Holographic entropy inequalities without unscaled contraction proofs

The proofs explained, from the basics

A companion to the paper *Holographic entropy inequalities without unscaled contraction proofs*

Produced by large language models
not reviewed by the author or by any human expert

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Preface

Status

This companion, like the paper it explains, was produced by large language models, instances of the Claude models of Anthropic running as autonomous agents, in October and November 2026. The proofs it explains were checked only by other instances of the same models, and such checks may share the blind spots of the models that wrote the proofs. The companion was not reviewed by the author or by any human expert, and no human expert has checked the proofs. It is published together with the paper as an explanation of its proofs, and it may contain errors in its explanations, examples and remarks, in addition to any errors in the proofs themselves.

The paper *Holographic entropy inequalities without unscaled contraction proofs* (November 2026) states the proofs concisely, for specialists, and the code for the computations on which the proofs rely is published with it. This companion explains the same proofs from the basics. It is meant to be read on its own: it states everything it uses, proves what can be proved in a few pages, and cites the published literature for the rest.

What the paper claims

The Ryu–Takayanagi entropies of N boundary regions obey linear inequalities, and the standard way to prove such an inequality, due to Bao, Nezami, Ooguri, Stoica, Sully and Walter [**1**], is a contraction map, a map between two Boolean cubes that respects some prescribed values and does not increase a weighted Hamming distance. Sometimes a map exists only after the inequality has been multiplied by an integer $k \geq 2$. The paper asks for which inequalities $k = 1$ suffices, and how large k must be when it does not. It has four results.

The first concerns balanced inequalities, those in which every party carries the same total coefficient on both sides. The paper gives a valid balanced inequality B_7^* on seven parties, a hub A and two triangles $\{B, C, D\}$ and $\{E, F, G\}$, that has no contraction map without scaling, while $2B_7^*$ has one. The failure is visible at a single point of the cube: an integer map would have to choose three edges of the two triangles that together touch all six vertices, which is impossible because each triangle needs two of its edges, while the value $\frac{1}{2}$ on every edge satisfies all the conditions. On the full cube the failure is certified, and on the smaller realizable domain it rests on this written proof.

The second concerns superbalanced inequalities, a smaller class that, by a theorem of He, Hubeny and Rangamani [**2**], contains every facet of the holographic entropy cone other than subadditivity and its relabelings. An eight-party inequality Q_8, built on the Fano plane, and seven-party inequalities $E_0, \ldots, E_5$, where E_0 comes from Q_8 by an exact splitting and the deletion of a party and each E_{s+1} from E_s by subtracting a positive multiple of a sum of instances of the monogamy of mutual information, are valid and superbalanced, and none of them has an unscaled map. After each split the remaining inequality still has no unscaled map, while the subtracted instances have unscaled maps. A smaller superbalanced example has six parties: an inequality U_6 built on two triangles, with the purifier in the role of the hub of B_7^*, fails at a single point by the same covering argument, proved by hand, while $2U_6$ has an integer map on its realizable domain. Also, $3Q_8$ has no integer map, while $2E_0$, $2E_1$, $2E_3$ and $2E_4$ have integer maps on the full cube, so

these four inequalities need scaling by exactly 2. Non-existence is certified by DRAT proofs, which a program independent of the SAT solvers checks.

The third result follows the chain one step further. A decomposition of $24E_0$ into valid inequalities contains a superbalanced seven-party inequality F_7 that defines a facet of the seven-party cone, and the face of the cone on which E_0 vanishes lies in this facet. The inequality F_7 has an unscaled map on its realizable domain, but none that respects its symmetry group, which has order 6. The facet property is derived from 126 explicit graph models, which are certified. A shorter decomposition writes $2E_s$, for every s, as the sum of two seven-party inequalities V_7' and P_7 and twice a sum of instances of the monogamy of mutual information. Both V_7' and P_7 have unscaled maps on their realizable domains, while their half-sum has none.

The fourth measures the scaling. For star inequalities, whose left-hand side consists of single-party entropies and the entropy of all parties, the least scaling is the least k for which an explicit polytope contains a point of the grid $\frac{1}{k}\mathbb{Z}^m$. Read as a coin-weighing problem, this gives a general construction: a list of weighings that determines the weights of the coins, and whose weighings can be given positive weights such that every coin lies on each pan with the same total weight, gives a star inequality whose least scaling is the total weight of the coins, written in coprime integers. An explicit family D_j on $N = 4j + 7$ parties needs scaling by exactly $2^{j+2} = 2^{(N+1)/4}$, and a seven-party star Z_7 needs scaling by 8. With the weighing matrices of Alon and Vũ [**3**], the construction gives N-party star inequalities, which are implied by subadditivity and are not balanced, that need scaling by $N^{(1/4-o(1))N}$, while no N-party star inequality with a finite least scaling needs more than $N^{(1/2+o(1))N}$. The theorem on D_j, the general construction and the two bounds rest on written proofs, and the lower bound also on the theorem of Alon and Vũ. The first two members of D_j are also certified at some of the scalings, and $k_{\min}(Z_7) = 8$ is certified, as are five further examples on 9 to 17 parties.

The line of argument is the following. The entropies of N regions obey linear inequalities, and the valid ones cut out the holographic entropy cone HEC_N, which, by a theorem of Bao et al. [**1**], is also the set of entropy vectors of graph models, weighted graphs in which entropies are minimal cuts. A contraction map between Boolean cubes proves such an inequality, but it may exist only after the inequality has been multiplied by an integer k, and the companion follows the paper's two questions, when $k = 1$ suffices and how large k must be. Written in count form, the existence of a map is a finite integer feasibility problem on the cube, or on the smaller realizable domain of Avis and Hernández-Cuenca [**4**], and two lemmas make it a proof tool in both directions: a real-valued map on the realizable domain proves validity, and the absence of an integer map on a small subdomain, certified by a SAT solver with a checked DRAT proof, excludes integer maps on the cube and, when the LHS terms that contain the party used to build the subdomain form an antichain, on the realizable domain. With these tools, balance does not suffice for an unscaled map (B_7^*), and superbalance does not suffice either (U_6, Q_8 and $E_0, \ldots, E_5$). Following the superbalanced failure one step further reaches the facet F_7, which has an unscaled map on its realizable domain, though none that respects its symmetry. For star inequalities the least scaling is a lattice-point problem, which gives Z_7 and, read as coin weighing, D_j and star inequalities, implied by subadditivity and not balanced, whose least scaling grows faster than exponentially with N. The paper leaves open whether some facet needs $k > 1$, and the least scaling of the superbalanced failures is known only for E_0, E_1, E_3 and E_4, where it is 2, and on the realizable domain for U_6, where it is 2 as well.

The paper assigns each of its statements one of three statuses, and the companion uses them in the same sense, in a line "*Status:* ..." after each main statement of Chapters 7–11. In the paper's words, a statement is *certified* if it rests on an explicit certificate, a DRAT proof, an exact map, a list of graph models or an exact identity, checked by a program independent of the one that produced it (Chapter 6). A DRAT proof on a subdomain to which every map on a larger domain

restricts, such as the subdomain $\mathcal{U}(Q)$ of Chapter 5, counts as a certificate on that larger domain. It is *derived* if it follows from certified statements by a short proof given in the text, and *argued* if it rests on a written proof that was checked only by other instances of the language models that wrote the paper. Section 6.7 discusses the steps that the paper regards as least secure. The lemmas of Part I, on which the method rests, are proved in full in this companion and carry no status line.

Who this is for

This companion is written for a reader who knows holography at the level of a graduate course, the Ryu–Takayanagi formula in particular, and has not worked with the holographic entropy cone, contraction maps, SAT solvers or DRAT proofs.

Two ways to read

Part I builds the method, starting from the Ryu–Takayanagi formula for several regions: graph models and the cone, balance and superbalance, contraction maps in the count form used throughout, and the two lemmas on the realizable domain. Part II explains the certificates and proves the failures for balanced and superbalanced inequalities. Part III treats the facet F_7, the least scaling of star inequalities and the open questions, and Appendix A collects the symbols. A reader who wants the main line first can read Sections 4.1–4.3 (contraction maps and their count form), Section 5.3 (the realizable-domain criterion), Chapter 7 (the balanced failure, proved by hand) and Chapter 12 (the status of every result). A reader who is willing to trust the computer checks can skim Chapter 6 from the SAT encoding up to Section 6.4.

Each chapter opens with a blue box, *Where we are*, that says what the chapter does and what it needs. Cyan boxes, *Physics aside*, read a step in the language of holography; they are interpretation and are never used in a proof. In form this companion is shorter than the companions to the author's other notes of this kind. It works through examples in the text and has no exercises, and its sources are listed in the bibliography only.

Conventions

Parties are $A, B, C, \ldots$, the purifier is O, and $[N] = \{1, \ldots, N\}$ is the set of parties. A term written ABC in an inequality stands for $S(ABC)$, and $2ABC$ for $2S(ABC)$. Terms that contain O are replaced by the entropy of the complementary set of parties (Chapter 1), so every inequality is a vector $Q \in \mathbb{Z}^{2^N-1}$ and reads $Q \cdot S \geq 0$, and kQ is the inequality with all coefficients multiplied by k. We write $\mathrm{MMI}(X{:}Y{:}Z)$ for the coefficient vector of the instance $S(XY)+S(XZ)+S(YZ) \geq S(X)+S(Y)+S(Z)+S(XYZ)$ of the monogamy of mutual information, with parties concatenated, so that $\mathrm{MMI}(BC{:}DE{:}FG)$ has $X = \{B, C\}$. The Iverson bracket $[P]$ is 1 if the statement P holds and 0 otherwise. The companion keeps the paper's letters, except in a few places where a letter of the paper has a second meaning. Each such change is pointed out where the new letter is introduced. The programs that check the certificates, the certificates themselves, and a program that regenerates the Boolean formulas and the integer maps are in the code repository accompanying the paper.

What this companion does and does not establish

This companion explains the proofs of the paper, and it does not certify them. Writing a proof out in more detail makes it easier to check, but it does not make it checked. The worked examples test special cases with small numbers. They can expose an error in a formula or a number, but they do not replace the computations of the paper, which can only be checked by rerunning them or by redoing them by other means. One statement about the method, the proposition on changing the purifier (Proposition 5.22), is not in the paper. For the full cube it is an observation of Grimaldi, Headrick, Hubeny and Shteyner [**5**], the statement for the realizable domain is added here, and the

proposition is proved in full in this companion. Chapter 12 puts the status of every result in one place, and Section 6.7 names the steps that the paper regards as least secure.

Part I

Entropy vectors, the cone and contraction maps

CHAPTER 1

Entropy vectors of several boundary regions

Where we are. This chapter fixes the objects whose inequalities the paper studies: the Ryu–Takayanagi entropies of all unions of N disjoint boundary regions, collected into one vector. It then writes down the first linear inequalities they obey, recalls how the cut-and-paste argument proves them, and shows that the monogamy of mutual information is a property of holographic states only.

1.1. The Ryu–Takayanagi formula for N regions

Consider a holographic state with a time-symmetric bulk slice Σ, a Riemannian manifold whose conformal boundary $\partial\Sigma$ is the time slice of the boundary theory. For a region $\mathsf{A} \subseteq \partial\Sigma$, Ryu and Takayanagi proposed that the entanglement entropy of A is, at leading order in $1/G_N$,

$$S(\mathsf{A}) = \min_{\gamma} \frac{\mathrm{Area}(\gamma)}{4G_N} \ , \tag{1.1}$$

where the minimum runs over surfaces $\gamma \subseteq \Sigma$ that are homologous to A, that is, for which there is a bulk region $W \subseteq \Sigma$ whose boundary consists of A and γ [**6, 7**]. A minimizer is an RT surface of A, and the corresponding region W is its homology region. Areas of surfaces that reach the conformal boundary diverge and are regulated by a cutoff near $\partial\Sigma$, and every statement below concerns the regulated quantities.

For states without a time-reflection symmetry, Hubeny, Rangamani and Takayanagi replaced the minimal surface on a slice by an extremal surface in the full spacetime [**8**]. This companion works with the formula (1.1) on a time-symmetric slice, and the covariant formula is not used anywhere below.

Now fix N disjoint regions $A_1, \dots, A_N$ of $\partial\Sigma$, the *parties*, written $A, B, C, \dots$ in examples, and let $O = \partial\Sigma \setminus (A_1 \cup \cdots \cup A_N)$ be the rest of the boundary, the *purifier*. For a nonempty set $I \subseteq [N] = \{1, \dots, N\}$ we write $S(I)$ for the RT entropy (1.1) of the union $\bigcup_{i\in I} A_i$, and $S(ABC)$ for $S(\{A, B, C\})$. The $2^N - 1$ numbers $S(I)$ form the *entropy vector*

$$S = \big(S(I)\big)_{\emptyset \neq I \subseteq [N]} \in \mathbb{R}^{2^N-1} \ . \tag{1.2}$$

Unions that contain the purifier need no extra coordinates, as Bao et al. observe [**1**]. A surface γ is homologous to a region A through W exactly when it is homologous to the complement $\partial\Sigma \setminus \mathsf{A}$ through $\Sigma \setminus W$, so a region and its complement share their RT surfaces and have the same entropy, as for a pure state of the whole boundary. The complement of $\bigcup_{i\in I} A_i$ together with O is $\bigcup_{i\in[N]\setminus I} A_i$, so

$$S\big(I \cup \{O\}\big) = S\big([N] \setminus I\big) \qquad \text{for every } I \subseteq [N] \ , \tag{1.3}$$

with $S(\emptyset) = 0$, and in particular $S(O) = S([N])$. We call (1.3) *purification*. Every entropy that involves the purifier is thereby one of the coordinates of (1.2), and from now on every term of an inequality is written as the entropy of a nonempty subset of $[N]$: a term that contains O is replaced by the entropy of the complementary set of parties. Chapter 2 proves the analog of (1.3) for graph models, which is the form in which it is used.

1.2. Linear entropy inequalities

A linear entropy inequality is written

$$Q: \qquad \sum_{l=1}^{L} \alpha_l S(I_l) \;\geq\; \sum_{r=1}^{R} \beta_r S(J_r)\ , \tag{1.4}$$

with L left-hand side (LHS) terms, R right-hand side (RHS) terms, positive integers α_l, β_r and nonempty sets $I_l, J_r \subseteq [N]$, the I_l distinct, the J_r distinct and no I_l equal to a J_r. Rational coefficients can always be cleared by multiplying with a common denominator, and like terms can always be combined, so these conventions lose nothing. The same letter Q denotes the coefficient vector in $\mathbb{Z}^{2^N-1}$, with entry α_l at I_l, $-\beta_r$ at J_r and 0 elsewhere, so that (1.4) reads

$$Q \cdot S = \sum_{l=1}^{L} \alpha_l S(I_l) - \sum_{r=1}^{R} \beta_r S(J_r) \;\geq\; 0\ . \tag{1.5}$$

We write $|\alpha| = \sum_l \alpha_l$ and $|\beta| = \sum_r \beta_r$, and kQ for the inequality with all coefficients multiplied by a positive integer k. The letters L and R always count terms.

For pairwise disjoint nonempty $X, Y, Z \subseteq [N]$, with $XY = X \cup Y$, the first three examples are *subadditivity* (SA), *strong subadditivity* (SSA) and the *monogamy of mutual information* (MMI),

$$\begin{aligned} &S(X) + S(Y) \;\geq\; S(XY)\ , \qquad S(XY) + S(YZ) \;\geq\; S(Y) + S(XYZ)\ , \\ &S(XY) + S(XZ) + S(YZ) \;\geq\; S(X) + S(Y) + S(Z) + S(XYZ)\ . \end{aligned} \tag{1.6}$$

Each choice of X, Y, Z gives an *instance*. We write MMI(X:Y:Z) for the coefficient vector of the instance of MMI, with parties concatenated, so that MMI(BC:DE:FG) has $X = \{B, C\}$, $Y = \{D, E\}$ and $Z = \{F, G\}$. In terms of the tripartite information $I_3(X{:}Y{:}Z) = S(X) + S(Y) + S(Z) - S(XY) - S(XZ) - S(YZ) + S(XYZ)$, MMI reads $I_3(X{:}Y{:}Z) \leq 0$, so MMI(X:Y:Z) $\cdot\, S = -I_3(X{:}Y{:}Z)$.

Subadditivity and strong subadditivity hold for the von Neumann entropies of all quantum states. For RT entropies the next section recalls their geometric proof. MMI was proved for RT entropies by Hayden, Headrick and Maloney [**9**], and Section 1.4 shows that it fails for some quantum states.

1.3. Cut and paste

The proofs of these inequalities for RT entropies all follow one pattern, which the contraction method of Chapter 4 turns into a combinatorial statement. We recall it here as motivation. The arguments of this section are the standard geometric ones and assume that the regions involved have piecewise smooth boundaries. Nothing later depends on them, since Chapters 2–5 prove everything for graph models.

For a bulk region $W \subseteq \Sigma$ write $|\partial W|$ for the area of the part of its boundary that lies in the bulk. If the part of ∂W on the conformal boundary is the region A, we call W a *candidate region* for A. Then the bulk part of ∂W is a surface homologous to A, so $|\partial W| \geq 4G_N S(\mathsf{A})$ by (1.1), with equality when W is the homology region of an RT surface.

Subadditivity. Let W_X and W_Y be homology regions of RT surfaces of X and Y. The union $W_X \cup W_Y$ is a candidate region for XY, and every bulk point of its boundary lies on ∂W_X or on ∂W_Y, so

$$4G_N S(XY) \leq |\partial(W_X \cup W_Y)| \leq |\partial W_X| + |\partial W_Y| = 4G_N\big(S(X) + S(Y)\big)\ . \tag{1.7}$$

Strong subadditivity. Headrick and Takayanagi proved SSA for RT entropies [**7**]. With W_1 and W_2 the homology regions of RT surfaces of XY and YZ, the union $W_1 \cup W_2$ is a candidate region for XYZ and the intersection $W_1 \cap W_2$ one for Y. A piece of ∂W_1 that does not lie on ∂W_2 belongs

to exactly one of $\partial(W_1 \cup W_2)$ and $\partial(W_1 \cap W_2)$, according to whether W_2 lies on neither side of it or on both, and a piece shared by ∂W_1 and ∂W_2 belongs to both or to neither. Hence the areas satisfy the submodular inequality

$$|\partial(W_1 \cup W_2)| + |\partial(W_1 \cap W_2)| \leq |\partial W_1| + |\partial W_2| \ , \tag{1.8}$$

and minimality gives $S(XYZ) + S(Y) \leq S(XY) + S(YZ)$.

Words. Following Bao et al. [**1**], both arguments can be organized by labeling every bulk point p with the bitstring that records which homology regions of the LHS contain it. In the proof of SSA the point p gets $([p \in W_1], [p \in W_2]) \in \{0,1\}^2$. Near the boundary region X the word is 10, near Y it is 11, near Z it is 01 and near O it is 00. The candidate region for Y consists of the points with word 11, and the one for XYZ of the points with words 10, 11, 01. So the construction is a rule $f\colon \{0,1\}^2 \to \{0,1\}^2$ that tells, from the word of a point alone, to which candidate regions of the RHS terms $S(Y)$ and $S(XYZ)$ it belongs: $f(00) = 00$, $f(10) = 01$, $f(01) = 01$ and $f(11) = 11$.

