma}$ is analytically a proper submersion over $\tilde{\Sigma}$, and the pull-backs $\tilde{\mathcal{Z}}_i$ of the $\mathcal{Z}_i$ are flat over $\tilde{\Sigma}$, also analytically [34, Exposé XII, Propositions 3.1 and 3.2]. So a local coordinate of $\tilde{\Sigma}$ vanishing at a point is a non-zero-divisor on $\mathcal{O}_{\tilde{\mathcal{Z}}_i}$, and the derived restriction of $\mathcal{O}_{\tilde{\mathcal{Z}}_i}$ to the fiber over that point is $\mathcal{O}_{Z_{i,\mathfrak{s}}}$, where $\mathfrak{s}$ is the image of the point in $\mathcal{Q}$. By Lemma 7.2(b), z composed with the map $\tilde{\Sigma}\to\mathcal{Q}$ is a flat section over $\tilde{\Sigma}$. So z equals the given class either everywhere or nowhere on $\mathcal{Q}$, and the image of $\mathcal{Q}$ in $\mathcal{M}$ is closed. Hence $\mathcal{M}^{\mathrm{alg}}$ is a countable union of closed algebraic subsets of $\mathcal{M}$, and it contains the image of $\mathcal{U}$, a non-empty open subset. This is the standard argument for Hodge loci, see e.g. [38, Section 4.2].

By Baire's theorem (see e.g. [23, Tag 0CQN]), one of these closed subsets, $\mathcal{M}_1$ say, has non-empty interior $\mathcal{M}_1^\circ$ in $\mathcal{M}(\mathbb{C})$. Then $\mathcal{M}_1=\mathcal{M}$. Indeed, $\mathcal{M}(\mathbb{C})$ is a connected complex manifold, being the image of $\mathcal{D}$, and $\mathcal{M}_1^\circ$ is closed in it: a point of the closure of $\mathcal{M}_1^\circ$ lies in $\mathcal{M}_1$, and near it $\mathcal{M}_1$ is the common zero set of finitely many regular functions on a connected coordinate ball around the point, each of which vanishes on the non-empty open intersection of the ball with $\mathcal{M}_1^\circ$, hence on the ball. Thus $\mathcal{M}^{\mathrm{alg}}=\mathcal{M}(\mathbb{C})$, which proves the proposition, since every member of the family is a fiber $\mathfrak{X}_t$. □

Lemma 7.4. *Assume Hypothesis 7.1. For even $d\geq 3$, $\kappa_3(\mathcal{E})$ has components only of K_d-types $(6,0)$, $(0,6)$ and $(3,3)$, and its component of types $(6,0)\oplus(0,6)$ is $\mp\frac{d+1}{2d}x$ with $x=\frac{1}{2}(\omega_{X_6}+\bar{\omega}_{X_6})$, which is primitive in $H^6(X_6,\mathbb{Z})$ and hence in the lattice of integral Weil classes. In particular, for p inert in K_d with $p\nmid 2d(d+1)$ it is $\lambda_6\omega_{X_6}+\bar{\lambda}_6\bar{\omega}_{X_6}$ with $\lambda_6=\mp\frac{d+1}{4d}\in O^\times_{K_d,(p)}$.*

Proof. By [2, Lemma 8.2.1], $\mathrm{ch}(\mathcal{I}_1)=1+\Theta-\frac{d}{2}\Theta^2-d[\mathrm{pt}]$, and $\mathrm{ch}(\mathcal{I}_2)=\mathrm{ch}(\mathcal{I}_1)$ because a translate of $-\mathrm{AJ}(C)$ has the same class $\Theta^2/2$ and genus as one of $\mathrm{AJ}(C)$. With $\rho_1=\exp(\sqrt{-d}\Theta)$, $\rho_2=\bar{\rho}_1$ and $\Theta^3=6[\mathrm{pt}]$ one checks $\mathrm{ch}(\mathcal{I}_1)=\alpha_d\rho_1+\bar{\alpha}_d\rho_2$ with $\alpha_d=\frac{1}{2}(1+1/\sqrt{-d})$. Let τ be the involution of $H^\bullet(X_3)$ in [2, (1.2.3)], which exchanges ρ_1 and ρ_2, and let ξ be the map that Markman calls ϕ'

[2, Proposition 6.4.1], so that $\mathrm{ch}\Phi(\mathcal{I}_2 \boxtimes \mathcal{I}_1) = \xi(\mathrm{ch}(\mathcal{I}_1) \otimes \tau(\mathrm{ch}(\mathcal{I}_1)))$ [2, proof of Lemma 8.3.1]. Put $\tilde{\xi} = \exp(-c_1(\mathcal{P})/2) \cup \xi$, with $\mathcal{P}$ the Poincaré bundle on X_6. By [2, (6.1.4) and Proposition 6.1.2], $\tilde{\xi}(gc \otimes gc') = \bar{g}(\tilde{\xi}(c \otimes c'))$ for $g \in \mathrm{Spin}(\Gamma_K)$, where g acts on $H^\bullet(X_3, K_d)$ by the spin representation and $\bar{g} \in \mathrm{SO}(\Gamma_K)$ is its image, acting on $H^\bullet(X_6, K_d) = \bigwedge^\bullet \Gamma_K$. Since κ does not change when a class is multiplied by the exponential of a class of degree 2, $\kappa(\Phi(\mathcal{I}_2 \boxtimes \mathcal{I}_1)) = \kappa(\mathfrak{c})$ with

$$\mathfrak{c} = \tilde{\xi}\big(\mathrm{ch}(\mathcal{I}_1) \otimes \tau(\mathrm{ch}(\mathcal{I}_1))\big) = \tfrac{d+1}{4d}\big(\tilde{\xi}(\rho_1 \otimes \rho_1) + \tilde{\xi}(\rho_2 \otimes \rho_2)\big) + \alpha_d^2 \tilde{\xi}(\rho_1 \otimes \rho_2) + \bar{\alpha}_d^2 \tilde{\xi}(\rho_2 \otimes \rho_1)\ ,$$

as $\alpha_d \bar{\alpha}_d = \frac{d+1}{4d}$. The stabilizer in $\mathrm{Spin}(\Gamma_K)$ of the lines $K_d\rho_1$ and $K_d\rho_2$ acts on $\rho_1 \otimes \rho_1$ and $\rho_2 \otimes \rho_2$ through the determinants of its actions on $\Gamma_{1,K}$ and $\Gamma_{2,K}$, and trivially on $\rho_1 \otimes \rho_2$ and $\rho_2 \otimes \rho_1$ [2, Section 6.4]. It contains an element that acts on $\Gamma_{1,K}$ by 4 and on $\Gamma_{2,K}$ by $1/4$ [2, Remark 2.4.3], hence by 4^{i-j} on the classes of K_d-type (i,j). By the equivariance of $\tilde{\xi}$, $\tilde{\xi}(\rho_1 \otimes \rho_1)$ has type $(6,0)$, $\tilde{\xi}(\rho_2 \otimes \rho_2)$ has type $(0,6)$, and the other two terms have only types (j,j). Hence $c_1(\mathfrak{c})$ has type $(1,1)$, the parts of $\mathfrak{c}$ of degree less than 6 have types (j,j), and $\kappa_3(\mathfrak{c})$ has only types $(6,0)$, $(0,6)$ and $(3,3)$, with $(6,0)$-part $\frac{d+1}{4d}\tilde{\xi}(\rho_1 \otimes \rho_1)$. Without the normalization by κ this fails: the degree-6 part of $\mathrm{ch}\Phi(\mathcal{I}_2 \boxtimes \mathcal{I}_1)$ has components of every K_d-type.