MMI. The same bookkeeping proves MMI for $X = A$, $Y = B$, $Z = C$. Let W_1, W_2, W_3 be homology regions of RT surfaces of the LHS terms AB, AC and BC, give each bulk point p the word $x(p) = ([p \in W_1], [p \in W_2], [p \in W_3]) \in \{0,1\}^3$, and let Σ_u be the set of points with word u. The eight sets Σ_u, some of them possibly empty, partition Σ. A homology region of an RT surface of a union of parties contains the bulk points next to the interior of each of its parties and none next to the other parties. Hence the bulk points next to A have the word 110, those next to B the word 101, those next to C the word 011, and those next to O the word 000. These four words record which LHS terms contain the party, and Chapter 3 calls them the occurrence words. Now define candidate regions for the four RHS terms,

$$W'_A = \Sigma_{110} \ , \qquad W'_B = \Sigma_{101} \ , \qquad W'_C = \Sigma_{011} \ , \qquad W'_{ABC} = \Sigma \setminus \Sigma_{000} \ . \tag{1.9}$$

By the previous paragraph their parts on the conformal boundary are A, B, C and ABC, and Figure 1 shows the pieces in an example. Up to sets of zero area, every bulk point of $\partial W_1 \cup \partial W_2 \cup \partial W_3$ lies on the common boundary of exactly two pieces Σ_u and Σ_v, and it lies on ∂W_l exactly when $u_l \neq v_l$. Writing $a(u,v)$ for the area of the common boundary of Σ_u and Σ_v and $d_1(u,v)$ for the Hamming distance, the number of positions in which u and v differ, the LHS areas are therefore

$$|\partial W_1| + |\partial W_2| + |\partial W_3| = \sum_{\{u,v\}} a(u,v) d_1(u,v) \ , \tag{1.10}$$

with the sum over unordered pairs of distinct words. Each candidate region in (1.9) is a union of pieces. Let $f(u) \in \{0,1\}^4$ record to which of W'_A, W'_B, W'_C, W'_{ABC} the piece Σ_u belongs. Then

$$\begin{aligned} &f(110) = 1001 \ , \qquad f(101) = 0101 \ , \qquad f(011) = 0011 \ , \qquad f(000) = 0000 \ , \\ &f(100) = f(010) = f(001) = f(111) = 0001 \ , \end{aligned} \tag{1.11}$$

and the same counting gives $\sum_J |\partial W'_J| = \sum_{\{u,v\}} a(u,v) d_1(f(u), f(v))$. Section 4.2 checks that $d_1(f(u), f(v)) \leq d_1(u,v)$ for all $u, v \in \{0,1\}^3$. With minimality of the RT surfaces of A, B, C and ABC,

$$\begin{aligned} 4G_N\big(S(A) + S(B) + S(C) + S(ABC)\big) &\leq \sum_{\{u,v\}} a(u,v) d_1\big(f(u), f(v)\big) \leq \sum_{\{u,v\}} a(u,v) d_1(u,v) \\ &= 4G_N\big(S(AB) + S(AC) + S(BC)\big) \ . \end{aligned} \tag{1.12}$$

This is the proof of Hayden, Headrick and Maloney [**9**], whose regions are those of (1.9), organized as the special case for MMI of the proof by contraction of Bao, Nezami, Ooguri, Stoica, Sully and Walter [**1**] and written on the bulk slice. Its ingredients are the words of bulk points, a map f that is fixed on the words of the parties and of the purifier, and the condition that f does not increase Hamming distances. Chapter 4 states and proves the general theorem on graph models, where the pieces become sets of vertices and the areas $a(u,v)$ become sums of edge weights.

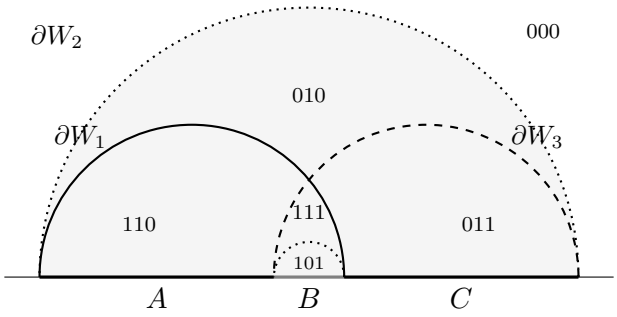


FIGURE 1. The pieces Σ_u for three adjacent intervals A, B, C of lengths 2, 0.6, 2 on the boundary of the Poincaré half-plane, for which the RT surface of AC is the connected one, the large arc together with the small arc over B. The word u records membership in the homology regions W_1, W_2, W_3 of AB (solid), AC (dotted) and BC (dashed). The pieces 100 and 001 are empty here. The candidate region $W'_A = \Sigma_{110}$ is bounded by A, part of ∂W_1 and part of ∂W_3. It is anchored on A, and its bulk boundary is not minimal.

1.4. MMI is not an inequality for all quantum states

Subadditivity and strong subadditivity constrain every quantum state, so their geometric proofs above are consistency checks of the RT formula. MMI is different, as Hayden, Headrick and Maloney point out [**9**]. The following example is that of Bao et al. [**1**], and its marginal on three qubits is the classically correlated state with $I_3 = 1$ of [**9**]. Consider four qubits A, B, C, O in the state

$$|\mathrm{GHZ}_4\rangle = \frac{1}{\sqrt{2}}\big(|0000\rangle + |1111\rangle\big) \ . \tag{1.13}$$

Tracing out any nonempty proper subset of the four qubits leaves, on the remaining one, two or three qubits, the density matrix $\frac{1}{2}(|0\cdots0\rangle\langle 0\cdots 0| + |1\cdots1\rangle\langle 1\cdots 1|)$, which has the two nonzero eigenvalues $\frac{1}{2}$, $\frac{1}{2}$. So every nonempty proper subset of $\{A, B, C, O\}$ has entropy 1 bit, in particular $S(A) = S(B) = S(C) = S(AB) = S(AC) = S(BC) = S(ABC) = 1$. Then

$$\begin{aligned} \mathrm{MMI}(A{:}B{:}C)\cdot S &= S(AB) + S(AC) + S(BC) - S(A) - S(B) - S(C) - S(ABC) \\ &= 3 - 3 - 1 = -1 < 0 \ , \end{aligned} \tag{1.14}$$

and MMI fails. So MMI is a property of RT entropies that general quantum states do not share, and the GHZ entropy vector $(1, 1, 1, 1, 1, 1, 1)$ of three parties is not, at leading order in $1/G_N$, the entropy vector of any three boundary regions in any time-symmetric holographic state.

CHAPTER 2

Graph models and the holographic entropy cone

Where we are. The RT formula minimizes an additive cost, the area, over bulk regions anchored on a boundary region. A graph model keeps exactly this structure in discrete form, and every validity and facet statement in the paper and in this companion is proved for graph models. This chapter defines them, states the theorem that identifies their entropy vectors with the RT entropy vectors, and proves the facts about the resulting cone that the later chapters use: the facet criterion, the positive-combination lemma, the action of relabelings, and the isolated-party lemma.

2.1. Graph models

DEFINITION 2.1 (Graph models [**1**]). A *graph model* Γ on N parties consists of a finite set V of vertices, a finite set of edges, each an unordered pair $e = \{w, w'\}$ of distinct vertices with a weight $w_e \geq 0$, a set $\partial V \subseteq V$ of *boundary vertices*, and a labeling $b\colon \partial V \to [N] \cup \{O\}$. The vertices in $V \setminus \partial V$ are *bulk vertices*. For $I \subseteq [N]$, an *I-cut* is a set $W \subseteq V$ with

$$W \cap \partial V = b^{-1}(I)\ , \tag{2.2}$$

its weight $C(W)$ is the total weight of the edges with exactly one endpoint in W, and the entropy of I is

$$S(I) = \min\big\{C(W)\colon W \text{ an } I\text{-cut}\big\}\ . \tag{2.3}$$

An I-cut attaining the minimum is a *min cut* of I. The *entropy vector* of Γ is $S_\Gamma = (S(I))_{\emptyset \neq I \subseteq [N]} \in \mathbb{R}^{2^N-1}$.

The paper writes a graph model as G and its entropy vector as S_G. Here G is a party, so we write Γ and S_Γ. Vertices are written w, w', where the paper writes v (and u_e, v_e for the endpoints of an edge), because the letters u, v are kept, as in the paper, for points of the Boolean cubes of Chapter 4. The weight of an edge e is always w_e, with the edge as subscript, which distinguishes it from a vertex w.

Condition (2.2) says that an I-cut contains every boundary vertex labeled by a party in I, no boundary vertex labeled by a party outside I, and no boundary vertex of O. Bulk vertices may lie on either side. A label may have several boundary vertices or none. There are finitely many I-cuts, since V is finite, so the minimum in (2.3) exists. For $I = \emptyset$ the set $W = \emptyset$ is an I-cut of weight 0, so $S(\emptyset) = 0$.

EXAMPLE 2.4 (One bulk vertex). Let $N = 3$, and let Γ have four boundary vertices, one for each of A, B, C, O, which we denote by their labels, one bulk vertex w_0, and four edges of weight 1 joining w_0 to A, B, C and O. An I-cut is $b^{-1}(I)$ or $b^{-1}(I) \cup \{w_0\}$. For a single party, $\{A\}$ cuts one edge and $\{A, w_0\}$ cuts three, so $S(A) = 1$. For two parties, $\{A, B\}$ and $\{A, B, w_0\}$ both cut two edges, so $S(AB) = 2$ with two min cuts. For three parties, $\{A, B, C\}$ cuts three edges and $\{A, B, C, w_0\}$ cuts one, the edge to O, so $S(ABC) = 1$. In the order $(A, B, C, AB, AC, BC, ABC)$,

$$S_\Gamma = (1, 1, 1, 2, 2, 2, 1)\ , \tag{2.5}$$

and $\mathrm{MMI}(A{:}B{:}C) \cdot S_\Gamma = 6 - 3 - 1 = 2$. This is the star graph model of Bao et al. for the extreme ray $(1,1,1;2,2,2;1)$ of HEC_3 [**1**], and He, Hubeny and Rangamani draw the same graph for the entropy vector of a four-party perfect tensor [**2**].

Physics aside 2.6. For a holographer a graph model is a discretized time-symmetric bulk slice. A party is a boundary region and the purifier the rest of the boundary, an edge weight is an element of area in units of $4G_N$, an I-cut W is a bulk region anchored on the regions of I, its weight $C(W)$ is the area of its bulk boundary, a min cut is the homology region of an RT surface, and (2.3) is the RT formula (1.1). The cut-and-paste arguments of Section 1.3 become statements about sets of vertices. This remark, and every later remark titled "Physics", is interpretation: nothing in any proof depends on it. The proofs use only that $S(I)$ is the minimum of an additive cost over vertex sets anchored on I.

2.2. Purification

The analog of (1.3) holds in every graph model. For a set $Y \subseteq [N] \cup \{O\}$ that contains O, define a Y-cut by $W \cap \partial V = b^{-1}(Y)$, its entropy $S(Y)$ as the least weight of a Y-cut, and write $Y^c = ([N] \cup \{O\}) \setminus Y$, a subset of $[N]$.

Lemma 2.7 (Purification). *In every graph model, $C(W) = C(V \setminus W)$ for every $W \subseteq V$, and $S(Y) = S(Y^c)$ for every $Y \subseteq [N] \cup \{O\}$ that contains O.*

Proof. An edge has exactly one endpoint in W if and only if it has exactly one endpoint in $V \setminus W$, so $C(W) = C(V \setminus W)$. Since $b^{-1}(Y)$ and $b^{-1}(Y^c)$ partition ∂V, a set W satisfies $W \cap \partial V = b^{-1}(Y)$ exactly when $(V \setminus W) \cap \partial V = b^{-1}(Y^c)$. So $W \mapsto V \setminus W$ is a bijection from the Y-cuts onto the Y^c-cuts, and it preserves the weight. The two minima are equal. □

So the entropy of every union of labels is a coordinate of S_Γ, exactly as for RT entropies, and the convention of Section 1.1 applies: every term that contains O is replaced by the entropy of its complement. For $Y = [N] \cup \{O\}$ the lemma gives $S(Y) = S(\emptyset) = 0$.

2.3. The holographic entropy cone

A subset of $\mathbb{R}^d$ is a *convex cone* if it contains 0 and is closed under addition and under multiplication by nonnegative numbers.

Lemma 2.8 (Sums and multiples). *The entropy vectors of graph models on N parties form a convex cone in $\mathbb{R}^{2^N-1}$.*

Proof. The argument is the one Bao et al. give for bulk geometries, by disjoint unions and rescaled metrics [**1**]. The graph with no vertices has entropy vector 0. Let Γ_1 and Γ_2 be graph models on the same parties, and let Γ be their disjoint union, with the labelings of both. An I-cut of Γ is the union of an I-cut of Γ_1 and an I-cut of Γ_2, chosen independently, and no edge joins the two parts, so its weight is the sum of the two weights. Minimizing over the two parts separately gives $S_\Gamma = S_{\Gamma_1} + S_{\Gamma_2}$. Multiplying all edge weights of a graph model by the same nonnegative number multiplies every cut weight, and hence every entropy, by that number. □

Theorem 2.9 (Bao, Nezami, Ooguri, Stoica, Sully and Walter [**1**]). *The* holographic entropy cone $\mathrm{HEC}_N \subseteq \mathbb{R}^{2^N-1}$, *defined through the Ryu–Takayanagi formula from the entropy vectors* (1.2) *of N boundary regions, is the set of entropy vectors of graph models on N parties, and it is a rational polyhedral cone: there are finitely many vectors $Q_1, \ldots, Q_m \in \mathbb{Z}^{2^N-1}$ with $\mathrm{HEC}_N = \{S \colon Q_a \cdot S \geq 0$ for $a = 1, \ldots, m\}$.*

The companion uses this theorem only to connect the graph models with RT entropies. All statements below are proved for graph models. A further property of HEC_N that is used is the following [**4**].

THEOREM 2.10 (Full dimension [**4**]). HEC_N *is full-dimensional: its linear span is* $\mathbb{R}^{2^N-1}$.

The cone is known completely for $N \le 5$ [**1, 10**], and two infinite families of its facets, in the sense defined next, are known [**11, 12**].

2.4. Validity, faces and facets

An inequality Q of the form (1.4) is *valid* if $Q \cdot S_\Gamma \ge 0$ for every graph model Γ, that is, by Theorem 2.9, if it holds on HEC_N. A graph model with $Q \cdot S_\Gamma = 0$ is *tight* for Q. For a valid $Q \ne 0$, the *face* of Q is

$$F_Q = \mathrm{HEC}_N \cap Q^\perp \,, \qquad Q^\perp = \left\{ S \in \mathbb{R}^{2^N-1} \colon Q \cdot S = 0 \right\} , \tag{2.11}$$

the set of entropy vectors on which Q is saturated. The dimension of a cone is the dimension of its linear span. Since $Q^\perp$ is a hyperplane, $\dim F_Q \le 2^N - 2$, one less than $\dim \mathrm{HEC}_N = 2^N - 1$ (Theorem 2.10). The valid inequality Q *defines a facet* of HEC_N if $\dim F_Q = 2^N - 2$, the largest possible value.

PROPOSITION 2.12 (Facet criterion). *A valid* $Q \ne 0$ *defines a facet of* HEC_N *if and only if there are* $2^N - 2$ *tight graph models whose entropy vectors are linearly independent.*

PROOF. This is a standard fact of polyhedral theory, used in this form e.g. by Avis and Hernández-Cuenca [**4**]. By Theorem 2.9, the elements of F_Q are exactly the entropy vectors of the tight graph models. The dimension of the span of F_Q is the largest number of linearly independent elements of F_Q, and the statement follows. □

The criterion turns the facet property into a finite certificate: a list of $2^N - 2$ explicit graph models, each tight for Q, whose entropy vectors have rank $2^N - 2$. Together with a proof that Q is valid, such a list proves that Q defines a facet. Chapter 9 uses it with $2^7 - 2 = 126$ graph models.

2.5. Positive combinations

The paper proves that an inequality is not a facet by writing it as a positive combination of valid inequalities, and it proves that the face of one inequality lies in the face of another in the same way. Both rest on the following lemma.

LEMMA 2.13 (Positive combinations). *Let* $Q = \sum_{a=1}^{m} \gamma_a X_a$ *with real* $\gamma_a > 0$ *and valid* $X_a \in \mathbb{R}^{2^N-1}$.

(a) *Every graph model that is tight for* Q *is tight for every* X_a. *If* $Q \ne 0$ *and* $X_a \ne 0$, *then* $F_Q \subseteq F_{X_a}$.
(b) *If* Q *defines a facet of* HEC_N, *then every nonzero* X_a *is a positive multiple of* Q.

PROOF. (a) If $Q \cdot S_\Gamma = 0$, then $0 = \sum_a \gamma_a (X_a \cdot S_\Gamma)$ is a sum of nonnegative numbers, since each X_a is valid and each $\gamma_a > 0$. So every term vanishes, and $X_a \cdot S_\Gamma = 0$.

(b) Let $X_a \ne 0$. By (a), X_a vanishes on F_Q, hence on the linear span of F_Q. Since Q defines a facet, this span has dimension $2^N - 2$ and lies in the hyperplane $Q^\perp$, so it equals $Q^\perp$. A linear form that vanishes on the hyperplane $Q^\perp$ is a multiple of Q, so $X_a = \nu Q$ for a real ν, and $\nu \ne 0$ because $X_a \ne 0$. If $\nu < 0$, then $X_a \cdot S = \nu (Q \cdot S) \le 0$ for every $S \in \mathrm{HEC}_N$, while $X_a \cdot S \ge 0$ by validity. So X_a would vanish on HEC_N, hence on its span, which is $\mathbb{R}^{2^N-1}$ by Theorem 2.10, and $X_a = 0$, a contradiction. Hence $\nu > 0$. □

Part (b) says that a facet is not a positive combination of valid inequalities unless all of them are positive multiples of it. The paper's typical use reads as follows. If $Q = \gamma_1 X_1 + \gamma_2 X_2$ with $\gamma_1, \gamma_2 > 0$, both X_a valid, and X_1 nonzero and not a positive multiple of Q, then Q is not a facet.

2.6. Relabelings

The $N+1$ labels $[N] \cup \{O\}$ play symmetric roles: the purifier is the label whose regions are left out of the coordinates. A permutation σ of $[N] \cup \{O\}$, an element of the symmetric group $\mathfrak{S}_{N+1}$, acts on graph models by relabeling the boundary vertices: $\sigma\Gamma$ has the labeling $\sigma \circ b$. We write σ for a generic relabeling, where the paper writes g, because the companion reserves g for one particular relabeling in Chapter 9.

For $I \subseteq [N]$, the I-cuts of $\sigma\Gamma$ are the sets W with $W \cap \partial V = (\sigma \circ b)^{-1}(I) = b^{-1}(\sigma^{-1}(I))$, the $\sigma^{-1}(I)$-cuts of Γ. So

$$S_{\sigma\Gamma}(I) = S_\Gamma\big(\sigma^{-1}(I)\big) \ , \tag{2.14}$$

where the right-hand side is read through Lemma 2.7 when $\sigma^{-1}(I)$ contains O. This defines an action of $\mathfrak{S}_{N+1}$ on $\mathbb{R}^{2^N-1}$, $(\sigma S)(I) = S(\sigma^{-1}(I))$ with purification, under which $\sigma S_\Gamma = S_{\sigma\Gamma}$. Hence HEC_N is invariant under relabelings [**1**]. On inequalities the action is $(\sigma Q) \cdot S = Q \cdot (\sigma^{-1} S)$, so that $(\sigma Q) \cdot (\sigma S) = Q \cdot S$. Concretely, $Q \cdot (\sigma^{-1} S) = \sum_I Q_I S(\sigma(I))$, so σQ is obtained from Q by renaming the labels in every term according to σ and then purifying the terms that contain O. Since HEC_N is invariant, σQ is valid if and only if Q is, and σQ defines a facet if and only if Q does. The *stabilizer* of Q is

$$\mathrm{Stab}(Q) = \big\{\sigma \in \mathfrak{S}_{N+1} \colon \sigma Q = Q\big\} \ . \tag{2.15}$$

For example, with $N = 2$, the relabeling that exchanges B and O turns subadditivity $S(A)+S(B) \geq S(AB)$ into $S(A) + S(O) \geq S(AO)$, which after purification reads $S(A) + S(AB) \geq S(B)$. This is the Araki–Lieb inequality [**13**], valid because subadditivity is, and the example with which Hubeny, Rangamani and Rota introduce the exchange of a party with the purifier [**14**].

2.7. The isolated-party lemma

Adding a party that is attached to nothing changes nothing, and in particular it does not change validity [**1, 4**]. This fact is used twice: in Chapter 8 to delete a party from an eight-party inequality, and in Chapter 11 to extend examples on 7, 9 and 15 parties to all larger N. For a graph model Γ on N parties, let Γ^+ be the graph model on $N+1$ parties obtained by adding one boundary vertex, labeled by the new party $N+1$, with no edges.

Lemma 2.16 (Isolated party). *For every $I \subseteq [N+1]$,*

$$S_{\Gamma^+}(I) = S_\Gamma\big(I \setminus \{N+1\}\big) \ , \tag{2.17}$$

with $S(\emptyset) = 0$. Consequently:

(a) Deletion. *For an inequality Q on $N+1$ parties, let Q^- be the N-party inequality obtained by replacing every term $S(I)$ by $S(I \setminus \{N+1\})$, dropping terms $S(\emptyset)$ and combining like terms. Then $Q^- \cdot S_\Gamma = Q \cdot S_{\Gamma^+}$ for every N-party graph model Γ, and if Q is valid, so is Q^-.*

(b) Padding. *For an inequality Q on N parties, let Q^+ be the same inequality read as an inequality on $N+1$ parties, so that the new party lies in no term. Then Q^+ is valid if and only if Q is.*

Proof. Let h be the added vertex. A set W' is an I-cut of Γ^+ exactly when $W' \setminus \{h\}$ is an $(I \setminus \{N+1\})$-cut of Γ and $h \in W'$ holds exactly when $N+1 \in I$. Since h lies on no edge, W' and $W' \setminus \{h\}$ have the same weight, and the minima agree. This is (2.17).

(a) By (2.17), the value of each term $S(I)$ of Q on S_{Γ^+} is the value of $S(I \setminus \{N+1\})$ on S_Γ, so $Q \cdot S_{\Gamma^+} = Q^- \cdot S_\Gamma$. If Q is valid, the left-hand side is nonnegative for every Γ.

(b) The inequality Q is the deletion of $N+1$ from Q^+, so Q^+ valid implies Q valid by (a). Conversely, let Q be valid and let Γ' be a graph model on $N+1$ parties. Relabel the boundary vertices of the party $N+1$ as O, which gives a graph model Γ on N parties. For $I \subseteq [N]$, an I-cut contains the boundary vertices of the parties in I and avoids all other boundary vertices, in particular those that carried the label $N+1$, under either labeling. So Γ and Γ' have the same I-cuts and the same $S(I)$ for every $I \subseteq [N]$. All terms of Q^+ are subsets of $[N]$, hence $Q^+ \cdot S_{\Gamma'} = Q \cdot S_\Gamma \geq 0$. □

Deletion of a party other than $N+1$ is defined in the same way, after a relabeling. Corollary 3.12 adds that deletion preserves superbalance, and Corollary 4.24 that padding preserves the contraction maps of Chapter 4 for every scaling. Both need notions defined later.

CHAPTER 3

Bell pairs, balance and superbalance

Where we are. The simplest graph models are single edges. Evaluating an inequality on them gives a finite set of numbers, the Bell values, which every valid inequality must keep nonnegative. The two classes of the paper, balanced and superbalanced inequalities, are defined by the vanishing of some or all of these numbers. This chapter also introduces the occurrence words, which are the boundary data of every contraction map in Chapter 4.

3.1. Bell-pair vectors

For distinct labels $i, j \in [N] \cup \{O\}$, let Γ_{ij} be a graph model with one edge, of weight 1, between a boundary vertex labeled i and a boundary vertex labeled j. Any further vertices lie on no edge. Its entropy vector is the *Bell-pair vector* $r_{ij} = S_{\Gamma_{ij}}$ [**1, 2**], whose values Avis and Hernández-Cuenca state in the form of Lemma 3.1 [**4**].

LEMMA 3.1 (Bell-pair vectors). *For every nonempty $I \subseteq [N]$, $r_{ij}(I) = 1$ if I contains exactly one of i and j, and $r_{ij}(I) = 0$ otherwise. In particular $r_{iO}(I) = [i \in I]$.*

PROOF. If I contains exactly one of i, j, every I-cut contains exactly one endpoint of the edge, so it has weight 1. If I contains both or neither, the I-cut $b^{-1}(I)$ contains both endpoints or neither, and has weight 0. Since $I \subseteq [N]$ never contains O, for $j = O$ the condition reads $i \in I$. □

The vector r_{ij} is also the entropy vector of a Bell pair of qubits shared between the regions i and j, which explains the name. Only the graph-model statement is used.

3.2. Occurrence words and the Bell-value formula

Let Q be the inequality (1.4). The *occurrence words* of a party $i \in [N]$, the occurrence vectors of Bao et al. [**1**], record which terms contain it,

$$x^{(i)} \in \{0,1\}^L \ , \quad x^{(i)}_l = [i \in I_l] \ , \qquad y^{(i)} \in \{0,1\}^R \ , \quad y^{(i)}_r = [i \in J_r] \ , \tag{3.2}$$

and for the purifier $x^{(O)} = 0$ and $y^{(O)} = 0$, since no term contains O. On $\{0,1\}^L$ we use the Hamming distance weighted by the LHS coefficients, and on $\{0,1\}^R$ the one weighted by the RHS coefficients,

$$d_\alpha(u,v) = \sum_{l=1}^{L} \alpha_l |u_l - v_l| \ , \qquad d_\beta(y,y') = \sum_{r=1}^{R} \beta_r |y_r - y'_r| \ , \tag{3.3}$$

and d_1 denotes the unweighted Hamming distance, the number of positions in which two words differ. Points of $\{0,1\}^L$ are written u, v, and 0 is the word with all bits 0.

For MMI(A:B:C), with the LHS terms in the order (AB, AC, BC) and the RHS terms in the order (A, B, C, ABC), the occurrence words are

$$\begin{aligned} &x^{(A)} = 110 \ , \quad x^{(B)} = 101 \ , \quad x^{(C)} = 011 \ , \quad x^{(O)} = 000 \ , \\ &y^{(A)} = 1001 \ , \quad y^{(B)} = 0101 \ , \quad y^{(C)} = 0011 \ , \quad y^{(O)} = 0000 \ , \end{aligned} \tag{3.4}$$

the words of the bulk points next to each region in Section 1.3 and the values prescribed for f there.

LEMMA 3.5 (Bell values). *For distinct $i, j \in [N] \cup \{O\}$, the* Bell value *of Q on the pair i, j is*

$$Q \cdot r_{ij} = d_\alpha\big(x^{(i)}, x^{(j)}\big) - d_\beta\big(y^{(i)}, y^{(j)}\big) \ . \tag{3.6}$$

PROOF. By Lemma 3.1, the term $S(I_l)$ takes the value 1 on r_{ij} exactly when I_l contains exactly one of i and j, that is, when the bits $x_l^{(i)} = [i \in I_l]$ and $x_l^{(j)} = [j \in I_l]$ differ. This holds also for $j = O$, since $x_l^{(O)} = 0$ and $O \notin I_l$. So $S(I_l)$ takes the value $|x_l^{(i)} - x_l^{(j)}|$, and likewise $S(J_r)$ takes the value $|y_r^{(i)} - y_r^{(j)}|$. Inserting these values into (1.5) gives

$$Q \cdot r_{ij} = \sum_{l=1}^{L} \alpha_l \big|x_l^{(i)} - x_l^{(j)}\big| - \sum_{r=1}^{R} \beta_r \big|y_r^{(i)} - y_r^{(j)}\big| \ , \tag{3.7}$$

which is (3.6). □

3.3. Balanced and superbalanced inequalities

Every r_{ij} is an entropy vector of a graph model, so a valid inequality has nonnegative Bell values. Bao et al. use this as the first test in their search for a contraction map: if the prescribed values on the occurrence words do not contract, the inequality is violated for a Bell pair [**1**]. With $j = O$, (3.7) reads

$$Q \cdot r_{iO} = \sum_{l\colon i \in I_l} \alpha_l - \sum_{r\colon i \in J_r} \beta_r \ , \tag{3.8}$$

the total coefficient of the terms that contain i on the left minus that on the right.

DEFINITION 3.9 (Balance and superbalance). The inequality Q is *balanced* if $Q \cdot r_{iO} = 0$ for every party $i \in [N]$, that is, if every party has the same total coefficient on both sides [**1**]. It is *superbalanced* if $Q \cdot r_{ij} = 0$ for all distinct $i, j \in [N] \cup \{O\}$.

The pairwise form of superbalance is that of Grimaldi, Headrick, Hubeny and Shteyner [**5**]. It has earlier forms. Bao et al. observed that their five-party inequalities are saturated by Bell pairs shared between any two regions, the purifying region included [**1**]. Hubeny, Rangamani and Rota introduced superbalance and showed that a balanced inequality is superbalanced exactly when it stays balanced under every purification [**14**]. He, Hubeny and Rangamani use it in this form, and they proved that an inequality is superbalanced exactly when its expansion in the coordinates of the K-basis, the basis formed by the entropy vectors of perfect tensors, contains no Bell-pair term [**2**]. Avis and Hernández-Cuenca note that it suffices to ask that Q and every inequality obtained from it by exchanging a party with the purifier be balanced, and they prove that this holds exactly when, for every party and every pair of parties, the entries of Q on the terms that contain it sum to zero, by a computation that in Bell values is the one we repeat in (3.10) [**4**]. This exchange form and the pairwise form of Definition 3.9 agree. A relabeling σ carries the Bell graph Γ_{ab} to a Bell graph of $\sigma(a)$ and $\sigma(b)$, so $\sigma r_{ab} = r_{\sigma(a)\sigma(b)}$ by Section 2.6. Let σ exchange a party j with O. Then $\sigma^{-1} = \sigma$, and

$$(\sigma Q) \cdot r_{iO} = Q \cdot \big(\sigma r_{iO}\big) = Q \cdot r_{ij} \quad (i \neq j) \ , \qquad (\sigma Q) \cdot r_{jO} = Q \cdot r_{Oj} \ . \tag{3.10}$$

So σQ is balanced for every such σ, and Q itself is balanced, exactly when all Bell values of Q vanish.

THEOREM 3.11 (He, Hubeny and Rangamani [**2**]). *Every facet of* HEC_N *other than subadditivity and its relabelings, which include the Araki–Lieb inequality, is superbalanced.*

This theorem is the reason why the paper asks its question for superbalanced inequalities. The paper uses it only as motivation.

COROLLARY 3.12 (Deletion preserves superbalance). *If an inequality Q on $N+1$ parties is superbalanced, so is its deletion Q^- of Lemma 2.16(a).*

PROOF. Let $i, j \in [N] \cup \{O\}$ be distinct. Adding an isolated boundary vertex labeled $N+1$ to the Bell graph Γ_{ij} on N parties gives a Bell graph of i, j on $N+1$ parties. By Lemma 2.16(a), the Bell value of Q^- on i, j equals that of Q on i, j, which vanishes. □

3.4. Examples

MMI is superbalanced [**2, 4**]. Take MMI(X:Y:Z), with LHS terms XY, XZ, YZ and RHS terms X, Y, Z, XYZ, and use (3.7): the Bell value on i, j is the sum of the coefficients of the terms that contain exactly one of them. For $i \in X$ and $j \in Y$ these terms are XZ and YZ on the left and X and Y on the right, so the value is $1+1-1-1=0$. For $i \in X$ and j outside $X \cup Y \cup Z$, the purifier included, they are XY and XZ on the left and X and XYZ on the right, so the value is again 0. If i and j lie in the same one of X, Y, Z, or both outside $X \cup Y \cup Z$, every term contains both or neither, and the value is 0. The remaining cases follow by exchanging the roles of X, Y and Z.

Subadditivity is balanced but not superbalanced. Take $S(X) + S(Y) \geq S(XY)$. A party in X or Y has coefficient 1 on each side, and a party outside $X \cup Y$ has coefficient 0 on each side, so the inequality is balanced. For $i \in X$ and $j \in Y$, the terms that contain exactly one of them are X and Y, both on the left, while XY contains both, so

$$d_\alpha\big(x^{(i)}, x^{(j)}\big) - d_\beta\big(y^{(i)}, y^{(j)}\big) = 2 - 0 = 2\ . \tag{3.13}$$

So subadditivity is not superbalanced, in agreement with Theorem 3.11, which exempts its instances.

CHAPTER 4

Proof by contraction and its count form

Where we are. This chapter states and proves the theorem of Bao et al. that a contraction map proves an inequality, the graph-model form of the cut-and-paste argument of Section 1.3. It then introduces the count form in which the paper writes every map, shows that on the full cube this form loses nothing against the other ways of presenting an inequality, and defines the least scaling $k_{\min}$. The chapter ends with the question of the paper and its place in the literature.

4.1. The contraction theorem

Let Q be the inequality (1.4), with occurrence words (3.2) and distances (3.3).

THEOREM 4.1 (Proof by contraction, Bao et al. [**1**]). *Suppose* $f\colon \{0,1\}^L \to \{0,1\}^R$ *satisfies* $f(x^{(i)}) = y^{(i)}$ *for every* $i \in [N] \cup \{O\}$, *in particular* $f(0) = 0$, *and*

$$d_\beta\big(f(u), f(v)\big) \le d_\alpha(u, v) \qquad \text{for all } u, v \in \{0,1\}^L \ . \tag{4.2}$$

Then Q *is valid.*

A map with these properties is a *contraction map* for Q, and the conditions $f(x^{(i)}) = y^{(i)}$ are its boundary conditions.

PROOF. Fix a graph model Γ. We show $\sum_r \beta_r S(J_r) \le \sum_l \alpha_l S(I_l)$ in six steps.

(C1) For each l choose a min cut W_l of I_l, so $C(W_l) = S(I_l)$. Any choice works here.

(C2) Give every vertex w the word $x(w) \in \{0,1\}^L$ with $x(w)_l = [w \in W_l]$. It records which of the chosen LHS cuts contain w, as the words of bulk points did in Section 1.3.

(C3) A boundary vertex w labeled by a party i lies in the I_l-cut W_l exactly when $i \in I_l$, by (2.2), so $x(w) = x^{(i)}$. A boundary vertex of O lies in no W_l, so $x(w) = 0 = x^{(O)}$.

(C4) For each r let $U_r = \{w \in V\colon f(x(w))_r = 1\}$. For a boundary vertex of a label i, $f(x(w))_r = f(x^{(i)})_r = y_r^{(i)} = [i \in J_r]$ by (C3) and the boundary conditions of f. So $U_r \cap \partial V = b^{-1}(J_r)$, U_r is a J_r-cut, and $S(J_r) \le C(U_r)$.

(C5) An edge $e = \{w, w'\}$ is cut by U_r exactly when $f(x(w))_r \ne f(x(w'))_r$, and by W_l exactly when $x(w)_l \ne x(w')_l$. So

$$C(U_r) = \sum_e w_e \big|f(x(w))_r - f(x(w'))_r\big| \ , \qquad C(W_l) = \sum_e w_e \big|x(w)_l - x(w')_l\big| \ , \tag{4.3}$$

where the sums run over the edges $e = \{w, w'\}$ of Γ.

(C6) Multiply (C4) by β_r, sum over r and insert the first formula of (4.3). Exchanging the two sums, the inner sum over r becomes d_β of the images of the two endpoints. Then apply (4.2) edge by edge, exchange the sums back, and use the second formula of (4.3) together with (C1):

$$\begin{aligned} \sum_r \beta_r S(J_r) &\le \sum_e w_e \sum_r \beta_r \big|f(x(w))_r - f(x(w'))_r\big| = \sum_e w_e d_\beta\big(f(x(w)), f(x(w'))\big) \\ &\le \sum_e w_e d_\alpha\big(x(w), x(w')\big) = \sum_l \alpha_l \sum_e w_e \big|x(w)_l - x(w')_l\big| = \sum_l \alpha_l S(I_l) \ . \end{aligned} \tag{4.4}$$

Since Γ was arbitrary, Q is valid. □

Three features of the proof are used later. First, it evaluates f only at the words $x(w)$ that occur in the graph model, and (4.2) only at pairs of words joined by an edge. Since we do not know in advance which words occur, the theorem asks for all of $\{0,1\}^L$. Chapter 5 shows that, with suitably chosen min cuts, only the words of a smaller domain occur. Second, the proof never uses that the sets I_l are distinct, or that the J_r are, so it applies verbatim to an inequality written with repeated terms, as in Section 4.3. Third, the theorem gives a sufficient condition for validity, and whether a map exists for a given valid inequality is the subject of the paper.

Physics aside 4.5. The proof is the graph version of the cut-and-paste arguments of Section 1.3 [**7, 9**]. The cuts W_l are the homology regions of the LHS, the word $x(w)$ is the address of the bulk point w in the arrangement of these regions, and f is a rule that decides from the address alone to which candidate regions U_r of the RHS the point belongs. The condition (4.2) says that across every bulk edge the reassembled surfaces cost at most what the original surfaces cost there. The regions U_r are only candidates, so their total area bounds the RHS entropies from above.

4.2. Edges suffice on the full cube, and two examples

On the full cube the condition (4.2) needs to be checked only for pairs of words that differ in one position [**1**].