The element $g = \exp(\sqrt{-d}\Theta)$ of $\mathrm{Spin}(\Gamma_K)$ acts on $H^\bullet(X_3, K_d)$ as cup product with ρ_1, and $\bar{g}(w,y) = (w - \sqrt{-d}\Theta^\sharp y, y)$ [2, (2.4.4)]. So $\tilde{\xi}(\rho_1 \otimes \rho_1) = \bar{g}(\tilde{\xi}(1 \otimes 1))$. By [2, Section 1.3], Φ is the inverse of the composite of the equivalence $\mathrm{id} \times \Phi_{\mathcal{P}} \colon D^b(X_6) \to D^b(X_3 \times X_3)$, whose kernel is the exterior product of the structure sheaf of the diagonal of $X_3 \times X_3$ and $\mathcal{P}$, with the push-forward along the automorphism $(x,y) \mapsto (x+y, y)$ of $X_3 \times X_3$. The former maps $\mathcal{O}_{X_3 \times \{0\}}$ to $\mathcal{O}_{X_3} \boxtimes \mathcal{P}|_{X_3 \times \{0\}} = \mathcal{O}_{X_3 \times X_3}$, as the Poincaré bundle is trivial on $X_3 \times \{0\}$ (see e.g. [1, Chapter III, Section 1 and Summary 6.11], X_3 being a Jacobian), and the push-forward fixes $\mathcal{O}_{X_3 \times X_3}$. Hence $\Phi(\mathcal{O}_{X_3 \times X_3}) = \mathcal{O}_{X_3 \times \{0\}}$, and $\xi(1 \otimes 1) = \pm[X_3 \times \{0\}]$, which $c_1(\mathcal{P})$ annihilates; so $\tilde{\xi}(1 \otimes 1) = \pm y_1 \wedge \cdots \wedge y_6$ for a $\mathbb{Z}$-basis (y_i) of $H^1(X_3^\vee, \mathbb{Z})$, and $\tilde{\xi}(\rho_1 \otimes \rho_1) = \pm\omega_{X_6}$. Thus the Weil component of $\kappa_3(\Phi(\mathcal{I}_2 \boxtimes \mathcal{I}_1))$ is $\pm\frac{d+1}{4d}(\omega_{X_6} + \bar{\omega}_{X_6}) = \pm\frac{d+1}{2d}x$. Dualizing changes the sign of κ_3, and $\mathcal{E}$ is, up to an even shift, the dual of $\Phi(\mathcal{I}_2 \boxtimes \mathcal{I}_1)$ [2, Theorem 1.4.1(2)]. Finally, expanding $\omega_{X_6} = \bigwedge_i (y_i - \sqrt{-d}\Theta^\sharp y_i)$ shows that x is an integral combination of wedge products of the y_i and the $\Theta^\sharp y_i$, with coefficients $(-d)^k$ and with coefficient 1 on $y_1 \wedge \cdots \wedge y_6$. So x is integral and primitive, and $\lambda_6 = \mp\frac{d+1}{4d}$ is a unit at p when $p \nmid 2d(d+1)$.

The derivation uses [2, Proposition 6.1.2]. An exact computation through the cohomological Fourier–Mukai transform of Orlov's equivalence, which does not use it, gives the same types and the same content $\frac{d+1}{2d}$ for $d = 4, 6, 8, 10, 12, 28$ (Appendix A). □

7.2. **The Weil cycle on the CM fourfold.**

Proposition 7.5. *Assume Hypothesis 7.1. Let X_c and $K = \mathbb{Q}(\sqrt{-d_0})$, d_0 squarefree, be as in Proposition 6.1, with the polarization of Lemma 6.2, and let $\omega = \bigwedge^4$ of an $O_{K,(p)}$-basis of the ι-eigenlattice of $H^1(X_c, \mathbb{Z}_{(p)}) \otimes O_K$. Then there is an algebraic cycle Z on X_c with integer coefficients whose class has Weil component $\lambda\omega + \bar{\lambda}\bar{\omega}$ with $\lambda \in O_{K,(p)}^\times$, and whose other components are of K-types $(i, 4-i)$ with $i \neq 0, 4$.*

Proof. Put $d = 4d_0$, so $d \geq 4$ is even and $K_d = K$. Since p is odd and unramified in K, $p \nmid 2d$. Also $p \nmid d+1$, since otherwise $-d \equiv 1$ would be a square modulo p, so $-d_0$ would be a square modulo p and p would split in K. The polarization ψ_6 of X_6 has determinant d^6 on $\Gamma_{\mathbb{Z}}$, so $\Gamma_{\mathbb{Z}}$ is unimodular at p.

Schoen's degeneration at p. We follow Schoen's proof of [39, Proposition 10]. Let $\Lambda_c = H_1(X_c, \mathbb{Z})$ with the Hermitian form whose trace form $\mathrm{Tr}_{K/\mathbb{Q}}(\cdot/2\sqrt{-d})$ is the Riemann form of the polarization of Lemma 6.2; it is unimodular at p by Lemma 6.2, since $2\sqrt{-d}$ is a unit at p. By [2, (2.4.2) and (3.1.2)], $\psi_6 = \mathrm{Tr}_{K_d/\mathbb{Q}}(\mathcal{H}/2\sqrt{-d})$ is the trace form of $\mathcal{H}$ in the same sense. Here Λ_c is a homology lattice and $\Gamma_{\mathbb{Z}}$ a cohomology lattice. For every member $X_{6,t}$ of the family, $x \mapsto \psi_6(x, \cdot)$ maps $\Gamma_{\mathbb{Z}}$