LEMMA 4.6 (Edges suffice). *If $f\colon \{0,1\}^L \to \{0,1\}^R$ satisfies* (4.2) *for all u, v with $d_1(u,v) = 1$, it satisfies* (4.2) *for all $u, v \in \{0,1\}^L$.*

PROOF. Let u and v differ in the positions $l_1, \dots, l_m$, and flip these bits of u one at a time, which gives words $u = u_0, u_1, \dots, u_m = v$ with $d_1(u_{s-1}, u_s) = 1$. Each position changes once along this walk, so $d_\alpha(u,v) = \sum_{s=1}^m d_\alpha(u_{s-1}, u_s)$. By the triangle inequality for d_β and the assumption,

$$d_\beta\big(f(u), f(v)\big) \le \sum_{s=1}^m d_\beta\big(f(u_{s-1}), f(u_s)\big) \le \sum_{s=1}^m d_\alpha(u_{s-1}, u_s) = d_\alpha(u,v)\ . \tag{4.7}$$

□

EXAMPLE 4.8 (Subadditivity). For $S(X) + S(Y) \ge S(XY)$ take $L = 2$ with the terms X, Y and $R = 1$ with the term XY, all coefficients 1. A party in X has the words $x^{(i)} = 10$ and $y^{(i)} = 1$, a party in Y has 01 and 1, and every other party and the purifier have 00 and 0. The logical or, $f(u) = u_1 \vee u_2$, has $f(10) = f(01) = 1$ and $f(00) = 0$, so it satisfies the boundary conditions, and changing one bit of u changes $u_1 \vee u_2$ by at most 1. By Lemma 4.6 and Theorem 4.1, subadditivity is valid. In the proof, U is the union $W_X \cup W_Y$ of the two LHS cuts, as in (1.7).

EXAMPLE 4.9 (The MMI map). For MMI(A:B:C), with the orders of (3.4), all coefficients are 1, so $d_\alpha = d_\beta = d_1$. The boundary conditions fix f on the four words 110, 101, 011, 000, and they determine f on the other four. The word 100 is at distance 1 from 000, 110 and 101, so $f(100)$ must be at distance at most 1 from 0000, 1001 and 0101. A word at distance at most 1 from 0000 has at most one bit 1, the word 0000 is at distance 2 from 1001, and the only words with one bit 1 at distance 1 from both 1001 and 0101 have that bit in the last position. So $f(100) = 0001$, and likewise $f(010) = f(001) = 0001$. The word 111 is at distance 1 from 110, 101 and 011, so $f(111)$ is at distance at most 1 from 1001, 0101 and 0011. These three words are pairwise at distance 2, so $f(111)$ is none of them and is at distance exactly 1 from each. A word with one bit 1 at distance 1 from a word with two bits 1 lies inside it, and the only bit common to the three words is the last. A word with three bits 1 at distance 1 from each would contain all three, whose union 1111 has

four bits. Hence $f(111) = 0001$, and the only candidate, the unique extension noted in [**1, 4**], is

$$f(u) = \begin{cases} 1001,\ 0101,\ 0011,\ 0000 & \text{for } u = 110,\ 101,\ 011,\ 000\ , \\ 0001 & \text{for } u = 100,\ 010,\ 001,\ 111\ , \end{cases} \tag{4.10}$$

the map (1.11) found in the bulk. It is a contraction map. Each of the 12 edges of $\{0,1\}^3$ joins a word with an odd number of bits 1, which f sends to 0001, to a word with an even number, which f sends to one of 0000, 1001, 0101, 0011, each at distance 1 from 0001 (Figure 1). By Lemma 4.6 and Theorem 4.1, MMI(A:B:C) is valid. For an instance MMI(X:Y:Z) with arbitrary pairwise disjoint X, Y, Z, the occurrence words of the parties in X, in Y and in Z and of all other labels are again 110, 101, 011 and 000, so the same map proves every instance.

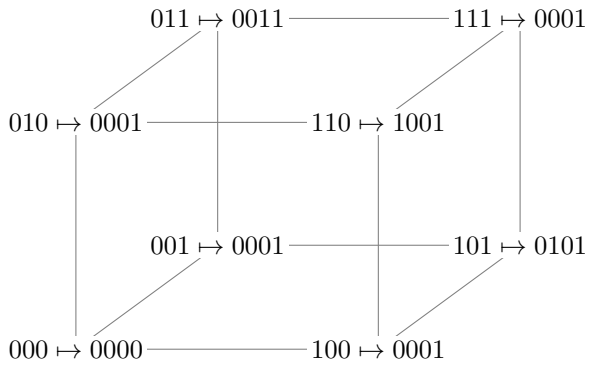


FIGURE 1. The contraction map of MMI(A:B:C) on the cube $\{0,1\}^3$ of the LHS terms (AB, AC, BC), with the RHS terms in the order (A, B, C, ABC). The words 110, 101, 011 and 000 are the occurrence words of A, B, C and O. Along every edge the image changes in exactly one position, as in the hypercube picture of [**1**].

4.3. Presentations and count maps

The same inequality can be written in several ways before a map is sought. A term $\beta_r S(J_r)$ can be kept as one coordinate of weight β_r, *merged*, or *split* into β_r coordinates of weight 1, one for each copy of $S(J_r)$, and likewise on the left [**1, 15**]. Grimaldi, Headrick, Hubeny and Shteyner classify the presentations by these choices on each side [**5**], and Bao, Furuya and Naskar split both sides into unit terms [**16**]. The coefficient vector is the same in every presentation, so by the second feature after Theorem 4.1 a map in any presentation proves the same inequality. What changes is the cube, the occurrence words and the distance, and therefore whether a map exists [**4, 5**]. The paper merges the left-hand side and splits the right-hand side, the presentation of Problem 3 of Avis and Hernández-Cuenca [**4**], used by Bao and Naskar [**17**] and discussed by Grimaldi et al. [**5**], and it records a map on the split side by counts.

In the presentation with split right-hand side, the RHS cube is $\{0,1\}^{|\beta|}$ with coordinates (r,s), $1 \le r \le R$, $1 \le s \le \beta_r$, the distance is d_1, and the occurrence word of a party i is

$$\overline{y}^{(i)} \in \{0,1\}^{|\beta|}\ , \qquad \overline{y}^{(i)}_{r,s} = y^{(i)}_r \quad \text{for } s = 1, \dots, \beta_r\ , \tag{4.11}$$

with $\overline{y}^{(O)} = 0$.

DEFINITION 4.12 (Count maps). Let $\Omega \subseteq \{0,1\}^L$ contain the occurrence words $x^{(i)}$ of all parties $i \in [N]$ and the word 0. A *real map for Q on Ω* is a map $n = (n_r)_{r=1}^R \colon \Omega \to \prod_{r=1}^R [0, \beta_r]$ such that

$$n_r\big(x^{(i)}\big) = \beta_r[i \in J_r] \quad \text{for all } i \in [N] \text{ and all } r\ , \qquad n(0) = 0\ , \tag{4.13}$$

and

$$\sum_{r=1}^{R} \bigl|n_r(u) - n_r(v)\bigr| \le d_\alpha(u,v) \qquad \text{for all } u,v \in \Omega\ . \tag{4.14}$$

It is an *integer map*, or an *unscaled map*, if $n_r(u) \in \{0,1,\dots,\beta_r\}$ for all u and r. For an integer $k \ge 1$, a map for kQ is a map for the inequality with coefficients $k\alpha_l$ and $k\beta_r$.

The conditions (4.13) are the *pins*, the condition $n(0) = 0$ is the pin of the purifier, and (4.14) is the *contraction condition*. The proof of Lemma 4.6 applies verbatim with $\sum_r |n_r(u) - n_r(v)|$ in place of $d_\beta(f(u), f(v))$, so on the full cube the contraction condition needs to be checked only for pairs at Hamming distance 1. We call $\Omega = \{0,1\}^L$ the *full cube*, and say that Q has an *unscaled contraction proof* on a domain if it has an integer map there. The value $n_r(u)$ counts in how many of the β_r copies of $S(J_r)$ the point u is placed, which is the content of the next lemma.

Lemma 4.15 (Sorting: count maps are split-RHS contraction maps). *Let Ω be as in Definition 4.12. The inequality Q has an integer map on Ω if and only if there is a map $f\colon \Omega \to \{0,1\}^{|\beta|}$ with $f(x^{(i)}) = \overline{y}^{(i)}$ for all $i \in [N]$, $f(0) = 0$, and $d_1(f(u), f(v)) \le d_\alpha(u,v)$ for all $u, v \in \Omega$.*

Proof. Given f, put $n_r(u) = \sum_{s=1}^{\beta_r} f(u)_{r,s}$, the number of copies of J_r that contain u. Then $n_r(u) \in \{0,\dots,\beta_r\}$. The pins hold because $f(x^{(i)})_{r,s} = y_r^{(i)} = [i \in J_r]$ for each of the β_r copies, and $n(0) = 0$ because $f(0) = 0$. For the contraction condition, the triangle inequality gives

$$\begin{aligned} \sum_r \bigl|n_r(u) - n_r(v)\bigr| &= \sum_r \Bigl|\sum_{s=1}^{\beta_r} \bigl(f(u)_{r,s} - f(v)_{r,s}\bigr)\Bigr| \le \sum_{r,s} \bigl|f(u)_{r,s} - f(v)_{r,s}\bigr| \\ &= d_1\bigl(f(u), f(v)\bigr) \le d_\alpha(u,v)\ . \end{aligned} \tag{4.16}$$

Conversely, given an integer map n, put the point u into the first $n_r(u)$ copies of J_r,

$$f(u)_{r,s} = [s \le n_r(u)]\ . \tag{4.17}$$

For two points u, v, the bits of the copy (r,s) differ exactly when $\min(n_r(u), n_r(v)) < s \le \max(n_r(u), n_r(v))$, which happens for $|n_r(u) - n_r(v)|$ values of s. So $d_1(f(u), f(v)) = \sum_r |n_r(u) - n_r(v)| \le d_\alpha(u,v)$. The pins of n give $f(x^{(i)})_{r,s} = [s \le \beta_r [i \in J_r]] = [i \in J_r] = \overline{y}^{(i)}_{r,s}$, and $f(0) = 0$. □

On the full cube the lemma connects count maps with the theorem: by Theorem 4.1 applied to the presentation with split right-hand side, an integer map on $\{0,1\}^L$ proves that Q is valid. Two further observations show that on the full cube this presentation is the most permissive one.

A merged-RHS map gives a split-RHS map. If $g\colon \{0,1\}^L \to \{0,1\}^R$ satisfies $g(x^{(i)}) = y^{(i)}$ and $d_\beta(g(u), g(v)) \le d_\alpha(u,v)$, repeat each bit r of $g(u)$ exactly β_r times: $f(u)_{r,s} = g(u)_r$. Then $d_1(f(u), f(v)) = \sum_r \beta_r |g(u)_r - g(v)_r| = d_\beta(g(u), g(v))$, and the boundary conditions become $f(x^{(i)}) = \overline{y}^{(i)}$ [**5**]. The same construction works when the left-hand side is split.

A split-LHS map gives a merged-LHS map. Let $\delta_\alpha\colon \{0,1\}^L \to \{0,1\}^{|\alpha|}$ repeat bit l exactly α_l times. Then $d_1(\delta_\alpha(u), \delta_\alpha(v)) = d_\alpha(u,v)$, and δ_α sends the occurrence words of the merged presentation to those of the split one. If h is a map on the split cube $\{0,1\}^{|\alpha|}$, with unit distance, that satisfies the boundary conditions and the contraction condition into the RHS cube of some presentation, then $h \circ \delta_\alpha$ satisfies them on $\{0,1\}^L$, since the distance between the images of u and v is at most $d_1(\delta_\alpha(u), \delta_\alpha(v)) = d_\alpha(u,v)$.

Combining the two observations with Lemma 4.15: if Q has an unscaled map on the full cube in any presentation, with each coefficient on either side kept, split into unit copies or divided into several parts, it has an integer count map on $\{0,1\}^L$, because the two observations apply to every

such division. We record the contrapositive, which is how the non-existence results of Parts II and III are to be read.

(4.18) no integer count map for Q on $\{0,1\}^L$
$\implies$ no unscaled contraction map for Q on the full cube in any presentation.

The second observation needs h on all of $\{0,1\}^{|\alpha|}$, which is why (4.18) is a statement about the full cube. On the realizable domain of Chapter 5 the paper always merges the left-hand side. Section 4.5 recalls that the converse of the first observation fails.

Example 4.19 (Merged and split right-hand side). This is the example of Grimaldi, Headrick, Hubeny and Shteyner, a sum of two instances of MMI, with their argument for the merged right-hand side [**5**]. Let $N = 4$ and $Q = \mathrm{MMI}(A{:}B{:}C) + \mathrm{MMI}(A{:}B{:}D)$, which reads

$$2AB + AC + AD + BC + BD \;\geq\; 2A + 2B + C + D + ABC + ABD\ . \tag{4.20}$$

Order the LHS terms as (AB, AC, AD, BC, BD), with $\alpha = (2,1,1,1,1)$, and the RHS terms as (A, B, C, D, ABC, ABD), with $\beta = (2,2,1,1,1,1)$. Then $x^{(A)} = 11100$, $y^{(A)} = 100011$, $x^{(B)} = 10011$ and $y^{(B)} = 010011$.

No map with merged right-hand side. Suppose $g\colon \{0,1\}^5 \to \{0,1\}^6$ satisfies $g(x^{(i)}) = y^{(i)}$ and $d_\beta(g(u), g(v)) \leq d_\alpha(u,v)$. The point $z_A = 11000$ differs from $x^{(A)}$ only in the bit of AD, of weight 1, so $g(z_A)$ is at d_β-distance at most 1 from $y^{(A)}$ and agrees with it in the two bits of weight 2: $g(z_A)_A = 1$ and $g(z_A)_B = 0$. Likewise $z_B = 10010$ differs from $x^{(B)}$ only in the bit of BD, so $g(z_B)_A = 0$ and $g(z_B)_B = 1$. Then $d_\beta(g(z_A), g(z_B)) \geq 2 + 2 = 4$, while $d_\alpha(z_A, z_B) = 2$, from the bits of AC and BC. So no such g exists.

An integer count map. For the fully split presentation of a sum, Grimaldi et al. take the product of maps of the summands, and they observe for this example that the two copies of a RHS term then disagree at some points where the two copies of AB agree [**5**]. We write their construction with merged left-hand side and in counts. Let f be the MMI map (4.10), and evaluate it on the two triples of LHS bits that belong to the two instances, $f(u_{AB}, u_{AC}, u_{BC})$ for $\mathrm{MMI}(A{:}B{:}C)$ and $f(u_{AB}, u_{AD}, u_{BD})$ for $\mathrm{MMI}(A{:}B{:}D)$. Their four output bits refer to (A, B, C, ABC) and (A, B, D, ABD). Add the two outputs term by term: n_A and n_B are the sums of the two bits of A and of B, which lie in $\{0,1,2\}$, and n_C, n_D, n_{ABC}, n_{ABD} are single bits. The pins hold because the restriction of an occurrence word of Q to each triple of bits is the occurrence word of the same label in that instance, where f satisfies the boundary conditions. Flipping the bit of AB, of weight 2, moves each of the two outputs by at most 1, and flipping any other bit, of weight 1, moves only one of them by at most 1. So n is an integer map on $\{0,1\}^5$, and Q has an unscaled contraction proof in the count form. At z_A it takes the value $n(z_A) = (1,0,0,0,1,1)$: the point lies in one of the two copies of $S(A)$, which the merged right-hand side does not allow.

Physics aside 4.21. In the count form, $n_r(w) = n_r(x(w))$ says in how many of the β_r copies of the region J_r the bulk point w is placed. With a merged right-hand side all β_r copies must be the same candidate region. With a split right-hand side each copy may be a different candidate region, and by the sorting step (4.17) they can always be taken nested, copy s being $\{w\colon n_r(w) \geq s\}$. So $\beta_r S(J_r)$ is bounded by the sum of the areas of β_r nested candidate surfaces, which is a weaker requirement than one candidate surface whose area, counted β_r times, does the job.

4.4. Scaling

A map for kQ has ranges $[0, k\beta_r]$, pins $k\beta_r[i \in J_r]$ and distance kd_α on the left. Since $kQ \cdot S$ and $Q \cdot S$ have the same sign, a map for kQ proves Q. Four facts about scaling follow directly from Definition 4.12.

(K1) If n is a real map for Q, then kn is a real map for kQ, since the ranges, the pins and both sides of (4.14) scale by k. Conversely, n/k is a real map for Q if n is one for kQ. So the existence of a real map does not depend on the scaling, while the existence of an integer map may.

(K2) If n is an integer map for k_1Q and n' an integer map for k_2Q on the same domain, then $n+n'$ is an integer map for $(k_1+k_2)Q$. The pins add to $(k_1+k_2)\beta_r[i \in J_r]$, the values lie in $\{0,\dots,(k_1+k_2)\beta_r\}$, and

$$\sum_r \big|(n_r+n'_r)(u)-(n_r+n'_r)(v)\big| \le \sum_r \big|n_r(u)-n_r(v)\big|+\sum_r \big|n'_r(u)-n'_r(v)\big| \le (k_1+k_2)d_\alpha(u,v)\,. \tag{4.22}$$

So the set of k for which kQ has an integer map on a given domain is closed under addition.

(K3) If n is a real map with rational values and M is a common denominator of all its values, then Mn is an integer map for MQ, by (K1).

(K4) A map on a domain restricts to every smaller domain that still contains the occurrence words and 0, since the pins are conditions at points of the smaller domain and its contraction conditions are among those of the larger one.

Definition 4.23 (Least scaling). For a valid inequality Q, $k_{\min}(Q)$ is the least integer $k \ge 1$ such that kQ has an integer map on the full cube $\{0,1\}^L$, and $k^{\mathcal{A}}_{\min}(Q)$ is the least $k \ge 1$ such that kQ has an integer map on the realizable domain $\mathcal{A}(Q)$ of Definition 5.1 below. Either is ∞ if no such k exists.

By (K4), $k^{\mathcal{A}}_{\min}(Q) \le k_{\min}(Q)$. In this language the paper asks whether $k_{\min}(Q)=1$, or at least $k^{\mathcal{A}}_{\min}(Q)=1$, for every valid Q in a given class, and how large $k_{\min}$ and $k^{\mathcal{A}}_{\min}$ can be. When a real map exists, which by (K1) and (K3) happens exactly when some kQ has an integer map (a real map can be chosen with rational values, because its conditions are finitely many linear inequalities with rational coefficients once each absolute value is expanded), the obstruction to an unscaled map is one of integrality, and Chapter 7 makes it visible at a single point of the cube.

Corollary 4.24 (Padding preserves maps). *Let Q be an inequality on N parties and Q^+ the same inequality on $N+1$ parties, as in Lemma 2.16(b). For every $k \ge 1$ and every domain Ω as in Definition 4.12, the integer maps for kQ^+ on Ω are exactly the integer maps for kQ on Ω. In particular $k_{\min}(Q^+) = k_{\min}(Q)$.*

Proof. The two inequalities have the same terms, hence the same cube $\{0,1\}^L$, the same distance d_α and the same ranges, and the old parties have the same occurrence words and pins. The new party lies in no term, so its occurrence word is 0 and its pin is $n_r(0) = k\beta_r \cdot 0 = 0$, which is the pin of the purifier and is already imposed for kQ. □

4.5. The question of the paper and the literature

Whether a valid inequality has a contraction proof thus depends on how it is presented: a coefficient can be kept as a weight or split into unit copies, on either side, and the whole inequality can be multiplied by a positive integer k before a map is sought [**1, 4, 5**]. Bao, Furuya and Naskar state that every valid holographic entropy inequality with rational coefficients has a contraction map, and in the arXiv version of their paper of 14 July 2026 they specify that a map may be sought for any presentation obtained by merging or expanding terms and by uniformly rescaling all coefficients [**16**]. Grimaldi, Headrick, Hubeny and Shteyner remark that, empirically, when the fully split presentation of an inequality has no map, multiplying the inequality by an integer does not create one; their discussion concerns primitive, that is facet-defining, inequalities [**5**]. Much work has gone into understanding and constructing contraction maps [**18, 19, 20, 17, 21, 22, 5, 23**]; see also [**24**].

The failures of unscaled maps that the paper cites from the literature are failures of the presentation with merged right-hand side that disappear after splitting [**4, 5**]. Example 4.19

reproduces the example of Grimaldi et al. So the converse of the first observation of Section 4.3 fails, which is why the paper asks for maps in the count form, with split right-hand side. The paper reports that for the facets at small N it found no failure in this presentation. For $N \leq 4$ every facet of HEC_N is an instance of subadditivity or MMI [**1**], and these have unscaled maps. For $N = 5$ the paper reports an unscaled map on the full cube for a representative of each of the eight orbits of facets under relabelings [**10**]. These maps are not among the paper's certificates.

The results of Parts II and III concern unscaled maps in the count form. They are consistent with the statement of Bao, Furuya and Naskar in the form of their arXiv version of 14 July 2026, and they show that the uniform rescaling allowed there cannot be dropped, and, by the star results of Chapter 11, that for star inequalities, which are implied by subadditivity and are not balanced, it may have to grow faster than exponentially with N. None of the inequalities of the paper without an unscaled map is known to be a facet, and the one facet treated, F_7, has $k^{\mathcal{A}}_{\min}(F_7) = 1$, so the results do not bear on the remark of Grimaldi et al.

CHAPTER 5

The realizable domain and the two lemmas

Where we are. The contraction theorem asks for a map on all 2^L words, while a graph model uses only the words of its vertices. If the LHS min cuts are chosen minimal, the words that occur satisfy two simple conditions, and a map is needed only on the words that satisfy them, the realizable domain. This chapter proves the two lemmas that the paper uses throughout. The realizable-domain criterion (Section 5.3) proves validity from a real-valued map on the realizable domain, and the restriction lemma (Section 5.5) transfers the non-existence of integer maps from a small subdomain to the full cube and to the realizable domain. The last section shows what happens to both domains when another label plays the purifier.

5.1. The realizable domain

DEFINITION 5.1 (Realizable domain). The *realizable domain* $\mathcal{A}(Q) \subseteq \{0,1\}^L$ of the inequality (1.4) is the set of words $x \in \{0,1\}^L$ such that

(R1) if $x_l = x_{l'} = 1$, then $I_l \cap I_{l'} \neq \emptyset$, and
(R2) if $x_l = 1$ and $I_l \subseteq I_{l'}$, then $x_{l'} = 1$.

This is the domain of Avis and Hernández-Cuenca [**4**], called the realizable subset by Grimaldi et al. [**5**]. Avis and Hernández-Cuenca describe its elements through the antichains of their minimal elements; the pairwise conditions (R1) and (R2) also appear in [**20, 17**]. Condition (R1) says that a word may not place a point in the regions of two disjoint LHS terms, and (R2) that a point in the region of a term must also lie in the region of every larger term. Section 5.2 shows that minimal min cuts have exactly these two properties.

Four facts follow from the definition. First, $0 \in \mathcal{A}(Q)$, since both conditions concern coordinates equal to 1. Second, every occurrence word lies in $\mathcal{A}(Q)$: if $x_l^{(i)} = x_{l'}^{(i)} = 1$, then $i \in I_l \cap I_{l'}$, and if $x_l^{(i)} = 1$ and $I_l \subseteq I_{l'}$, then $i \in I_{l'}$, so $x_{l'}^{(i)} = 1$. So $\mathcal{A}(Q)$ is a domain in the sense of Definition 4.12. Third, $\mathcal{A}(kQ) = \mathcal{A}(Q)$, since the domain depends only on the sets I_l, not on the coefficients. Fourth, padding does not change it: the inequality Q^+ of Lemma 2.16(b) has the same LHS sets, so $\mathcal{A}(Q^+) = \mathcal{A}(Q)$ and, with Corollary 4.24, $k^{\mathcal{A}}_{\min}(Q^+) = k^{\mathcal{A}}_{\min}(Q)$.

The definition refers to the sets $I_l \subseteq [N]$ of (1.4), that is, to the terms after every term containing O has been replaced by its complement. Section 5.6 gives an example in which the written terms and the purified sets give different domains.

For subadditivity $S(X) + S(Y) \geq S(XY)$ the two LHS sets are disjoint, so (R1) excludes the word 11 and $\mathcal{A} = \{00, 10, 01\}$. For MMI($A$:$B$:$C$) any two of the LHS sets AB, AC, BC intersect and none contains another, so $\mathcal{A} = \{0,1\}^3$.

5.2. Minimal min cuts

Fix a graph model. The proof of the criterion needs three properties of minimal min cuts, which Avis and Hernández-Cuenca state and prove, building on earlier work cited there [**4**]. The proofs below follow theirs; in (c) their argument for a single vertex is applied to the whole intersection. All three rest on the submodularity of the cut weight.

LEMMA 5.2 (Submodularity). *For all $X, Y \subseteq V$,*

$$C(X \cap Y) + C(X \cup Y) \leq C(X) + C(Y) \ . \tag{5.3}$$

PROOF. Each vertex lies in exactly one of the four sets $X \cap Y$, $X \setminus Y$, $Y \setminus X$ and $V \setminus (X \cup Y)$, the outside. For each type of edge, the table counts how many of the sets $X \cap Y$ and $X \cup Y$ cut it, and how many of X and Y.

edge between	cut by $X \cap Y$, by $X \cup Y$	cut by X, by Y
$X \cap Y$ and $X \setminus Y$	$1 + 0$	$0 + 1$
$X \cap Y$ and $Y \setminus X$	$1 + 0$	$1 + 0$
$X \cap Y$ and the outside	$1 + 1$	$1 + 1$
$X \setminus Y$ and $Y \setminus X$	$0 + 0$	$1 + 1$
$X \setminus Y$ and the outside	$0 + 1$	$1 + 0$
$Y \setminus X$ and the outside	$0 + 1$	$0 + 1$
two endpoints in the same set	$0 + 0$	$0 + 0$

In every row the left count is at most the right count. Multiplying by w_e and summing over all edges gives (5.3). □

The inequality is strict exactly when some edge of positive weight joins $X \setminus Y$ to $Y \setminus X$. It is the graph version of (1.8).

LEMMA 5.4 (Minimal min cuts). *Fix a graph model and nonempty $I, I' \subseteq [N]$.*

(a) *The intersection of all min cuts of I is a min cut W_I of I, the* minimal min cut. *No proper subset of W_I is a min cut of I.*
(b) Nesting. *If $I \subseteq I'$, then $W_I \subseteq W_{I'}$.*
(c) Disjointness. *If $I \cap I' = \emptyset$, then $W_I \cap W_{I'} = \emptyset$.*

PROOF. (a) Let W and W' be min cuts of I. Since $W \cap \partial V = W' \cap \partial V = b^{-1}(I)$, both $W \cap W'$ and $W \cup W'$ are I-cuts, so each has weight at least $S(I)$. With submodularity,

$$2S(I) \leq C(W \cap W') + C(W \cup W') \leq C(W) + C(W') = 2S(I) \ . \tag{5.5}$$

So both inequalities are equalities, and since each of the two weights on the left is at least $S(I)$, each equals $S(I)$. In particular $W \cap W'$ is a min cut. There are finitely many min cuts, so intersecting them two at a time shows that their intersection W_I is a min cut. It is contained in every min cut, so a proper subset of W_I is not a min cut.

(b) The set $W_I \cap W_{I'}$ meets ∂V in $b^{-1}(I) \cap b^{-1}(I') = b^{-1}(I)$, so it is an I-cut, and $W_I \cup W_{I'}$ meets ∂V in $b^{-1}(I')$, so it is an I'-cut. By submodularity,

$$S(I) + S(I') \leq C\big(W_I \cap W_{I'}\big) + C\big(W_I \cup W_{I'}\big) \leq C(W_I) + C(W_{I'}) = S(I) + S(I') \ . \tag{5.6}$$

As in (a), $W_I \cap W_{I'}$ has weight $S(I)$ and is a min cut of I. By (a) it contains W_I, that is, $W_I \subseteq W_{I'}$.

(c) Let $X = W_I \cap W_{I'}$. It contains no boundary vertex, since $b^{-1}(I) \cap b^{-1}(I') = \emptyset$. Suppose $X \neq \emptyset$, and let a_1, a_2 and a_3 be the total weights of the edges from X to $W_I \setminus W_{I'}$, to $W_{I'} \setminus W_I$ and to the outside $V \setminus (W_I \cup W_{I'})$. The set $W_I \setminus X$ has the same boundary vertices as W_I, so it is an I-cut. Compare its cut with that of W_I edge by edge. Edges that do not touch X, and edges inside X, are cut by both sets or by neither. An edge from X to $W_I \setminus W_{I'}$ was inside W_I and is now cut. An edge from X to $W_{I'} \setminus W_I$ or to the outside was cut and now has both endpoints outside. So

$$C\big(W_I \setminus X\big) = S(I) + a_1 - a_2 - a_3 \ . \tag{5.7}$$

The set $W_I \setminus X$ is a proper subset of W_I, so by (a) it is not a min cut, and $a_1 - a_2 - a_3 > 0$. Exchanging the roles of I and I' gives $a_2 - a_1 - a_3 > 0$. Adding the two inequalities gives $-2a_3 > 0$, which contradicts $a_3 \geq 0$. Hence $X = \emptyset$. □

Minimality is needed in (c). Let $N = 2$, with boundary vertices labeled A, B and O, one bulk vertex w, and two edges of weight 1, from A to w and from B to w, so that O lies on no edge. Both $\{A\}$ and $\{A, w\}$ are min cuts of A, of weight 1, and both $\{B\}$ and $\{B, w\}$ are min cuts of B. The maximal min cuts $\{A, w\}$ and $\{B, w\}$ share the vertex w, while the minimal ones $\{A\}$ and $\{B\}$ are disjoint. In Example 2.4 the min cut of AB is not unique either: $W_{AB} = \{A, B\}$, while $\{A, B, w_0\}$ is also a min cut.

Physics aside 5.8. Part (b) is entanglement wedge nesting on a time-symmetric slice, the reading that Grimaldi et al. and Bao and Naskar give for the excluded words [**5, 17**]: a larger boundary region has a larger homology region. Part (c) says that the smallest homology regions of disjoint boundary regions do not overlap. Choosing minimal min cuts resolves a degenerate RT surface, two minimal surfaces of equal area, consistently for all regions at once, always toward the smaller homology region. In the example above, w is a bulk point that each of A and B may include at no cost, and the minimal choice excludes it from both.

5.3. The realizable-domain criterion

LEMMA 5.9 (Realizable-domain criterion). *Let Q be as in* (1.4). *Suppose there is a map $n\colon \mathcal{A}(Q) \to \prod_{r=1}^{R}[0, \beta_r]$ with $n_r(x^{(i)}) = \beta_r[i \in J_r]$ for every party $i \in [N]$ and every r, with $n(0) = 0$, and with $\sum_r |n_r(u) - n_r(v)| \le d_\alpha(u, v)$ for all $u, v \in \mathcal{A}(Q)$. Then Q is valid.*

In the language of Definition 4.12, a real map for Q on $\mathcal{A}(Q)$ proves that Q is valid. The proof follows that of Theorem 4.1, with two changes: the LHS cuts are the minimal ones, so that every vertex receives a word of $\mathcal{A}(Q)$, and the RHS cuts are threshold sets of the real-valued counts.

PROOF. Fix a graph model Γ. Write $W_l = W_{I_l}$ for the minimal min cut of the l-th LHS term, and give each vertex w the word $x(w) \in \{0,1\}^L$ with $x(w)_l = [w \in W_l]$.

Step 1: the words lie in $\mathcal{A}(Q)$, and boundary vertices carry the occurrence words. If $x(w)_l = x(w)_{l'} = 1$, then $w \in W_l \cap W_{l'}$, so this intersection is not empty, and $I_l \cap I_{l'} \neq \emptyset$ by Lemma 5.4(c). This is (R1). If $x(w)_l = 1$ and $I_l \subseteq I_{l'}$, then $w \in W_l \subseteq W_{l'}$ by Lemma 5.4(b), so $x(w)_{l'} = 1$. This is (R2). As in (C3) of the proof of Theorem 4.1, a boundary vertex of a party i has $x(w) = x^{(i)}$, and a boundary vertex of O has $x(w) = 0$. Put $n_r(w) = n_r(x(w))$ for every vertex w. By the pins, a boundary vertex of a party i has $n_r(w) = \beta_r$ if $i \in J_r$ and $n_r(w) = 0$ otherwise, and a boundary vertex of O has $n_r(w) = 0$.

Step 2: layer cake. Fix r and a threshold $t \in [0, \beta_r)$, and let

$$U_r(t) = \big\{w \in V\colon n_r(w) > t\big\}\ . \tag{5.10}$$

By Step 1, a boundary vertex lies in $U_r(t)$ exactly when it is labeled by a party in J_r, since $\beta_r > t \ge 0$. So $U_r(t)$ is a J_r-cut, and

$$C\big(U_r(t)\big) \ge S(J_r) \qquad \text{for every } t \in [0, \beta_r)\ . \tag{5.11}$$

An edge $e = \{w, w'\}$ is cut by $U_r(t)$ exactly when one endpoint has $n_r > t$ and the other $n_r \le t$, that is, when t lies in the interval $[\min(n_r(w), n_r(w')), \max(n_r(w), n_r(w')))$. This interval lies in $[0, \beta_r)$, because all values of n_r lie in $[0, \beta_r]$, and its length is $|n_r(w) - n_r(w')|$. The function $t \mapsto C(U_r(t)) = \sum_e w_e[e \text{ is cut by } U_r(t)]$ is a step function with finitely many steps, since n_r takes finitely many values on V. Integrating (5.11) over $t \in [0, \beta_r)$ and integrating each indicator over its interval gives the layer-cake bound

$$\boxed{\beta_r S(J_r) \le \int_0^{\beta_r} C\big(U_r(t)\big)dt = \sum_{e=\{w,w'\}} w_e\big|n_r(w) - n_r(w')\big|\ .} \tag{5.12}$$

Step 3: summing. By Step 1 the words of both endpoints of every edge lie in $\mathcal{A}(Q)$, so the contraction condition applies to them. Summing (5.12) over r, using the contraction condition edge by edge, and then the formula (4.3) for the cuts W_l, whose weights are $S(I_l)$,

$$\begin{aligned}\sum_{r=1}^{R}\beta_r S(J_r) &\le \sum_e w_e \sum_{r=1}^{R}\big|n_r(w)-n_r(w')\big| \le \sum_e w_e d_\alpha\big(x(w),x(w')\big)\\ &= \sum_{l=1}^{L}\alpha_l \sum_e w_e\big|x(w)_l - x(w')_l\big| = \sum_{l=1}^{L}\alpha_l S(I_l)\ .\end{aligned} \tag{5.13}$$

Since Γ was arbitrary, Q is valid. □

For an integer map the integral is a sum over integer thresholds: the sets $\{w\colon n_r(w)\ge s\}$, $s=1,\dots,\beta_r$, are J_r-cuts, and an edge is cut by exactly $|n_r(w)-n_r(w')|$ of them. These are the nested copies of Physics aside 4.21. The ranges $[0,\beta_r]$ in the hypothesis are a normalization. If a map satisfies the pins and the contraction condition with values outside the ranges, replacing each value by the nearest point of $[0,\beta_r]$ keeps the pins, which lie in the ranges, and does not increase any $|n_r(u)-n_r(v)|$.

Physics aside 5.14. The layer-cake bound (5.12) is a discrete coarea formula. In the continuum, for a function ϕ on the bulk slice, $\int|\nabla\phi| = \int dt$ Area$(\{\phi=t\})$. Here n_r is a height function on the bulk that equals β_r on the boundary regions of J_r and 0 on the rest of the boundary, each level set is a candidate surface for J_r, and the total variation of n_r bounds β_r times the RT area. The contraction condition bounds the total variation of all the height functions together by the area of the LHS surfaces.

Relation to the literature. For maps of the form $n_r=\beta_r f_r$ with $f\colon\mathcal{A}(Q)\to\{0,1\}^R$, for which $\sum_r|n_r(u)-n_r(v)| = d_\beta(f(u),f(v))$, Lemma 5.9 is the theorem of Avis and Hernández-Cuenca on their domain [**4**]. Grimaldi, Headrick, Hubeny and Shteyner record that a contraction map on the realizable domain proves validity [**5**]. For maps with rational values, which are all the validity certificates of the paper, Lemma 5.9 also follows from the proof of the theorem of Avis and Hernández-Cuenca. If M is a common denominator of n, then Mn is an integer map for MQ on $\mathcal{A}(MQ)=\mathcal{A}(Q)$, by (K3). By Lemma 4.15 with $\Omega=\mathcal{A}(Q)$, it gives a $\{0,1\}$-valued contraction map into the cube of the split right-hand side of MQ, and that proof applies to this presentation [**4**], although the theorem is stated for distinct terms. The proof of Lemma 5.9 given above covers real values directly.

Problem 3 of Avis and Hernández-Cuenca. They ask whether the existence of a contraction map on their domain becomes a necessary condition for validity once every RHS term $\beta_r S(J_r)$ with $\beta_r\ge 2$ is replaced by β_r unit terms [**4**]. By Lemma 4.15 with $\Omega=\mathcal{A}(Q)$, a $\{0,1\}$-valued map in that presentation, with merged left-hand side, is the same as an integer count map on $\mathcal{A}(Q)$. So in the formulation with merged left-hand side and no scaling, Problem 3 asks whether $k^{\mathcal{A}}_{\min}(Q)=1$ for every valid Q. In this formulation the examples of the paper give a negative answer: B_7^* (Chapter 7) and U_6, Q_8, $E_0,\dots,E_5$ (Chapter 8) are valid and have $k^{\mathcal{A}}_{\min}>1$ (that $k^{\mathcal{A}}_{\min}>1$ is certified for U_6, Q_8 and $E_0,\dots,E_5$, and for B_7^* it rests on the written proof of Chapter 7).

5.4. Irreducible pairs

On the full cube the contraction condition can be checked on edges (Lemma 4.6). On a subset of the cube, such as $\mathcal{A}(Q)$, the intermediate words of the walk in that proof may be missing, and the condition can only be reduced to the following pairs.