into $H_1(X_{6,t},\mathbb{Z}) = \mathrm{Hom}(\Gamma_\mathbb{Z},\mathbb{Z})$ with cokernel of order d^6, prime to p; it is a morphism of Hodge structures up to a Tate twist, as ψ_6 is a polarization, and it carries ψ_6 to the dual form, the Riemann form of the polarization. As η is skew-adjoint for ψ_6, since it is for $(\cdot,\cdot)$ [2, Section 2.4], this map carries $-\eta$ to the push-forward by the endomorphism of $X_{6,t}$ whose pull-back is η. So in this paragraph $\sqrt{-d}$ acts on $\Gamma_\mathbb{Q}$ by $-\eta$, with the Hermitian form $-\bar{\mathcal{H}}$, whose trace form in the same sense is ψ_6 and which, like $\mathcal{H}$, has signature $(3,3)$ and discriminant -1. This change of K-structure exchanges the K-types (i,j) and (j,i), so it maps Weil classes to Weil classes; outside this paragraph, K-types on the members of the family refer to η. Choose a Hermitian K-space $\Gamma'_\mathbb{Q}$ of dimension 2 and signature $(1,1)$ whose determinant makes $\Lambda_c\otimes\mathbb{Q}\oplus\Gamma'_\mathbb{Q}$ isometric to $\Gamma_\mathbb{Q}=\Gamma_\mathbb{Z}\otimes\mathbb{Q}$. Such a space exists by the classification of Hermitian spaces, see e.g. [39, Section 7]. Since $\det\Lambda_c$ and $\det\Gamma_\mathbb{Z}$ are units at p and norms from K_p have even valuation, $\det\Gamma'_\mathbb{Q}$ has even valuation at p. So by (R8) $\Gamma'_\mathbb{Q}$ contains an O_K-lattice $\Gamma'_\mathbb{Z}$ unimodular at p, and after scaling away from p it gives an abelian surface X_2 of Weil type for K, polarized by the trace form of the Hermitian form. By (R8), there is an isometry $\Psi_p\colon(\Lambda_c\oplus\Gamma'_\mathbb{Z})\otimes\mathbb{Z}_p\to\Gamma_\mathbb{Z}\otimes\mathbb{Z}_p$. Fix a global isometry $\Psi_0\colon\mathcal{V}=\Lambda_c\otimes\mathbb{Q}\oplus\Gamma'_\mathbb{Q}\to\Gamma_\mathbb{Q}$. By (R8), applied to $g_p=\Psi_0^{-1}\Psi_p$ and the stabilizer Υ of $(\Lambda_c\oplus\Gamma'_\mathbb{Z})\otimes\mathbb{Z}_p$ in $\mathrm{U}_\mathcal{V}(\mathbb{Q}_p)$, there is $g\in g_p\Upsilon\cap\mathrm{U}_\mathcal{V}(\mathbb{Q})$, and $\Psi=\Psi_0 g$ satisfies $\Psi((\Lambda_c\oplus\Gamma'_\mathbb{Z})\otimes\mathbb{Z}_{(p)})=\Gamma_\mathbb{Z}\otimes\mathbb{Z}_{(p)}$. Then $N\Psi$ maps $\Lambda_c\oplus\Gamma'_\mathbb{Z}$ into $\Gamma_\mathbb{Z}$, and so into the first homology of the members of the family, for an integer N prime to p, and this gives an isogeny $f\colon X_c\times X_2\to X'_6$ of degree prime to p compatible with the actions of K. As Ψ is an isometry of Hermitian forms, ψ_6 pulls back under $N\Psi$ to N^2 times the polarization of $X_c\times X_2$, so the complex structure of $X_c\times X_2$ transported by Ψ commutes with η and is polarized by ψ_6, and the target of f is a member X'_6 of the family of $(\Gamma_\mathbb{Z},\eta,\psi_6)$. By Proposition 7.3 there is an algebraic cycle z on X'_6 with integer coefficients and class $2(8d)^{8d}\kappa_3(\mathcal{E})$. By Lemma 7.4 its Weil component is $\lambda'_6\omega_{X'_6}+\bar\lambda'_6\bar\omega_{X'_6}$ with $\lambda'_6=2(8d)^{8d}\lambda_6\in O^\times_{K,(p)}$, because $2(8d)^{8d}$ is prime to p and ω_{X_6}, being defined by the lattice $(\Gamma_\mathbb{Z},\eta)$, continues flatly to $\omega_{X'_6}$.

Pull-back and push-forward. The pull-back f^* is an isomorphism on $H^1(\cdot,\mathbb{Z}_{(p)})$ that commutes with the actions of $O_{K,(p)}$ by pull-back, so it maps $\Gamma_1\otimes\mathbb{Z}_{(p)}$, on which η acts by $\sqrt{-d}$, onto the ι-eigenlattice of $H^1(X_c\times X_2,\mathbb{Z}_{(p)})\otimes O_K$, the sum of those of X_c and X_2. Hence $f^*\omega_{X'_6}=\det_K(f^*)\cdot\omega\otimes\omega'$, where ω' is the analog of ω on X_2 and $\det_K(f^*)\in O^\times_{K,(p)}$ is the determinant of this map in the chosen bases. The class $\omega'+\bar\omega'\in H^2(X_2,\mathbb{Z}_{(p)})$ is of type $(1,1)$, and $N'(\omega'+\bar\omega')$ is integral for some integer N' prime to p, hence the class of a divisor by the Lefschetz theorem on $(1,1)$-classes (see e.g. [38, Theorem 2]). Put $Z=\mathrm{pr}_{X_c*}(f^*z\cdot\mathrm{pr}_{X_2}^*(N'(\omega'+\bar\omega')))$. A class of K-type (i,j) on $X_c\times X_2$, cupped with $\omega'+\bar\omega'$, which has types $(2,0)$ and $(0,2)$, and pushed forward, contributes to type $(4,0)$ on X_c only if (i,j) is $(6,0)$, $(5,1)$ or $(4,2)$. By Lemma 7.4 only $(6,0)$ occurs among these, and its contribution is $\lambda\omega$ with $\lambda=N'\lambda'_6\det_K(f^*)\int_{X_2}\omega'\wedge\bar\omega'$. The integral is a unit because $\omega'\wedge\bar\omega'$ generates $H^4(X_2,O_{K,(p)})$, $\Gamma'_\mathbb{Z}$ being unimodular at p. The $(0,4)$-component is the conjugate. □

7.3. **Specialization.** Choose $\beta\in O_K\cap\mathrm{End}(X_c)$ whose reduction generates $(O_K/p)^\times\cong\mathbb{F}_{p^2}^\times$. This is possible because $\mathrm{End}(X_c)\cap K$ has conductor prime to p. On the component of K-type $(i,4-i)$ of $H^4(X_c)$, $[\beta]^*$ acts by $\beta^i\bar\beta^{4-i}\equiv\zeta^{i+p(4-i)}$ mod p, $\zeta=\beta$ mod p. For $0\le i<j\le 4$, $\beta^i\bar\beta^{4-i}\equiv\beta^j\bar\beta^{4-j}$ would mean $(p+1)\mid(j-i)$, impossible for $p\ge5$. So the Lagrange interpolation polynomial Q with value 1 at $\bar\beta^4$ and β^4 and value 0 at $\beta^i\bar\beta^{4-i}$ for $i=1,2,3$ has coefficients in $O_{K,(p)}$, invariant under conjugation because conjugation exchanges $\beta^i\bar\beta^{4-i}$ and $\beta^{4-i}\bar\beta^i$, hence in $\mathbb{Z}_{(p)}$. Put $P=Q([\beta]^*)$. On cycles P is a $\mathbb{Z}_{(p)}$-combination of the pull-backs $[\beta^j]^*$, and in Betti and in crystalline cohomology it is the projector onto the Weil part, the latter by Proposition 6.1(d) and Lemma 4.4(iii).