LEMMA 5.15 (Irreducible pairs). *Let $\Omega \subseteq \{0,1\}^L$, and call two distinct points $u, v \in \Omega$* reducible *in Ω if some $z \in \Omega \setminus \{u, v\}$ lies in the box between them, that is, $z_l = u_l$ whenever $u_l = v_l$ (a point on a Hamming path between u and v in the terms of* [**17**]*), and* irreducible *otherwise. If $n\colon \Omega \to \mathbb{R}^R$ satisfies* (4.14) *for every irreducible pair, it satisfies* (4.14) *for all $u, v \in \Omega$.*

PROOF. By induction on $d_\alpha(u, v)$, which takes finitely many values. If u, v is irreducible, there is nothing to prove. If u, v is reducible through z, then in every position where $u_l \neq v_l$ the bit z_l equals one of them, and where $u_l = v_l$ it equals both. So $|u_l - v_l| = |u_l - z_l| + |z_l - v_l|$ for every l, and

$$d_\alpha(u, v) = d_\alpha(u, z) + d_\alpha(z, v) \ . \tag{5.16}$$

Since $z \neq u$, $z \neq v$ and all $\alpha_l > 0$, both terms on the right are smaller than $d_\alpha(u, v)$, so the induction hypothesis applies to the pairs u, z and z, v. By the triangle inequality in $\mathbb{R}^R$,

$$\sum_r \big|n_r(u) - n_r(v)\big| \le \sum_r \big|n_r(u) - n_r(z)\big| + \sum_r \big|n_r(z) - n_r(v)\big| \le d_\alpha(u, z) + d_\alpha(z, v) = d_\alpha(u, v) \ . \tag{5.17}$$

□

On the full cube the irreducible pairs are exactly the pairs at Hamming distance 1: if u and v differ in two or more positions, flipping one of these bits of u gives a third point of the box, and if they differ in one position, the box contains only u and v. On a subset a pair at larger distance can be irreducible, because the intermediate points of its box may be missing. For this reason every check of a map on a realizable domain tests all pairs, or all irreducible pairs. The lemma is needed only to verify that a map exists. For a non-existence statement no reduction is needed: if the problem restricted to some of the pairs has no solution, the full problem has none either.

5.5. Restriction to a subdomain

To show that kQ has no integer map, the paper exhibits a small set of points on which already no map exists. In most of its examples a distinguished party, always labeled A, carries the obstruction, and the set is

$$\mathcal{U}(Q) = \big\{x^{(1)}, \dots, x^{(N)}, 0\big\} \cup \big[0, x^{(A)}\big] \ , \qquad \big[0, x^{(A)}\big] = \big\{x \in \{0,1\}^L \colon x_l \le x_l^{(A)} \text{ for all } l\big\} \ , \tag{5.18}$$

the occurrence words and 0 together with the 2^{L_A} words supported on the L_A LHS terms that contain A. It is a domain in the sense of Definition 4.12.

LEMMA 5.19 (Restriction). *Let $k \ge 1$.*

(a) *If kQ has no integer map on $\mathcal{U}(Q)$, then kQ has no integer map on $\{0,1\}^L$.*
(b) *The inclusion $\mathcal{U}(Q) \subseteq \mathcal{A}(Q)$ holds if and only if the LHS terms that contain A form an antichain, that is, no two of them are nested. In that case, if kQ has no integer map on $\mathcal{U}(Q)$, then kQ has no integer map on $\mathcal{A}(Q)$ either.*

PROOF. (a) A map on $\{0,1\}^L$ restricts to $\mathcal{U}(Q)$ by (K4), so a map on the cube would give one on $\mathcal{U}(Q)$.

(b) The occurrence words and 0 lie in $\mathcal{A}(Q)$ (Section 5.1), so the inclusion holds exactly when $[0, x^{(A)}] \subseteq \mathcal{A}(Q)$. Let $x \in [0, x^{(A)}]$. Every l with $x_l = 1$ has $A \in I_l$, so any two such terms intersect in A, and (R1) holds. If $x_l = 1$ and $I_l \subseteq I_{l'}$, then $A \in I_l \subseteq I_{l'}$, so $I_{l'}$ is also a term containing A. If the terms containing A form an antichain, $I_l \subseteq I_{l'}$ forces $l' = l$, and (R2) holds. Conversely, if $I_l \subsetneq I_{l'}$ are two terms containing A, the word with a single 1, in position l, lies in $[0, x^{(A)}]$ and violates (R2). Under the antichain condition, a map on $\mathcal{A}(Q)$ restricts to $\mathcal{U}(Q)$ by (K4), and the last statement follows as in (a). □

By Lemma 5.15, the contraction condition on $\mathcal{U}(Q)$ needs to be imposed only on the pairs irreducible in $\mathcal{U}(Q)$ when a map is to be checked, and for a non-existence statement any subset of the conditions may be imposed. The size of $\mathcal{U}(Q)$ is $N+1+2^{L_A}$ minus the number of points counted twice. The words 0 and $x^{(A)}$ always lie in the interval, and the occurrence word of another party i lies in it exactly when every LHS term that contains i also contains A. So if no other occurrence word lies in the interval and the occurrence words of different parties are distinct, $|\mathcal{U}(Q)| = 2^{L_A} + N - 1$.

Remark 5.20 (Why the interval of A). The interval $[0, x^{(A)}]$ contains the words of all bulk points that lie only in LHS regions containing A, so it probes how a map must treat the neighborhood of the region A. For Q_8^A and the superbalanced inequalities built from it in Chapter 8 the paper certifies the obstruction on $\mathcal{U}(Q)$, and A is chosen as the party that carries the structure of the inequality. The obstructions of B_7^* and U_6 lie at points outside the interval (Sections 7.4 and 8.1). The choice of $\mathcal{U}(Q)$ is a choice of certificate, and by Lemma 5.19 a failure on $\mathcal{U}(Q)$ implies the failure on the full cube and, under the antichain condition, on the realizable domain.

5.6. Changing the purifier

The coordinates of the entropy vector single out one of the $N+1$ labels, the purifier, and so do the occurrence words, the pin $n(0) = 0$ and the realizable domain. This section shows that the existence of maps does not depend on this choice, and that the realizable domain must be computed from the purified sets. Grimaldi, Headrick, Hubeny and Shteyner note that writing an inequality with a different purifier flips bits in the two cubes and that, since bit flips are isometries of the cubes, the existence of a contraction map is preserved [**5**]. Proposition 5.22 below states this for count maps and every scaling k and adds the realizable domain, part (c). The paper does not state the proposition and fixes the printed presentation instead.

The purified sets. Take $N = 2$ and the inequality $S(A)+S(O) \geq S(AO)$, which after purification reads $S(A) + S(AB) \geq S(B)$, the Araki–Lieb inequality of Section 2.6. Its purified LHS sets are $\{A\} \subseteq \{A, B\}$, so (R2) excludes the word 10, and $\mathcal{A} = \{00, 01, 11\}$. Read literally, the written terms $\{A\}$ and $\{O\}$ are disjoint, and (R1) would exclude 11 and admit 10. The purified sets are the right ones, because the proof of Lemma 5.9 uses the minimal min cuts of the sets $I_l \subseteq [N]$, which avoid the boundary vertices of O: by Lemma 5.4(b) the minimal min cut of A lies inside that of AB, so the word 10 never occurs, while the vertices of W_A carry the word 11. The domain $\{00, 01, 11\}$ is the translate of the domain $\{00, 10, 01\}$ of $S(A) + S(B) \geq S(AB)$ by the occurrence word 01 of B, as Proposition 5.22 below predicts. Every statement in the paper about $\mathcal{A}(Q)$ refers to the presentation in which Q is printed, with O as the purifier and the terms purified.

Exchanging a party with the purifier. Let $j \in [N]$ and let $\sigma \in \mathfrak{S}_{N+1}$ exchange j and O. By Section 2.6, σQ has the LHS terms

$$I'_l = I_l \ \ (j \notin I_l)\ , \qquad I'_l = \big([N] \setminus I_l\big) \cup \{j\} \ \ (j \in I_l)\ , \tag{5.21}$$

with the same coefficients α_l, and RHS terms J'_r obtained from the J_r in the same way, with the same β_r. Indeed, for $j \in I_l$ the set $\sigma(I_l)$ contains O, and its complement in $[N] \cup \{O\}$ is $([N] \setminus I_l) \cup \{j\}$. So σQ has the same cube $\{0,1\}^L$, the same distance d_α and the same ranges. Write $x'^{(p)}$ and $y'^{(p)}$ for the occurrence words of σQ, let $\oplus$ be addition modulo 2 in each position, and let $T(u) = u \oplus x^{(j)}$.

Proposition 5.22 (Changing the purifier). *With this notation:*

(a) $T(x^{(p)}) = x'^{(\sigma(p))}$ *for every label* $p \in [N] \cup \{O\}$, *and* $d_\alpha(T(u), T(v)) = d_\alpha(u, v)$.

(b) *For every* $k \geq 1$ *and every domain* Ω, *a map* n *is a real map for* kQ *on* Ω *if and only if the map* n' *on* $T(\Omega)$ *defined by* $n'_r(T(u)) = n_r(u)$ *for* $j \notin J_r$ *and* $n'_r(T(u)) = k\beta_r - n_r(u)$ *for* $j \in J_r$ *is a real map for* $k\sigma Q$ *on* $T(\Omega)$, *and* n *is integer exactly when* n' *is.*

(c) $T(\mathcal{A}(Q)) = \mathcal{A}(\sigma Q)$.

Consequently $k_{\min}(\sigma Q) = k_{\min}(Q)$ *and* $k^{\mathcal{A}}_{\min}(\sigma Q) = k^{\mathcal{A}}_{\min}(Q)$ *for every* $\sigma \in \mathfrak{S}_{N+1}$.

PROOF. (a) Let $p \in [N] \setminus \{j\}$. In a position l with $j \notin I_l$ the bit of p is unchanged, and in a position with $j \in I_l$ it is $[p \in ([N] \setminus I_l) \cup \{j\}] = 1 - [p \in I_l]$. Since $x^{(j)}_l = [j \in I_l]$, this says $x'^{(p)} = x^{(p)} \oplus x^{(j)}$, and $\sigma(p) = p$. The label j of σQ lies in I'_l exactly when $j \in I_l$, so $x'^{(j)} = x^{(j)} = T(0) = T(x^{(O)})$, and $\sigma(O) = j$. Finally $T(x^{(j)}) = 0 = x'^{(O)}$, and $\sigma(j) = O$. A translation does not change $|u_l - v_l|$, so it preserves d_α.

(b) The same computation on the right gives $y'^{(\sigma(p))} = y^{(p)} \oplus y^{(j)}$. The pins of n at $x^{(p)}$ are $n_r(x^{(p)}) = k\beta_r y^{(p)}_r$. If $j \notin J_r$, then $y^{(j)}_r = 0$ and $n'_r(T(x^{(p)})) = k\beta_r y^{(p)}_r = k\beta_r y'^{(\sigma(p))}_r$. If $j \in J_r$, then $y^{(j)}_r = 1$ and $n'_r(T(x^{(p)})) = k\beta_r(1 - y^{(p)}_r) = k\beta_r y'^{(\sigma(p))}_r$. By (a) these are the pins of $k\sigma Q$, for all labels p including the purifier. The ranges are preserved, and $|n'_r(T(u)) - n'_r(T(v))| = |n_r(u) - n_r(v)|$, so with (a) the contraction conditions correspond. The construction is invertible and preserves integrality.

(c) Conditions (R1) and (R2) are conditions on pairs of positions l, l', and we compare them for Q at x and for σQ at $T(x)$. If neither I_l nor $I_{l'}$ contains j, the sets and the bits are unchanged. If both contain j, both bits are flipped, both sets are replaced by (5.21), and $I'_l \subseteq I'_{l'}$ holds exactly when $I_{l'} \subseteq I_l$. Condition (R1) holds trivially for both inequalities, since their two sets share j. Condition (R2) for σQ says that $T(x)_l = 1$ and $I'_l \subseteq I'_{l'}$ imply $T(x)_{l'} = 1$, that is, that $x_l = 0$ and $I_{l'} \subseteq I_l$ imply $x_{l'} = 0$, which is the contrapositive of (R2) for Q with l and l' exchanged. If $j \in I_l$ and $j \notin I_{l'}$, then $T(x)_l = 1 - x_l$, $T(x)_{l'} = x_{l'}$, $I'_l = ([N] \setminus I_l) \cup \{j\}$ and $I'_{l'} = I_{l'}$, so that

$$I'_l \cap I'_{l'} = I_{l'} \setminus I_l \ , \qquad I'_{l'} \subseteq I'_l \iff I_l \cap I_{l'} = \emptyset \ , \tag{5.23}$$

while $I'_l \subseteq I'_{l'}$ and $I_l \subseteq I_{l'}$ are both impossible, because j lies in the first set and not in the second. By (5.23), condition (R1) for σQ, that $x_l = 0$ and $x_{l'} = 1$ imply $I_{l'} \not\subseteq I_l$, is the contrapositive of (R2) for Q at the containment $I_{l'} \subseteq I_l$. Condition (R2) for σQ at the containment $I'_{l'} \subseteq I'_l$, that $x_{l'} = 1$ implies $x_l = 0$ when $I_l \cap I_{l'} = \emptyset$, is the contrapositive of (R1) for Q. So $x \in \mathcal{A}(Q)$ exactly when $T(x) \in \mathcal{A}(\sigma Q)$.

For the consequence, (b) with $\Omega = \{0,1\}^L$, and with $\Omega = \mathcal{A}(Q)$ together with (c), shows that kQ and $k\sigma Q$ have integer maps on the same domains. A relabeling that fixes O only renames parties and permutes the terms, which permutes the coordinates of both cubes and preserves everything. Every element of $\mathfrak{S}_{N+1}$ is a composition of a relabeling that fixes O and at most one exchange of a party with O. □

So the choice of the purifier among the $N + 1$ labels does not affect whether an integer map exists, on the full cube or on the realizable domain, as long as the realizable domain is computed from the purified sets of the presentation at hand. The statements of the paper about $\mathcal{A}(Q)$ are made in the printed presentation, and by the proposition, which is proved in this companion and is not in the paper, they transfer to every other choice of purifier.

Part II

Certificates, and inequalities without unscaled proofs

CHAPTER 6

Certificates

Where we are. Most statements of Chapters 7–8 and of Part III rest on computations, and this chapter says what those computations are and why each of them proves what the text needs. The method lemmas of Part I turn every question about maps into a finite problem: whether kQ has an integer map on a finite set of points is a question about finitely many integers subject to finitely many inequalities, and an inequality can be proved valid by exhibiting a map with real values on the realizable domain (Lemma 5.9). A computer can answer such finite questions. What makes the answer a proof is that it comes with an object that can be checked independently of the program that found it, and this object is what we call a certificate. We describe the four kinds of certificates in the paper, then the two tools behind the non-existence statements, SAT solvers and DRAT proofs, in enough detail that every step from "the checker accepted the file" to "no map exists" can be followed, and then say exactly where this chain is weakest.

6.1. What a certificate is

Every computer-assisted statement in the paper about maps or graph models is of one of four kinds. For each we say what is computed, why it proves the statement, and in which arithmetic.

(i) *Non-existence of a map*: kQ has no integer map on a given finite domain $\Omega \subseteq \{0,1\}^L$, possibly with a symmetry imposed on the map. What is computed is a Boolean formula that is satisfiable whenever such a map exists, together with a proof that the formula is unsatisfiable, in a format that a separate program checks clause by clause. The arithmetic is Boolean, and nothing is rounded. Sections 6.2–6.4 explain these certificates in full.

(ii) *An explicit map*: a table of values $n_r(u)$ for every point u of the domain and every RHS term r, as integers or as rational numbers with a common denominator. A checker verifies the pins, the ranges and the contraction condition in exact integer arithmetic. An integer map proves that kQ has an unscaled map, and any map proves validity, by the contraction theorem (Theorem 4.1) on the full cube and by the realizable-domain criterion (Lemma 5.9) on $\mathcal{A}(Q)$. Section 6.5 gives the details.

(iii) *A list of graph models with a rank*: finitely many graph models with integer edge weights, on which an inequality vanishes, whose entropy vectors are recomputed by the checker and whose rank is computed exactly. With validity, this proves that the inequality defines a facet (Section 6.6).

(iv) *A graph model violating an inequality*: one graph model with integer weights on which a given integer combination of inequalities is negative. Recomputing its entropy vector proves that the combination is invalid (Section 6.6).

Exact identities between integer coefficient vectors, such as the decomposition of B_7^* into instances of subadditivity and MMI in Chapter 7, are certificates as well, and checking one needs nothing beyond adding integer vectors.

Two words recur in the status lines. An *encoder* is the program that writes the Boolean formula of kind (i), and a *checker* is a program that verifies a certificate of kind (ii)–(iv). Programs are counted as independent when they were implemented separately, and the status lines give

the number of independent encoders or checkers that the paper reports for each statement. The certificates and the programs that check them are in the code repository accompanying the paper.

6.2. Boolean formulas and the map problem

Formulas in conjunctive normal form. Let $b_1, b_2, \dots$ be Boolean variables, each taking the value true or false. A *literal* is a variable b or its negation $\neg b$. A *clause* is a finite set of literals, read as their disjunction ("or"), and an assignment of truth values to the variables *satisfies* the clause if it makes at least one of its literals true. A clause with a single literal is a *unit clause*, and it can only be satisfied by making that literal true. The clause with no literals is the *empty clause*, written $\square$, and no assignment satisfies it. A formula in *conjunctive normal form* (CNF) is a finite set of clauses, read as their conjunction ("and"), and an assignment satisfies it if it satisfies every clause. The formula is *satisfiable* if some assignment satisfies it and *unsatisfiable* otherwise. An implication such as $(b_1 \wedge \neg b_2) \Rightarrow b_3$ is the clause $\neg b_1 \vee b_2 \vee b_3$, since it fails exactly when b_1 is true and b_2, b_3 are false.

A *SAT solver* is a program that decides whether a given CNF is satisfiable. If it is, the solver returns a satisfying assignment, and anyone can confirm the answer by checking the clauses one by one. If it is not, the solver's verdict "unsatisfiable" is by itself a claim about a search that nobody can repeat by hand. The paper decided its formulas with the SAT solvers CaDiCaL [**25**] and Kissat [**26**], and Section 6.3 explains how such a verdict is turned into a checkable proof.

A first example. The map problem at a single point already gives a small CNF. Take the three-party inequality $A + B + C + ABC \geq AB + AC$, which reappears as Q' in Chapter 10, with the LHS terms in the order (A, B, C, ABC), all coefficients 1, and the RHS terms in the order (AB, AC). The occurrence words are

$$\begin{aligned} &x^{(A)} = 1001\ , \qquad x^{(B)} = 0101\ , \qquad x^{(C)} = 0011\ , \qquad x^{(O)} = 0000\ , \\ &y^{(A)} = 11\ , \qquad y^{(B)} = 10\ , \qquad y^{(C)} = 01\ , \qquad y^{(O)} = 00\ . \end{aligned} \tag{6.1}$$

The point $u = 0001$ is at distance $d_\alpha(u, x^{(i)}) = 1$ from each of the four pinned points. For $k = 1$ the value $n(u)$ is a pair of bits (b_1, b_2), the values on AB and AC, and the contraction condition between u and the pinned point of i says that (b_1, b_2) differs from $y^{(i)}$ in at most one position. This excludes exactly the value opposite to $y^{(i)}$, so the four conditions are the four clauses

$$b_1 \vee b_2 \quad (i = A)\ , \qquad b_1 \vee \neg b_2 \quad (i = B)\ , \qquad \neg b_1 \vee b_2 \quad (i = C)\ , \qquad \neg b_1 \vee \neg b_2 \quad (i = O)\ . \tag{6.2}$$

Each of the four assignments violates exactly one of them, so the formula is unsatisfiable, and no integer map exists on any domain containing u, the occurrence words and 0. The general encoding below produces formulas of the same meaning with many more variables.