Proposition 7.6. *Assume Hypothesis 7.1. In the situation of Proposition 6.1,* $\mathcal{L}(X_{t_0})\subseteq\mathrm{Alg}^2(X_{t_0})$.

Proof. The fourfold X_c is defined over $\bar{\mathbb{Q}}$, and the cycle Z of Proposition 7.5, a priori defined over $\mathbb{C}$, is algebraically equivalent to a cycle defined over $\bar{\mathbb{Q}}$ (see e.g. [40]), which has the same class, and we replace Z by it. Let O be the localization at a prime above p of the ring of integers of a number field over which X_c, Z and β are defined, with an abelian scheme $\mathcal{X}/O$ of generic fiber X_c and special fiber X_{t_0} (Proposition 6.1(c)); β extends to $\mathcal{X}$. We replace O by its completion; this changes neither the special fiber nor the intersection numbers. Put $Z_1 = PZ$ and $Z_2 = [\beta]^* Z_1$, cycles with $\mathbb{Z}_{(p)}$-coefficients. Their Betti classes are $\lambda\omega + \bar{\lambda}\bar{\omega}$ and $\beta^4\lambda\omega + \overline{\beta^4\lambda}\bar{\omega}$. Since $\omega \wedge \omega = 0$, $\deg(Z_i \cdot Z_j) = u \operatorname{Tr}_{K/\mathbb{Q}}(\lambda_{(i)}\bar{\lambda}_{(j)})$ with $\lambda_{(1)} = \lambda$, $\lambda_{(2)} = \beta^4\lambda$ and $u = \int \omega \wedge \bar{\omega}$, and the determinant of this matrix is (1.4). Because p is unramified in K, $H^1(X_c, \mathbb{Z}_{(p)}) \otimes O_K$ is the sum of the two eigenlattices, so $\omega \wedge \bar{\omega}$ generates $H^8(X_c, O_{K,(p)})$ and $u \in \mathbb{Z}_{(p)}^\times$. And $\beta^4 \not\equiv \bar{\beta}^4 \bmod p$, since $\zeta^{4(p-1)} = 1$ would require $(p+1) \mid 4$. So the determinant is a unit at p.

By (R5) and (R4), the specialized cycles $\operatorname{sp} Z_i$ on X_{t_0} have $\operatorname{cl}(\operatorname{sp} Z_i) \in T^4(X_{t_0})$, $\operatorname{cl}(\operatorname{sp} Z_1) = P \operatorname{cl}(\operatorname{sp} Z)$ lies in the Weil part, hence in $\mathcal{L}$, and so does $\operatorname{cl}(\operatorname{sp} Z_2) = [\beta]^* \operatorname{cl}(\operatorname{sp} Z_1)$; and $\int \operatorname{cl}(\operatorname{sp} Z_i) \operatorname{cl}(\operatorname{sp} Z_j) = \deg(\operatorname{sp} Z_i \cdot \operatorname{sp} Z_j) = \deg(Z_i \cdot Z_j)$. The cup product pairs T^4 with itself into $T^8 = \mathbb{Z}_p[\mathrm{pt}]$, so $\mathcal{L}$ carries a $\mathbb{Z}_p$-valued symmetric form, and the span $\mathcal{L}'$ of $\operatorname{cl}(\operatorname{sp} Z_1), \operatorname{cl}(\operatorname{sp} Z_2)$ has unit discriminant. From $\operatorname{disc} \mathcal{L}' = [\mathcal{L} : \mathcal{L}']^2 \operatorname{disc} \mathcal{L}$ we get $\mathcal{L}' = \mathcal{L}$. □

No p-adic comparison theorem enters: the intersection numbers fix the p-adic normalization, given that the specialized classes are integral and lie in the Weil part.

8. Proof of the main theorem

Theorem 1.3 is proved at one supersingular point of a Mumford family and carried to every abelian fourfold with p-divisible group $\mathscr{G}_p$ by an isogeny of degree prime to p.

Lemma 8.1. *Let X, X' be supersingular abelian varieties over $\bar{\mathbb{F}}_p$ with $X[p^\infty] \cong X'[p^\infty]$. Then there is an isogeny $f\colon X' \to X$ of degree prime to p, and f^* maps $T^{2i}(X)$, $\operatorname{Alg}^i(X)$ and $S(X)$ isomorphically onto $T^{2i}(X')$, $\operatorname{Alg}^i(X')$ and $S(X')$.*

Proof. By (R3), $\operatorname{Hom}(X', X) \otimes \mathbb{Z}_p = \operatorname{Hom}(X'[p^\infty], X[p^\infty])$. Let f_0 be an isomorphism of p-divisible groups and $f \in \operatorname{Hom}(X', X)$ with $f - f_0 \in p \operatorname{Hom}(X'[p^\infty], X[p^\infty])$. Then $f[p^\infty] \in f_0(1 + p \operatorname{End}(X'[p^\infty]))$ is an isomorphism. So f is an isogeny of degree prime to p. Indeed, X and X' have the same dimension, since their p-divisible groups have the same height. The reduced identity component X_0' of $\ker f$ is an abelian subvariety of X', and $f[p^\infty]$ vanishes on $X_0'[p^\infty] \subseteq X'[p^\infty]$, which has height $2 \dim X_0'$; as $f[p^\infty]$ is injective, $X_0' = 0$. So $\ker f$ is finite, and its p-primary part, contained in $X'[p^\infty]$ and killed by $f[p^\infty]$, is trivial. Hence f^* is an isomorphism on M, so on T^{2i}, and maps $T^2(X)$ onto $T^2(X')$, hence $S(X)$ onto $S(X')$. On cohomology $f_* f^* = f^* f_* = \deg f \in \mathbb{Z}_p^\times$, and both maps preserve algebraic classes, so $f^* \operatorname{Alg}^i(X) = \operatorname{Alg}^i(X')$. □

Lemma 8.2. *For fixed (k, B, Λ), (h3) and (h4) hold for all but finitely many p. For every prime $p \geq 5$ there are $(k, B, \Lambda, \mathsf{K}^p)$ as in the introduction satisfying (h1)–(h4).*

Proof. Choose integers N_1, N_0 such that $\Xi^3 + N_1\Xi + N_0$ is irreducible modulo p. Every irreducible cubic over $\mathbb{F}_p$ can be brought to this form, since $3 \in \mathbb{F}_p^\times$. Changing N_1 by a multiple of p, we may assume $N_1 < 0$ so large that $-4N_1^3 - 27N_0^2 > 0$. Then $\Xi^3 + N_1\Xi + N_0$ is irreducible, with three real roots, and separable modulo p, so p is inert in $k = \mathbb{Q}[\Xi]/(\Xi^3 + N_1\Xi + N_0)$. Let B be the quaternion algebra over k ramified exactly at two of the three real places. Then $\operatorname{Cor}_{k/\mathbb{Q}} B$ has invariant $\frac{1}{2} + \frac{1}{2} = 0$ at ∞ and 0 at every finite place, so $\operatorname{Cor}_{k/\mathbb{Q}} B \cong \operatorname{Mat}_8(\mathbb{Q})$ (see e.g. [14, Section 2.1.2]), and B is split above p. The group $G_{\mathbb{Q}_p}$ is unramified, so it has hyperspecial subgroups (see e.g. [26, Section 16]). Concretely, take $\Lambda_p = \operatorname{Cor}(\Lambda_0)$ for an O_{k_p}-lattice $\Lambda_0 \subset \mathbb{U}_p$, extend Λ_p to a lattice $\Lambda \subset \mathbb{V}$, rescale ψ to be integral on Λ, and take any neat K^p. Then (h4) holds with $j = 0$.

For the first statement, fix a maximal order $\mathcal{O}_B \subset B$, and let $\mathrm{Nm}\colon B \to \mathrm{Cor}_{k/\mathbb{Q}} B = \mathrm{End}(\mathbb{V})$ be the norm map, a homogeneous polynomial map of degree 3 defined over $\mathbb{Q}$. For all but finitely many p, p is unramified in k, B is split at the places above p, which is (h3), so that $\mathcal{O}_B \otimes \mathbb{Z}_p = \mathrm{End}_{O_{k_p}}(\Lambda_0)$ for an O_{k_p}-lattice $\Lambda_0 \subset \mathbb{U}_p$, and $\mathrm{Nm}(\mathcal{O}_B \otimes \mathbb{Z}_p)$ preserves $\Lambda \otimes \mathbb{Z}_p$, because the coefficients of Nm in a $\mathbb{Z}$-basis of $\mathcal{O}_B$ are finitely many elements of $\mathrm{End}(\mathbb{V})$. Over W, Nm becomes $(x_1, x_2, x_3) \mapsto x_1 \otimes x_2 \otimes x_3$ on $\mathrm{Mat}_2(W)^3$. Its polarization is trilinear, its values on $\mathcal{O}_B \otimes \mathbb{Z}_p$ are integral combinations of values of Nm, and it takes the value $x_1 \otimes x_2 \otimes x_3$ at $(x_1, 0, 0)$, $(0, x_2, 0)$, $(0, 0, x_3)$. So the W-span of $\mathrm{Nm}(\mathcal{O}_B \otimes \mathbb{Z}_p)$ is $\mathrm{End}_W(\mathrm{Cor}(\Lambda_0) \otimes W)$, and as W is faithfully flat over $\mathbb{Z}_p$, the $\mathbb{Z}_p$-span is $\mathrm{End}_{\mathbb{Z}_p}(\mathrm{Cor}\,\Lambda_0) \cong \mathrm{Mat}_8(\mathbb{Z}_p)$. The lattices stable under $\mathrm{End}_{\mathbb{Z}_p}(\mathrm{Cor}\,\Lambda_0)$ are the $p^j \mathrm{Cor}(\Lambda_0)$, $j \in \mathbb{Z}$, which gives (h4). □

Theorem 8.3. *Assume (h1)–(h4), and let $t \in \mathscr{S}(\bar{\mathbb{F}}_p)$ be a point such that $X = \mathcal{A}_t$ is supersingular. Then $X[p^\infty] \cong \mathscr{G}_p$, so Theorem 5.2 applies to X. Assume moreover Hypothesis 7.1. Then there is an algebraic class $z \in \mathrm{Alg}^2(X)$ such that $S(X)$ and the classes U^*z, $U \in \mathrm{End}(X)$, span $T^4(X)$. The class z is obtained by specializing a Weil cycle on an abelian fourfold over $\bar{\mathbb{Q}}$ with complex multiplication to a supersingular point and pulling it back along an isogeny of degree prime to p. In particular $\mathrm{Alg}^2(X) = T^4(X)$, and the Künneth component in $H^2_{\mathrm{cris}}(X) \otimes H^2_{\mathrm{cris}}(X)$ of every class in $T^4(X \times X)$ is algebraic.*

Proof. The first assertion is Proposition 3.3. For the rest, let t_0 be as in Proposition 6.1. By Proposition 7.6 and (R3), $\mathcal{L}(X_{t_0}) \subseteq \mathrm{Alg}^2(X_{t_0})$, and Alg^2 contains S and is stable under $\mathrm{End}(X_{t_0})^*$. By Proposition 5.1(ii) and the paragraph preceding it, $\mathrm{Alg}^2(X_{t_0}) = T^4(X_{t_0})$, and any $z \in \mathcal{L} \setminus p\mathcal{L}$ has the stated property by Theorem 5.2. On $X_{t_0} \times X_{t_0}$, the Künneth $(2,2)$-components of algebraic classes are algebraic. Indeed, replacing U_1 by jU_1, $j = 0, \dots, 4$, multiplies the $(i, 4-i)$-component of $(U_1, U_2)^*x$ by j^i, and the Vandermonde matrix of $0, \dots, 4$ is invertible modulo $p \geq 5$. The components of products of divisor classes are products of components, which are divisor classes by (R3). So Proposition 5.1(iv) shows that T_{22} is algebraic. For an arbitrary supersingular t, Proposition 3.3 gives $\mathcal{A}_t[p^\infty] \cong X_{t_0}[p^\infty]$, and Lemma 8.1, applied to $\mathcal{A}_t$ and to $\mathcal{A}_t \times \mathcal{A}_t$, transports both statements. □

Proof of Theorem 1.3. By (R1), $M(X)$ is isomorphic to the model (1.2), and $T^4(X)$ and $S(X)$ depend only on $(M(X), F)$. So (a) follows from Proposition 4.3. For (b), Lemma 8.2 gives data satisfying (h1)–(h4) at p, and Proposition 6.1 a supersingular point t_0. By Proposition 3.3, $X_{t_0}[p^\infty] \cong \mathscr{G}_p \cong X[p^\infty]$. Lemma 8.1 uses no structure of Mumford's family, so it gives an isogeny $f\colon X \to X_{t_0}$ of degree prime to p, and $\mathrm{Alg}^2(X) = f^* \mathrm{Alg}^2(X_{t_0}) = f^*T^4(X_{t_0}) = T^4(X)$ by Theorem 8.3. The same applies to $f \times f$. □

At primes that are not inert in k, the quotient T^4/S vanishes at the supersingular points.