The order encoding. Fix an inequality Q as in Chapter 4, a scaling $k \geq 1$ and a finite domain $\Omega \subseteq \{0,1\}^L$ that contains the occurrence words and 0. In the paper Ω is the full cube, the realizable domain $\mathcal{A}(Q)$ or the subdomain $\mathcal{U}(Q)$ of (5.18). The unknowns of the map problem are the integers $n_r(u) \in \{0, 1, \dots, k\beta_r\}$ for $u \in \Omega$ and $r = 1, \dots, R$. The map problem is thus a finite constraint satisfaction problem [**17**], and the *order encoding* writes it as a CNF in the following steps, with each value in the unary form that Bailleux and Boufkhad use for cardinality constraints [**27**].

(E1) *Staircases.* For each $u \in \Omega$, each r and each $s = 1, \dots, k\beta_r$ there is a variable $b_{u,r,s}$, meant to be true exactly when $n_r(u) \geq s$. The clauses $\neg b_{u,r,s+1} \vee b_{u,r,s}$, that is $b_{u,r,s+1} \Rightarrow b_{u,r,s}$, force the bits of each pair (u, r) to read $1, \dots, 1, 0, \dots, 0$ as s increases. Such staircases correspond one to one to the values $n_r(u) = \sum_s b_{u,r,s} \in \{0, \dots, k\beta_r\}$.

(E2) *Counting differences.* For two staircases of heights $n = n_r(u)$ and $n' = n_r(v)$, the bits at threshold s differ exactly when $\min(n, n') < s \leq \max(n, n')$. So the number of thresholds at which $b_{u,r,s} \neq b_{v,r,s}$ is $|n_r(u) - n_r(v)|$, and summing over r gives the left side of the contraction condition (4.14).

(E3) *Difference variables.* For each pair u, v that is kept (see (E6)) and each r, s there is a variable $\Delta_{u,v,r,s}$ with the two clauses

$$\neg b_{u,r,s} \vee b_{v,r,s} \vee \Delta_{u,v,r,s} \ , \qquad b_{u,r,s} \vee \neg b_{v,r,s} \vee \Delta_{u,v,r,s} \ , \tag{6.3}$$

which say that $\Delta_{u,v,r,s}$ is true wherever the two bits differ. Where they agree, $\Delta_{u,v,r,s}$ is left free.

(E4) *The contraction condition.* For each kept pair, a cardinality constraint says that at most $kd_\alpha(u, v)$ of the variables $\Delta_{u,v,r,s}$ are true. With (E2) and (E3), every satisfying assignment then has $\sum_r |n_r(u) - n_r(v)| \le kd_\alpha(u, v)$.

(E5) *Pins.* At $u = x^{(i)}$ the unit clauses $b_{u,r,s}$ for $s \le k\beta_r[i \in J_r]$ and $\neg b_{u,r,s}$ for larger s fix the value $n_r(x^{(i)}) = k\beta_r[i \in J_r]$, and at $u = 0$ the unit clauses $\neg b_{0,r,s}$ fix $n(0) = 0$.

(E6) *Pairs that are dropped.* A pair that is reducible in Ω can be dropped, by the irreducible-pairs lemma (Lemma 5.15). A pair with $d_\alpha(u, v) \ge |\beta|$ can be dropped too, because any two values in the ranges satisfy $\sum_r |n_r(u) - n_r(v)| \le \sum_r k\beta_r = k|\beta| \le kd_\alpha(u, v)$.

A cardinality constraint "at most ν of the variables $p_1, \dots, p_m$ are true" is not a single clause, and it is written with auxiliary variables. The paper names three kinds of constructions, totalizers [**27**], sequential counters [**28**] and cardinality networks [**29**], used depending on the implementation. We describe the sequential counter of Sinz [**28**], with renamed variables, because it shows exactly which property the encoding needs. For $\nu \ge 1$ it has auxiliary variables $c_{h,q}$, $1 \le h \le m$, $1 \le q \le \nu$, meant as "at least q of $p_1, \dots, p_h$ are true", and the clauses

$$\begin{aligned} &p_h \Rightarrow c_{h,1} \ , \qquad c_{h-1,q} \Rightarrow c_{h,q} \ , \qquad (p_h \wedge c_{h-1,q-1}) \Rightarrow c_{h,q} \ \ (q \ge 2) \ , \\ &\neg(p_h \wedge c_{h-1,\nu}) \ , \end{aligned} \tag{6.4}$$

for all h for which the indices make sense (for $\nu = 0$ the constraint is the unit clauses $\neg p_h$). Sinz's clauses differ from these in two respects that the argument below does not use: they include the unit clauses $\neg c_{1,q}$ for $2 \le q \le \nu$, and the counter stops at $h = m - 1$, without the variables $c_{m,q}$. The property used below is that every assignment of $p_1, \dots, p_m$ with at most ν true variables extends to a satisfying assignment of (6.4). Indeed, set $c_{h,q}$ true exactly when at least q of $p_1, \dots, p_h$ are true. The first three kinds of clauses then hold by counting, and the last one holds because p_h together with ν true variables among $p_1, \dots, p_{h-1}$ would make $\nu + 1$ true variables. The converse also holds, although nothing below needs it: in a satisfying assignment the clauses force $c_{h,q}$ to be true whenever at least q of $p_1, \dots, p_h$ are true, so if p_h were the $(\nu + 1)$-th true variable, $c_{h-1,\nu}$ and p_h would both be true, against the last clause.

LEMMA 6.5 (Soundness of the encoding). *Suppose that every cardinality constraint in* (E4) *is encoded so that every assignment with at most $kd_\alpha(u, v)$ true inputs extends to a satisfying assignment of its clauses. If kQ has an integer map on Ω, then the CNF of* (E1)–(E6) *is satisfiable. Hence, if this CNF is unsatisfiable, kQ has no integer map on Ω.*

PROOF. Let n be an integer map for kQ on Ω. Set $b_{u,r,s}$ true exactly when $n_r(u) \ge s$. This satisfies the staircase clauses of (E1), and since n satisfies the pins (4.13) (scaled by k), it satisfies the unit clauses of (E5). For each kept pair set $\Delta_{u,v,r,s}$ true exactly where $b_{u,r,s} \ne b_{v,r,s}$, which satisfies (6.3). By (E2) the number of true $\Delta_{u,v,r,s}$ for the pair is $\sum_r |n_r(u) - n_r(v)|$, which is at most $kd_\alpha(u, v)$ by the contraction condition (4.14). By the assumption on the cardinality encoding, its auxiliary variables can be set so that all its clauses hold. Dropping pairs in (E6) only removes clauses. So all clauses are satisfied. □

Two features of this lemma matter for what follows. First, it runs in one direction only. It needs that every map gives a satisfying assignment, and it does not need that every satisfying assignment gives a map. For the same reason it does not use the irreducible-pairs lemma: any set of pairs may be dropped, since removing clauses can turn an unsatisfiable formula into a satisfiable

one but never the other way. The irreducible-pairs lemma matters only when a map is read off a satisfying assignment, and every such map is checked independently anyway (Section 6.5). Second, the lemma has a hypothesis about the cardinality encoding, and the whole construction has an implicit one: that the program writing the formula used the right weights, pins, points and scaling. Section 6.4 returns to this.

The split encoding. The sorting lemma (Lemma 4.15) gives a second formulation. Here the variables are the bits $f_{u,r,s}$, $1 \le s \le k\beta_r$, of the image $f(u) \in \{0,1\}^{k|\beta|}$ of each point in the presentation with split right-hand side, with no staircase clauses, a difference variable for each pair and each unit copy (r, s), and the same cardinality constraints and pins. A map n gives the sorted bits $f_{u,r,s} = [s \le n_r(u)]$, two sorted words of heights $n_r(u)$ and $n_r(v)$ differ in exactly $|n_r(u) - n_r(v)|$ copies, and the proof of Lemma 6.5 goes through word for word. The two encodings differ in their variables and clauses. The encoders that the paper counts as independent were implemented separately.

The equivariant encoding. A relabeling $\sigma \in \mathrm{Stab}(Q)$ that fixes the purifier sends each LHS set $I_l \subseteq [N]$ to a subset of $[N]$, and since σ fixes the coefficient vector this subset is again an LHS term, $\sigma(I_l) = I_{\sigma(l)}$ with $\alpha_{\sigma(l)} = \alpha_l$. Likewise $\sigma(J_r) = J_{\sigma(r)}$ with $\beta_{\sigma(r)} = \beta_r$. The relabeling acts on points by $(\sigma \cdot u)_{\sigma(l)} = u_l$, which preserves d_α, sends $x^{(i)}$ to $x^{(\sigma(i))}$ and 0 to 0, and maps the full cube and $\mathcal{A}(Q)$ to themselves, since it preserves intersections and inclusions of the sets I_l. A map is *equivariant* if $n_{\sigma(r)}(\sigma \cdot u) = n_r(u)$ for all σ, r and u. The equivariant encoding is the order encoding with the variables $b_{u,r,s}$ and $b_{\sigma\cdot u,\sigma(r),s}$ identified for every σ in the chosen group, and with the contraction condition imposed on one pair from each orbit of pairs. An equivariant map gives a satisfying assignment, since its values at identified variables agree, so an unsatisfiable formula excludes equivariant integer maps. Keeping one pair per orbit is a dropping of clauses, harmless by the first feature above, and it loses nothing in the other direction, because after the identification the conditions on all pairs of an orbit are the same condition.

6.3. DRAT proofs

A SAT solver of the kind used in the paper searches for an assignment by setting variables one at a time and deducing the consequences, and when it reaches a contradiction it derives a new clause, a *learned clause*, that every satisfying assignment must satisfy and that excludes the choices that led to the contradiction. We describe this only in outline, since nothing below depends on how the solver works. If the formula is unsatisfiable, the search ends by learning the empty clause. Writing the learned clauses to a file in the order in which they were learned gives a proof that a much simpler program can check, without repeating the search. The format of such files is DRAT, and the checker used in the paper is `drat-trim` [**30**]. We now define what the checker verifies and prove that a verified file implies unsatisfiability. Part (b) of Lemma 6.6 below is stated by Wetzler, Heule and Hunt [**30**], with a reference to earlier work, and we give proofs so that every step can be followed.

Unit propagation. Given a CNF F and a set of literals declared true, a clause of F all of whose literals are false except one, which is unassigned, forces that literal to be true. A clause all of whose literals are false is a *conflict*. *Unit propagation* repeats the forcing step until a conflict appears or nothing more is forced. In the example (6.2), declare $\neg b_1$ true. The clause $b_1 \vee b_2$ forces b_2, and then $b_1 \vee \neg b_2$ has both literals false, a conflict.

RUP and RAT clauses. For a clause C write $\neg C$ for the set of negations of its literals, so that declaring $\neg C$ true makes every literal of C false. A clause C has the *reverse unit propagation* (RUP) property for F if unit propagation on F, started from $\neg C$, reaches a conflict. It has the *resolution asymmetric tautology* (RAT) property for F with respect to a literal $\ell \in C$ if, for every clause $D \in F$ that contains $\neg\ell$, the clause $C \cup (D \setminus \{\neg\ell\})$ has the RUP property for F or contains a literal together with its negation. These are the properties used by Wetzler, Heule and Hunt,

who state RUP as asymmetric tautology and cite Van Gelder for the name [**30**]. A *DRAT proof* for F is a finite sequence of steps, each of which adds a clause or deletes one, defining formulas $F_0 = F, F_1, F_2, \ldots$, such that every added clause has the RUP or the RAT property for the formula before it, and such that at some step the empty clause is added. The empty clause has no literals, so it can only be added by the RUP property, which for $\square$ means that unit propagation on the current formula, started from nothing, reaches a conflict. The checker `drat-trim` checks, by running these unit-propagation tests, whether a given file is a DRAT proof for a given formula, testing the RAT property on the first literal of each added clause, and reports it as verified only if it is [**30**].

LEMMA 6.6 (What a DRAT proof proves). (a) *If C has the RUP property for F, every assignment that satisfies F satisfies C.* (b) *If C has the RAT property for F and F is satisfiable, then $F \cup \{C\}$ is satisfiable.* (c) *If F has a DRAT proof, F is unsatisfiable.*

PROOF. (a) Let τ satisfy F and suppose that τ makes every literal of C false. Then every literal of $\neg C$ is true under τ. When unit propagation forces a literal, the other literals of the forcing clause are false at that moment, and if they are false under τ, the forced literal is true under τ, because τ satisfies the clause. By induction over the propagation steps, every literal declared true during propagation from $\neg C$ is true under τ, so no clause ever has all its literals false under the declared literals, and no conflict is reached. This contradicts the RUP property, so τ satisfies C.

(b) Let τ satisfy F. If τ satisfies C there is nothing to show. Otherwise τ makes every literal of C false, the literal ℓ included. Let τ' agree with τ except that it makes ℓ true. Then τ' satisfies C. A clause of F that does not contain $\neg\ell$ is satisfied by τ through a literal that is neither ℓ, which τ makes false, nor $\neg\ell$, which the clause does not contain, so through a literal on another variable, and τ' leaves that literal unchanged. Let $D \in F$ contain $\neg\ell$. The clause $C \cup (D \setminus \{\neg\ell\})$ is satisfied by τ, by part (a) if it has the RUP property and trivially if it contains a literal and its negation. Since τ makes every literal of C false, it makes some literal of $D \setminus \{\neg\ell\}$ true. This literal is neither ℓ, which τ makes false, nor $\neg\ell$, which was removed, so it lies on another variable, and τ' makes it true as well and satisfies D. Hence τ' satisfies $F \cup \{C\}$.

(c) Deleting a clause removes a constraint and keeps a satisfiable formula satisfiable. By (a) and (b), adding a RUP or RAT clause does too. So if $F_0 = F$ were satisfiable, every F_i along the proof would be satisfiable. But some F_i contains the empty clause, which no assignment satisfies. Hence F is unsatisfiable. $\square$

A RUP clause is a consequence of the formula, and it is the kind of clause a solver learns. A RAT clause need not be a consequence. For instance, if $\neg b$ occurs in no clause of F, the unit clause b has the RAT property vacuously, and adding it removes every satisfying assignment with b false, but by part (b) it never turns a satisfiable formula into an unsatisfiable one. This suffices for the proof, which has to preserve satisfiability at each step.

For the example (6.2) a DRAT proof has two steps: add the unit clause b_1, then add $\square$. The first step has the RUP property, since propagation from $\neg b_1$ reaches the conflict found above. For the second, propagation in the enlarged formula starts from the unit clause b_1, which forces b_2 through $\neg b_1 \vee b_2$, and then $\neg b_1 \vee \neg b_2$ is a conflict. Read as a statement about maps, the first step says that every map for this inequality with $k = 1$ would have to take the value 1 on AB at the point 0001, and the second that it cannot.

Real proofs for the inequalities of the paper are files of the same kind, much longer. The checker `drat-trim` is a separate program from the solvers that wrote the proofs and shares no code with them. Every non-existence statement of the paper comes with a DRAT proof that it accepted, or follows from one by the restriction lemma or by adding parties that lie in no term, with the following exceptions. The first is that B_7^* has no integer map on its realizable domain, which rests on the hand proof of Chapter 7. The others concern star inequalities. Where no DRAT proof was produced for a star inequality and a scaling, the statement that no map exists rests on the argued

theorems of Chapters 10 and 11, that is, on the theorem on the scaling spectrum (Theorem 10.13) and the results built on it. Among them are the general statements of these theorems, the lower bound of Corollary 11.39 included, and the statements that kQ' has no integer map for odd $k \geq 5$, kD_0 none for $k \geq 5$ not divisible by 4, kD_1 none for $k \geq 3$ not divisible by 8, kD_j with $j \geq 2$ none for every k not divisible by 2^{j+2}, kZ_7 none for $k \geq 9$ not divisible by 8 and kH_9 none for $k \geq 19$ not divisible by 18, together with the same statements for the inequalities obtained from these by adding parties. Putting together Lemma 6.6, the soundness of the encoding and the restriction lemma (Lemma 5.19), the chain behind every such statement is

$$\boxed{\begin{array}{l}\text{DRAT proof verified} \implies \text{CNF unsatisfiable} \implies \text{no integer map on } \mathcal{U}(Q) \\ \quad \implies \text{no integer map on } \{0,1\}^L\text{, and none on } \mathcal{A}(Q) \\ \qquad \text{if the LHS terms containing } A \text{ form an antichain}\end{array}} \tag{6.7}$$

The first arrow is Lemma 6.6(c). The second is Lemma 6.5 and holds only if the encoder is faithful, which the next section discusses. The third is the restriction lemma, and when the formula is written for the full cube itself, the second arrow already gives the conclusion on $\{0,1\}^L$.

6.4. What a DRAT proof does not certify

A verified DRAT proof certifies that the file it refers to is an unsatisfiable formula. What the formula means lies outside this check. The second arrow of (6.7) rests on the encoder having written the formula that Lemma 6.5 describes, and no part of the DRAT check tests this. If the encoder wrote a formula that is stronger than the map problem, the formula could be unsatisfiable while maps exist, and the DRAT proof would still verify. The ways an encoder can make the formula too strong are the ways in which the proof of Lemma 6.5 can fail: a cardinality encoding that rejects an allowed number of true inputs, a wrong weight in d_α, a wrong pin, an extra point that does not belong to Ω, or a wrong scaling k. Errors in the other direction, a missing clause or a missing point, are harmless for non-existence statements, because they only make the formula weaker.

The paper reduces this risk by independent encoders: every non-existence statement was obtained from at least two separately implemented encoders, so that a spurious unsatisfiable verdict would need errors with the same effect in separately written programs. The exceptions are three statements whose DRAT proofs come from a single encoder: that E_5 has no integer map (Chapter 8), that $3Q_8$ has no integer map (Chapter 8), and that V_7 has no equivariant integer map on its realizable domain (Chapter 9).

Finally, the verdict "verified" rests on the correctness of the checker `drat-trim` itself, which the paper assumes.

6.5. Validity certificates: exact maps

A validity certificate for Q is a table of values $n_r(u)$ for all points u of the full cube or of $\mathcal{A}(Q)$ and all r, given as integers or as rational numbers with a common denominator M. It proves validity as follows. On $\mathcal{A}(Q)$, a real map with the pins and the contraction condition proves that Q is valid by the realizable-domain criterion (Lemma 5.9). On the full cube, the same lemma applies, and so does the contraction theorem: Mn is an integer map for MQ, by the sorting lemma it is a contraction map for the presentation of MQ with split right-hand side, so MQ is valid by Theorem 4.1, and hence so is Q.

A checker works from the list of terms of Q alone. It rebuilds the domain from Definition 5.1, or takes the full cube, and verifies that the points of the certificate are exactly this domain, so that a certificate on a smaller set cannot pass. It verifies the pins (4.13), the ranges $0 \leq n_r(u) \leq \beta_r$, and the contraction condition (4.14) on every pair of points of $\mathcal{A}(Q)$, or on every edge of the full cube, which suffices there by the irreducible-pairs lemma. All comparisons are made in exact integer

arithmetic after multiplication by M. Some checkers first screen all pairs in floating point with a rigorous bound on the rounding error and then check exactly every pair that the screen does not decide. Every validity certificate in the paper was checked by at least one program written independently of the code that produced the map.