Proposition 8.4 (computer-assisted)**.** *Let $p \geq 5$ be a prime that is unramified and not inert in k, assume (h3) and (h4), that is, B is split at every place of k above p and $\Lambda \otimes \mathbb{Z}_p = p^j \mathrm{Cor}(\Lambda_0)$ for an $O_k \otimes \mathbb{Z}_p$-lattice $\Lambda_0 \subset \mathbb{U}_p$, and let $t \in \mathscr{S}(\bar{\mathbb{F}}_p)$ be a point such that $X = \mathcal{A}_t$ is supersingular. Then $T^4(X) = S(X)$, and the Künneth component in $H^2_{\mathrm{cris}}(X) \otimes H^2_{\mathrm{cris}}(X)$ of $T^4(X \times X)$ is spanned by the corresponding components of products of divisor classes on $X \times X$.*

Proof. Lemma 3.1 holds, with the same proof, for p unramified in k. Then k_p is a product of unramified extensions of $\mathbb{Q}_p$, the Λ_i are indexed by the three embeddings $O_{k_p} \to W$, and σ permutes these embeddings within the places of k above p. So the argument of Proposition 3.3 applies with this action on the slots. If p splits completely in k, σ fixes each slot. The two slots with $b_i \in \mathrm{GL}_2(W)$ form a unit-root crystal of rank 4, which has a basis of fixed vectors by (R2), and the third is a supersingular crystal of rank 2 and Hodge slopes $0, 1$. So by Lemma 3.2 with $r = 1$, $(M, F) \cong (W^2, A\sigma) \otimes (W^4, \sigma)$, the Dieudonné module of E_0^4. If $p = \mathfrak{p}_1\mathfrak{p}_2$ with residue degrees 1

and 2 and the non-scalar slot is the one of $\mathfrak{p}_1$, the same argument applies. If it is a slot of $\mathfrak{p}_2$, the slot of $\mathfrak{p}_1$ is a unit-root crystal of rank 2, and the other two form a crystal $(W^2 \otimes W^2, (A \otimes 1)\sigma')$, $\sigma'(x \otimes y) = \sigma y \otimes \sigma x$, by Lemma 3.2 with $r = 2$. In each case (M, F) is monomial as in Section 2.2, and the exact computation of Appendix A gives $T^4 = S$ and that the Künneth component of $T^4(X \times X)$ is spanned by the components of $S(X \times X)$. No algebraicity input beyond (R3) is used. □

Remark 8.5. By (4.1), $\mathscr{G}_p$ is the p-divisible group of the monomial module with cycles of exponent words 01 and 000111, so $\mathscr{G}_p \cong E_0[p^\infty] \times \mathscr{H}$, where the Dieudonné module of $\mathscr{H}$ is generated by one element y with $F^3 y = V^3 y$. Its a-number is 2.

Remark 8.6. For a very general complex fiber $X_\mathbb{C}$ of Mumford's family, $H^4(X_\mathbb{C} \times X_\mathbb{C}, \mathbb{Q})$ contains Hodge classes that are not in the span of products of divisor classes [6]. Modulo that span they form a space of dimension 2 in the Künneth $(2,2)$-component. Indeed, the Hodge group is a $\mathbb{Q}$-form of an almost direct product of three copies of SL_2 [6], which by Mumford's construction acts on H^1 through $\mathbb{U}_1 \otimes \mathbb{U}_2 \otimes \mathbb{U}_3$ (Section 1), and $\bigwedge^2(\mathbb{U}_1 \otimes \mathbb{U}_2 \otimes \mathbb{U}_3)$ is the sum of the trivial representation and the three pairwise non-isomorphic irreducible representations $\mathrm{Sym}^2\mathbb{U}_i \otimes \mathrm{Sym}^2\mathbb{U}_j$, so the $(2,2)$-component contains a 4-dimensional space of Hodge classes, in which products of divisor classes span a 2-dimensional subspace. Each of the other Künneth components of H^4 contains a 1-dimensional space of Hodge classes, spanned by a product of divisor classes. If an integer multiple $N\delta$ of such a class δ were algebraic, the cycles would specialize, and for primes $p \nmid N$ outside a finite set the integral crystalline class δ_t at a point t of characteristic p would be algebraic. Conversely, a supply of non-algebraic δ_t at infinitely many primes would show that δ is not algebraic. Making these two statements precise needs, besides (R5), a comparison between the Betti cohomology of the complex fibers and the crystalline cohomology of the special fibers that carries δ to an integral class δ_t, which we do not discuss. Theorem 8.3 and Proposition 8.4 show that, under Hypothesis 7.1, no such δ_t exists at supersingular points: the whole Künneth $(2,2)$-component of $T^4(X_t \times X_t)$ is algebraic there. This removes one possible route to non-algebraicity. It says nothing about the algebraicity of δ in characteristic 0.

9. Other supersingular Dieudonné modules

A *monomial* supersingular Dieudonné module of height 8 and dimension 4 is (M_w, F) for a multiset w of binary words, each with as many 0s as 1s, of total length 8 and considered up to cyclic rotation: M_w is the direct sum of the modules M_v of the words v of w, where M_v has basis e_i, $i \in \mathbb{Z}/|v|$, and $Fe_i = p^{a(i)}e_{i+1}$, with $a(0), a(1), \dots$ the letters of v. There are twenty such multisets, the cycle decompositions, which we write as sums of words. For example $01 + 01 + 01 + 01$ is the Dieudonné module of E_0^4, and $01 + 000111$ is that of $\mathscr{G}_p$. Each M_w is the Dieudonné module of an abelian fourfold that is a quotient of E_0^4 by a finite subgroup scheme. Indeed, for a word v of length $2m$, let (e'_i) be the basis of the module of the word $0101\cdots01$ of length $2m$, which is isomorphic to $M(E_0^m)$ by the next paragraph. Then $e_i \mapsto p^{c_i}e'_i$ with $c_{i+1} = c_i + \frac{1-(-1)^i}{2} - a(i)$ is well defined, because v has m letters 1, and commutes with F and V. So, after scaling, M_w is a sublattice of finite index in $M(E_0^4)$ stable under F and V, and by Dieudonné theory (R1) it is $M(E_0^4/H)$ for the finite subgroup scheme $H \subset E_0^4[p^\infty]$ whose Dieudonné module is the quotient (see e.g. [17, Corollary 3.16]). Section 2.2 applies to it.

Different cycle decompositions can give isomorphic modules. Let v be a word and let M be the module of the single word $vv\cdots v$ of length $m|v|$, with basis e_i, $i \in \mathbb{Z}/m|v|$. Then $\varphi_v = p^{-|v|/2}F^{|v|}$ is a $\sigma^{|v|}$-semilinear bijection of $M_0 = \bigoplus_{0 \le j < m} We_{j|v|}$, so by (R2) M_0 has a basis $e'_1, \dots, e'_m$ of φ_v-fixed vectors. In the basis $p^{-(a(0)+\cdots+a(i-1))}F^i e'_j$, $0 \le i < |v|$, $1 \le j \le m$, F has the matrix of $M_v^{\oplus m}$. So $01+01+01+01$, $01+01+0101$, $0101+0101$, $01+010101$ and 01010101 give isomorphic modules, as do $01+01+0011$ and $0011+0101$, and $0011+0011$ and 00110011. Modules whose

multisets of primitive words differ are not isomorphic, because their reductions modulo p are not, by Kraft's classification of Dieudonné modules killed by p with $\ker F = \operatorname{im} V$, which includes the reductions modulo p of Dieudonné modules of p-divisible groups (see e.g. [18, Section 9]). So the twenty cycle decompositions form fourteen isomorphism classes.

For a supersingular abelian fourfold X over $\bar{\mathbb{F}}_p$, let $\mathcal{P}$ be the Poincaré bundle on $X \times X^\vee$ and put

$$\widehat{S}(X) = \mathbb{Z}_p\text{-span of } \tfrac{1}{24}\operatorname{pr}_{X*}\left(\operatorname{pr}_{X^\vee}^* y \cdot c_1(\mathcal{P})^4\right), \qquad y \in S(X^\vee)\ . \tag{9.1}$$

This is the degree-4 part of the cohomological Fourier transform of $S(X^\vee)$ (compare [41]).

Lemma 9.2. *For $p \geq 5$, $\widehat{S}(X) \subseteq \mathrm{Alg}^2(X)$.*

Proof. By (R3), $S(X^\vee) \subseteq \mathrm{Alg}^2(X^\vee)$. Moreover $c_1(\mathcal{P})$ is the class of a divisor, 24 is a unit in $\mathbb{Z}_p$, and the cycle class is compatible with products, pull-backs and push-forwards (R4). □

The lattice $\widehat{S}(X)$ is computed from $(M(X), F)$ as follows. Let (e_i) be a basis of $M(X)$. Then $c_1(\mathcal{P}) = \sum_i e_i \otimes e_i^\vee$ with $e_i^\vee \in M(X^\vee)$, since $\mathcal{P}$ is trivial on $X \times 0$ and $0 \times X^\vee$, so that $c_1(\mathcal{P})$ has no Künneth components of types $(2,0)$ and $(0,2)$. By (R4), $\int c_1(\mathcal{P})^8$ is $\pm 8!$ times a unit of W times the determinant of the $e_i^\vee$ in a basis of $M(X^\vee)$, and by Riemann–Roch on $X \times X^\vee$ it is $\deg(c_1(\mathcal{P})^8) = 8!\chi(\mathcal{P}) = \pm 8!$. Indeed $\chi(\mathcal{P})^2$ is the degree of the homomorphism $X \times X^\vee \to X^\vee \times X^{\vee\vee}$ attached to $\mathcal{P}$, which is $(x, x^\vee) \mapsto (x^\vee, x)$ with x viewed in $X^{\vee\vee}$, an isomorphism because biduality is one (see e.g. [1, Chapter I, Theorems 11.1 and 8.9]). So this determinant is a unit, the $e_i^\vee$ form a basis of $M(X^\vee)$, and $c_1(\mathcal{P})^n/n! = \sum_{|I|=n} \pm e_I \otimes e_I^\vee$ is integral for every n. Hence the Fourier transform $\mathfrak{F}(y) = \operatorname{pr}_{X*}\left(\operatorname{pr}_{X^\vee}^* y \cdot c_1(\mathcal{P})^{8-n}/(8-n)!\right)$ maps $e_J^\vee$, $|J| = n$, to a unit multiple of e_{J^c}, where J^c is the complement of J, and is an isomorphism $\bigwedge^n M(X^\vee) \to \bigwedge^{8-n} M(X)$. Since $c_1(\mathcal{P})$ is the class of a divisor, $Fc_1(\mathcal{P}) = pc_1(\mathcal{P})$, and together with $\int Fx = p^4\sigma(\int x)$ on $X^\vee$ this gives $F\mathfrak{F}(y) = p^{4-n}\mathfrak{F}(Fy)$, so $\mathfrak{F}$ maps $T^n(X^\vee)$ onto $T^{8-n}(X)$. Now let (e_i) be the monomial basis. For $n = 2$ it follows that $T^2(X^\vee) \otimes W$ is monomial in the basis $(e_J^\vee)$, with the exponent of $T^6(X)$ at e_{J^c}. So $\widehat{S}(X) \otimes W$, the image of $S(X^\vee) \otimes W$, is the monomial lattice whose exponent at e_J, $|J| = 4$, is the minimum of $\nu_T(J') + \nu_T(J'')$ over the 6-sets J', J'' with $J' \cap J'' = J$ and $J' \cup J''$ the whole index set, ν_T being the exponents of $T^6(X)$.

Proposition 9.3 (computer-assisted)**.** *Let $p \geq 5$. For the twenty cycle decompositions, the indices $[T^4 : S]$ and $[T^4 : S + \widehat{S}]$ are as in Table 1. Consequently the integral crystalline Tate conjecture holds in codimension 2, without Hypothesis 7.1, for every abelian fourfold over $\bar{\mathbb{F}}_p$ whose Dieudonné module is isomorphic to one of the fifteen with $S + \widehat{S} = T^4$, which form nine isomorphism classes. It holds under Hypothesis 7.1 for $01 + 000111$ (Theorem 1.3). For the remaining four modules the index $[T^4 : S + \widehat{S}]$ does not change when $S + \widehat{S}$ is enlarged by the push-forwards and pull-backs of T^4 along homomorphisms to and from the abelian fourfolds whose Dieudonné modules are among the fifteen with $S + \widehat{S} = T^4$, and then closed under pull-backs and push-forwards by endomorphisms. Nor does it change when the same is done with T^4 of the fourfolds with p-divisible group $\mathscr{G}_p$, which is algebraic under Hypothesis 7.1.*

Proof. By Section 2.2 and the description of $\widehat{S}$ above, all lattices involved are monomial with exponents given by min-plus expressions. Push-forward along a homomorphism f acts on $\bigwedge^4$ through the complementary 4×4 minors of f^*, by Poincaré duality. The Hom-lattices are monomial in the matrix units (Section 2.2), and since 4! is a unit, polarization as in Section 2.2 reduces the spans over all homomorphisms to spans over monomial multiples of matrix units. The exponents are an exact integer computation (Appendix A). The consequences follow from $S \subseteq \mathrm{Alg}^2$, Lemma 9.2, and the fact that T^4 and S depend only on the Dieudonné module. □

module	a-number	self-dual	$\log_p[T^4 : S]$	$\log_p[T^4 : S + \widehat{S}]$
$01+01+01+01$, $01+01+0101$, $0101+0101$, $01+010101$, 01010101	4	yes	0	0
$01+01+0011$, $0011+0101$, 00101101	3	yes	0	0
$0011+0011$, 00110011	2	yes	0	0
00001111	1	yes	0	0
00110101, 00101011	3	no	0, 2	0
00011101, 00010111	2	no	2, 16	0
$01+000111$	2	yes	4	4
$01+001101$, $01+001011$	3	no	2, 10	2
00100111, 00011011	2	no	26	26

TABLE 1. The twenty cycle decompositions of monomial supersingular Dieudonné modules of height 8 and dimension 4, named by their words. They form fourteen isomorphism classes: the five decompositions of the first row give isomorphic modules, as do $01+01+0011$ and $0011+0101$, and $0011+0011$ and 00110011. "Self-dual" means isomorphic to the dual module, whose words are the complementary words, with 0 and 1 exchanged; it is necessary for the existence of a polarization of degree prime to p, because such a polarization induces an isomorphism $X[p^\infty] \cong X^\vee[p^\infty]$ and $M(X^\vee)$ is the dual module [17, Definition 3.6 and Proposition 3.23]. A row naming two modules that are not self-dual lists a module and its dual, which are not isomorphic but have the same a-number and the same index $[T^4 : S + \widehat{S}]$; where $[T^4 : S]$ differs, both values are given, in the order of the names. The indices are powers of p with exponents that are exact and independent of $p \geq 5$.

Remark 9.4. For each of the four open modules, the exact computation shows that one lattice of integral Weil classes of an inert imaginary quadratic action of signature $(2, 2)$, diagonal in the monomial basis, together with its endomorphism translates, would span T^4 with S. For the two modules of a-number 3, the lattice and $S + \widehat{S}$ already span T^4. None of the four is self-dual, so these modules carry no polarization of degree prime to p, and Lemma 6.2 and the Schoen step of Proposition 7.5 do not apply as written. What is missing is a p-integral Weil cycle with a unit Weil coefficient on a lift whose first homology is not a unimodular Hermitian lattice at p. The pair 00100111, 00011011 keeps the index p^{26} under all the enlargements of Proposition 9.3, the largest gap among the monomial modules. It is the natural test case, both for a new source of algebraic classes and for a possible failure, and the computations decide neither.

APPENDIX A. THE COMPUTATIONS

All computations are exact, with integers or, for the check of Lemma 7.4, in $\mathbb{Q}(\sqrt{-d})$, and done in Python. By Section 2.2, every lattice in Propositions 4.3, 5.1 and 9.3 and in the proof of Proposition 8.4 is monomial over W in a basis of monomials (of $\bigwedge^n M$, of $\mathrm{End}_W(M)$, or of $\bigwedge^4$ of a product), with exponents obtained from the integers $a(s)$ by the orbit rule (2.2) and by minima over products and over values of polarized forms; the programs evaluate these expressions. A lattice index is the sum of the differences of exponents, and inclusions are comparisons of exponents. Nothing depends on p, except that $p \geq 5$ is used for the polarization step (4! a unit) and for the Künneth extraction in the proof of Theorem 8.3.

Proposition 4.3. For the model (4.1), the program lists the two π-orbits, computes $\nu_T(J)$ for all 2-sets and 4-sets J and $\nu_S(J)$ for all 4-sets J, and returns the four sets of (4.2) with the stated exponents; for 6-sets it checks that the span of triple products has the exponents of T^6. The program also checks that the four F-compatible eigensets of Lemma 4.4(i) are these sets, and the numbers $|J \cap \{s_1 = 1\}|$.

Proposition 5.1. The program computes the exponents of $\mathrm{End}(M, F) \otimes W$ in the matrix units, the monomial spans of the values $D(U_1, \dots, U_4)(e_J)$ over monomial generators, and from them the inclusions and indices in (i)–(iii); for (iv) it does the same for pull-backs along (U_1, U_2) into the Künneth $(2,2)$-component of $\bigwedge^4(M \oplus M)$.

Check of Lemma 7.4. For $d = 4, 6, 8, 10, 12, 28$, a program computes, exactly over $\mathbb{Q}(\sqrt{-d})$, the image of $\mathrm{ch}(\mathcal{I}_1) \boxtimes \mathrm{ch}(\mathcal{I}_1)$ under the cohomological Fourier–Mukai transform of Orlov's equivalence, written on explicit integral bases of $H^\bullet(X_3 \times X_3)$ and $H^\bullet(X_3 \times X_3^\vee)$, normalizes by $\exp(-c_1/\mathrm{rk})$, determines Γ_1 as the annihilator of the decomposable degree-6 part of the image of $\rho_1 \boxtimes \rho_2$, decomposes κ_3 by K_d-types, and computes the content of its Weil component in the saturated lattice of integral Weil classes by Hermite normal form.

Proposition 8.4. The same computations for the two models of the proof give $[T^4 : S] = 1$ and that the Künneth $(2,2)$-component of T^4 of the square is spanned by the components of divisor products.

Proposition 9.3. For each of the twenty cycle decompositions the program computes T^4, S, $\widehat{S}$ by the min-plus formula of Section 9, the closures under endomorphisms (pull-backs through 4×4 minors and push-forwards through complementary minors), the push-forwards and pull-backs, along all monomial homomorphisms from and to these modules, of T^4 of the fifteen cycle decompositions with $S + \widehat{S} = T^4$, and of T^4 of E_0^4 together with that of the module of $\mathscr{G}_p$, and the duality between modules. For Remark 9.4 it computes the lattices of integral Weil classes of the diagonal imaginary quadratic actions of signature $(2,2)$, the span of their endomorphism translates with S, and their span with $S + \widehat{S}$.

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