The integer maps for $2E_0$, $2E_1$, $2E_3$ and $2E_4$ on the full cube (Section 8.10) are tables of integers, one value $n_r(u)$ for each point u of the cube and each RHS term r. The map for $2E_0$ is the sum of the map for $2E_1$ and four times an unscaled map of μ_0, by the construction of Section 8.10. The proof uses only these four tables. Each was checked as described above, in exact integer arithmetic, on all edges of its cube by two checkers independent of the program that found it, and on all pairs of points by at least one of them.

6.6. Graph-model certificates

A graph-model certificate is a finite graph with integer edge weights and labeled boundary vertices, a graph model Γ in the sense of Chapter 2. The checker recomputes the entropy vector. For each nonempty $I \subseteq [N]$ it computes $S_\Gamma(I)$ by a maximum-flow computation between the boundary vertices labeled by I and all other boundary vertices, which by the max-flow min-cut theorem, a classical result that we do not prove here, gives the least weight of an I-cut. With integer weights every cut weight is an integer, so the entropy vector is an integer vector computed without rounding.

Invalidity. If Q and X are integer vectors and Γ has $Q \cdot S_\Gamma = 0$ and $X \cdot S_\Gamma > 0$, then $(Q - \lambda X) \cdot S_\Gamma = -\lambda X \cdot S_\Gamma < 0$ for every $\lambda > 0$, so $Q - \lambda X$ is invalid for every $\lambda > 0$. (We write λ for the step of such a splitting, where the paper writes t or ε, because t is the threshold of Chapter 5 and ϵ will be an edge of a triangle in Chapter 7.) This is how the paper shows that a splitting step cannot be enlarged, in Chapter 8.

Facets. By the facet criterion of Chapter 2, a valid $Q \neq 0$ defines a facet of HEC_N exactly when there are $2^N - 2$ graph models with $Q \cdot S_\Gamma = 0$ whose entropy vectors are linearly independent. A facet certificate is a list of $2^N - 2$ such graph models. The checker recomputes their entropy vectors, checks $Q \cdot S_\Gamma = 0$ for each, and computes, modulo a prime p, the rank of the integer matrix $\mathbf{S}$ whose rows are the entropy vectors. This suffices, because reduction modulo p does not increase the rank: the rank of $\mathbf{S}$ over $\mathbb{Q}$ is the largest ρ for which some $\rho \times \rho$ minor of $\mathbf{S}$ is nonzero, and a minor is a polynomial with integer coefficients in the entries, so the minor of $\mathbf{S}$ mod p is the corresponding minor of $\mathbf{S}$ reduced modulo p. If $\mathbf{S}$ mod p has rank ρ, some $\rho \times \rho$ minor is nonzero modulo p, hence nonzero as an integer, hence the rank over $\mathbb{Q}$ is at least ρ. All rows lie in the hyperplane $Q^\perp$ of dimension $2^N - 2$, so rank $2^N - 2$ modulo p means rank exactly $2^N - 2$ over $\mathbb{Q}$. The modular computation is faster than exact rational elimination and certifies the same lower bound. One prime suffices for the argument, and the paper reports the rank modulo three different primes.

6.7. The least secure steps

The paper names two steps that it regards as least secure, and we repeat them here in one place together with a remark that the paper adds. The first are the SAT encodings: by Section 6.4, a DRAT proof certifies only that a formula is unsatisfiable, and the step from the map problem to the formula rests on the encoder, which for the three statements listed there is a single program. The second are the results on star inequalities of Chapters 10 and 11, among them the scaling spectrum of star inequalities (Theorem 10.13), on which the others rest, and the theorem on the family D_j (Theorem 11.18), whose general cases rest on written proofs and are therefore argued. The remark concerns validity: the validity of E_2 and E_5 (Chapter 8) and of F_7 (Chapter 9) rests on the realizable-domain criterion (Lemma 5.9), which is proved in Chapter 5 and which, for the rational certificates used, reduces to a published theorem of Avis and Hernández-Cuenca [**4**].

CHAPTER 7

A balanced inequality without an unscaled proof: B_7^*

Where we are. This chapter answers the first question of the paper for balanced inequalities. We give a valid balanced inequality on seven parties, B_7^*, that has no unscaled contraction map, and prove the failure completely by hand: at a single point of the cube, the contraction conditions against the pinned points ask for an edge cover of two disjoint triangles by three edges, which does not exist. Doubling the inequality removes the obstruction, and $2B_7^*$ has a map.

7.1. The inequality

Let $N = 7$ with parties $A, \dots, G$. Party A is a hub, and $\{B, C, D\}$ and $\{E, F, G\}$ are the vertex sets of two triangles with the six edges

$$\mathcal{E} = \{BC, BD, CD, EF, EG, FG\} \ . \tag{7.1}$$

We write ϵ for an edge, and $A\epsilon$ for the set of three parties consisting of A and the two endpoints of ϵ, so that $ABC = A\epsilon$ for $\epsilon = BC$. The inequality is

$$\begin{aligned} B_7^* : \quad & 2A + AO + 3\,(BO + CO + DO + EO + FO + GO) \\ & \geq \ ABC + ABD + ACD + AEF + AEG + AFG + 14\,O \ . \end{aligned} \tag{7.2}$$

Terms that contain the purifier are replaced by the entropy of the complementary set of parties. So AO stands for $S(BCDEFG)$, iO for $S([N] \setminus \{i\})$, the entropy of the six parties other than i, and O for $S(ABCDEFG)$. After this replacement the LHS terms are

(7.3)

$$I_A = \{A\} \ \ (\alpha = 2) \ , \qquad I_{AO} = \{B, \dots, G\} \ \ (\alpha = 1) \ , \qquad I_{iO} = [N] \setminus \{i\} \ \ (\alpha = 3, \ i \in \{B, \dots, G\}) \ ,$$

so $L = 8$ and $|\alpha| = 2 + 1 + 6 \cdot 3 = 21$. The RHS terms are the six $S(A\epsilon)$, $\epsilon \in \mathcal{E}$, with $\beta = 1$, and $S(ABCDEFG)$ with $\beta = 14$, so $R = 7$ and $|\beta| = 20$.

We order the LHS terms as $(A, \ AO, \ BO, \ CO, \ DO, \ EO, \ FO, \ GO)$ and the RHS terms as $(ABC, ABD, ACD, AEF, AEG, AFG, ABCDEFG)$. Party A lies in $\{A\}$ and in every $[N] \setminus \{i\}$, but not in $\{B, \dots, G\}$. A party $i \in \{B, \dots, G\}$ lies in $\{B, \dots, G\}$ and in $[N] \setminus \{j\}$ for $j \neq i$, but not in $\{A\}$ or in $[N] \setminus \{i\}$. The occurrence words are therefore

$$\begin{aligned} & x^{(A)} = (1,0,1,1,1,1,1) \ , \qquad x^{(B)} = (0,1,0,1,1,1,1) \ , \qquad x^{(C)} = (0,1,1,0,1,1,1) \ , \\ & x^{(D)} = (0,1,1,1,0,1,1) \ , \qquad \dots \ , \qquad x^{(G)} = (0,1,1,1,1,1,0) \ , \qquad x^{(O)} = 0 \ , \\ & y^{(A)} = (1,1,1,1,1,1,1) \ , \qquad y^{(B)} = (1,1,0,0,0,0,1) \ , \qquad y^{(C)} = (1,0,1,0,0,0,1) \ , \\ & y^{(D)} = (0,1,1,0,0,0,1) \ , \qquad y^{(E)} = (0,0,0,1,1,0,1) \ , \qquad y^{(F)} = (0,0,0,1,0,1,1) \ , \\ & y^{(G)} = (0,0,0,0,1,1,1) \ , \qquad y^{(O)} = 0 \ , \end{aligned} \tag{7.4}$$

where $x^{(i)}$ for $i \in \{B, \dots, G\}$ has a 0 in the position of A, a 1 in the position of AO, a 0 in the position of iO and a 1 in the five other positions. The first six entries of $y^{(i)}$ record the edges at i, and the last entry is 1 for every party. Figure 1 shows the structure.

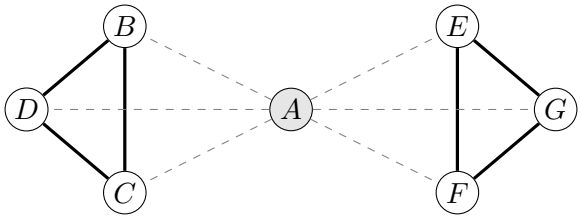


FIGURE 1. The structure of B_7^*. Each solid edge ϵ of the two triangles is an RHS term $S(A\epsilon)$, and the hub A (dashed) lies in all six. At the point z of Section 7.4, an integer map would have to choose three edges that touch all six vertices, which is impossible. Weight $\frac{1}{2}$ on every edge touches every vertex with total weight 1 and uses total weight 3.

7.2. Balance and Bell values

The Bell value of B_7^* on a pair of parties is given by (3.6), $B_7^* \cdot r_{ij} = d_\alpha(x^{(i)}, x^{(j)}) - d_\beta(y^{(i)}, y^{(j)})$. On a pair i, O it is the total coefficient of the LHS terms containing i minus that of the RHS terms containing i. For party A this is $2 + 6 \cdot 3 = 20$ on the left and $6 \cdot 1 + 14 = 20$ on the right. For party B it is $1 + 5 \cdot 3 = 16$ on the left, from $\{B, \dots, G\}$ and the five sets $[N] \setminus \{j\}$ with $j \neq B$, and $2 + 14 = 16$ on the right, from ABC, ABD and $S(ABCDEFG)$. The relabelings that permute $\{B, C, D\}$, permute $\{E, F, G\}$, or exchange the two triangles vertex by vertex fix A and O and map the inequality to itself, so every $i \in \{B, \dots, G\}$ behaves like B. Hence every party has the same total coefficient on both sides, and B_7^* is balanced.

On pairs of parties there are three types up to these relabelings. For B, C, inside one triangle, the LHS words differ in the positions of BO and CO, so $d_\alpha = 3 + 3 = 6$, and the RHS words differ at ABD and ACD, so $d_\beta = 2$. For B, E, in different triangles, again $d_\alpha = 6$, and the RHS words differ at ABC, ABD, AEF and AEG, so $d_\beta = 4$. For A, B, the LHS words differ at A (weight 2), AO (weight 1) and BO (weight 3), so $d_\alpha = 6$, and the RHS words differ at the four terms $A\epsilon$ with $B \notin \epsilon$, so $d_\beta = 4$. The Bell values are therefore 4 on the six pairs inside a triangle and 2 on the other fifteen pairs of parties, and B_7^* is balanced but not superbalanced. The positive Bell values on pairs of parties are what the obstruction below uses, as Section 7.6 explains.

7.3. The three properties

PROPOSITION 7.5. *The inequality B_7^* has the following properties.*

(i) *It is a nonnegative combination, with coefficients in $\frac{1}{2}\mathbb{Z}$, of* 43 *instances of subadditivity and* MMI. *In particular it is valid.*
(ii) *$2B_7^*$ has an integer map on the full cube $\{0,1\}^8$.*
(iii) *B_7^* has no integer map on the full cube, and none on its realizable domain $\mathcal{A}(B_7^*)$.*

Hence $k_{\min}(B_7^*) = k_{\min}^{\mathcal{A}}(B_7^*) = 2$.

Status: (i) certified (identity, 2 programs); (ii) certified (integer map, 2 checkers); (iii) on the full cube certified (DRAT, 3 encoders), on $\mathcal{A}(B_7^*)$ argued (the hand proof below); $k_{\min}(B_7^*) = 2$ derived; $k_{\min}^{\mathcal{A}}(B_7^*) = 2$ argued.

Part (i) is an identity between integer vectors, with 26 instances of subadditivity and 17 of MMI and coefficients in $\{\frac{1}{2}, 1, \frac{3}{2}, 2\}$, checked in exact rational arithmetic. Instances of subadditivity and MMI are valid, since they have contraction maps (Chapter 4), and so is every nonnegative combination of them. Part (ii) is an explicit integer map on the 256 points of $\{0,1\}^8$, whose pins, ranges and contraction condition on all 32,640 pairs of points were checked in exact arithmetic. It also proves validity a second time, by the contraction theorem applied to $2B_7^*$. Part (iii) on the full cube is certified by DRAT proofs as described in Chapter 6, and we now prove it by hand on

both domains. By Chapter 4, the absence of an integer map on the full cube excludes unscaled contraction maps in every presentation obtained by merging or splitting terms.

The last sentence of the proposition follows from (ii) and (iii). The map for $2B_7^*$ on $\{0,1\}^8$ restricts to a map on $\mathcal{A}(B_7^*) \subseteq \{0,1\}^8$, so $k_{\min}^{\mathcal{A}}(B_7^*) \le k_{\min}(B_7^*) \le 2$, and by (iii) neither is 1.

7.4. The hand proof of (iii)

The point. Let $z \in \{0,1\}^8$ be the word that contains every LHS term except $S(A)$,

$$z = (0,1,1,1,1,1,1,1)\ . \tag{7.6}$$

We compute its weighted distances d_α to the eight pinned points from (7.3) and (7.4). The words z and $x^{(A)}$ differ in the positions of A (weight 2) and AO (weight 1). The words z and $x^{(i)}$, $i \in \{B,\dots,G\}$, differ only in the position of iO (weight 3). The words z and 0 differ in every position except that of A. Hence

$$d_\alpha\big(z,x^{(A)}\big) = 2{+}1 = 3\ , \qquad d_\alpha\big(z,x^{(i)}\big) = 3 \quad (i \in \{B,\dots,G\})\ , \qquad d_\alpha(z,0) = 1{+}6{\cdot}3 = 19\ . \tag{7.7}$$

The pins. By (4.13) and (7.4), the values at the pinned points are

$$n\big(x^{(A)}\big) = (1,1,1,1,1,1,14)\ , \qquad n\big(x^{(i)}\big) = \big(([i\in\epsilon])_{\epsilon\in\mathcal{E}},14\big)\ , \qquad n(0) = (0,0,0,0,0,0,0)\ . \tag{7.8}$$

The candidate value. Write a candidate value at z as

$$n(z) = (\theta, 14-\tau)\ , \qquad \theta = (\theta_\epsilon)_{\epsilon\in\mathcal{E}} \in [0,1]^{\mathcal{E}}\ , \qquad \tau \in [0,14]\ , \tag{7.9}$$

with θ_ϵ the value on $S(A\epsilon)$ and $14-\tau$ the value on $S(ABCDEFG)$. (We write τ where the paper writes t, because t is the threshold of the realizable-domain criterion in Chapter 5.) The contraction condition (4.14) between z and a pinned point is the sum over the seven RHS terms of the absolute differences of the values, bounded by (7.7). Against $x^{(A)}$ the six edge terms contribute $|\theta_\epsilon - 1| = 1-\theta_\epsilon$ and the last term contributes $|(14-\tau)-14| = \tau$. Against $x^{(i)}$ the edge terms contribute $|\theta_\epsilon - [i\in\epsilon]|$ and the last term again τ. Against 0 the edge terms contribute θ_ϵ and the last term $14-\tau$. The eight conditions are therefore

$$\begin{aligned} &(A)\colon\ \sum_\epsilon (1-\theta_\epsilon) + \tau \le 3\ , \qquad (i)\colon\ \sum_\epsilon \big|\theta_\epsilon - [i\in\epsilon]\big| + \tau \le 3 \ \ (i\in\{B,\dots,G\})\ , \\ &(O)\colon\ \sum_\epsilon \theta_\epsilon + 14 - \tau \le 19\ . \end{aligned} \tag{7.10}$$

For each vertex $i \in \{B,\dots,G\}$ let $\mathcal{E}_i \subset \mathcal{E}$ be the set of the two edges at i.

Integer values. Suppose that $n(z)$ is an integer vector. Then θ is the indicator of a set $\mathcal{C} \subseteq \mathcal{E}$ of edges, and τ is an integer with $0 \le \tau \le 14$. In condition (A) the sum counts the edges outside $\mathcal{C}$, so (A) reads

$$6 - |\mathcal{C}| + \tau \le 3\ , \qquad \text{that is} \qquad |\mathcal{C}| \ge 3 + \tau \ge 3\ . \tag{7.11}$$

In condition (i) the term of an edge ϵ is 1 exactly when ϵ lies in one of $\mathcal{C}$ and $\mathcal{E}_i$ but not in the other, so the sum is the size of the symmetric difference $\mathcal{C} \triangle \mathcal{E}_i$,

$$\sum_\epsilon \big|\theta_\epsilon - [i\in\epsilon]\big| = |\mathcal{C}\triangle\mathcal{E}_i| = |\mathcal{C}| + |\mathcal{E}_i| - 2|\mathcal{C}\cap\mathcal{E}_i| = |\mathcal{C}| + 2 - 2|\mathcal{C}\cap\mathcal{E}_i|\ , \tag{7.12}$$

and (i) reads

$$|\mathcal{C}| + 2 - 2|\mathcal{C}\cap\mathcal{E}_i| + \tau \le 3\ , \qquad \text{that is} \qquad 2|\mathcal{C}\cap\mathcal{E}_i| \ge |\mathcal{C}| - 1 + \tau\ . \tag{7.13}$$

By (7.11), $|\mathcal{C}|$ is 3, 4, 5 or 6, and we treat the two cases $|\mathcal{C}| = 3$ and $|\mathcal{C}| \ge 4$.

Case $|\mathcal{C}| = 3$. Then (7.11) gives $\tau \le 0$, so $\tau = 0$, and (7.13) gives $2|\mathcal{C}\cap\mathcal{E}_i| \ge 2$, that is $|\mathcal{C}\cap\mathcal{E}_i| \ge 1$, for each of the six vertices i. So the three edges of $\mathcal{C}$ touch every vertex of both

triangles. But an edge of $\mathcal{E}$ lies inside one triangle and touches two of its three vertices, so at least two edges of $\mathcal{C}$ lie in each triangle, and $|\mathcal{C}| \geq 4$, a contradiction.

Case $|\mathcal{C}| \geq 4$. Then (7.13) gives $2|\mathcal{C} \cap \mathcal{E}_i| \geq 3 + \tau \geq 3$, so $|\mathcal{C} \cap \mathcal{E}_i| \geq 2$, and since $|\mathcal{E}_i| = 2$, $\mathcal{E}_i \subseteq \mathcal{C}$ for every vertex i. Every edge lies in some $\mathcal{E}_i$, so $\mathcal{C} = \mathcal{E}$. Then $|\mathcal{C}| = 6$ and $|\mathcal{C} \cap \mathcal{E}_i| = 2$, and the first form of (7.13) reads $6 + 2 - 4 + \tau = 4 + \tau \leq 3$, a contradiction.

So no integer value at z satisfies the conditions (A) and (i), and condition (O) is not needed:

(7.14)

at z, an integer value would be an edge cover of two disjoint triangles by three edges, and none exists

.

The conditions (7.10) are part of the contraction condition on every domain that contains z, the occurrence words and 0. Hence B_7^* has no integer map on any such domain.

The two domains. The full cube $\{0,1\}^8$ contains z. For the realizable domain we check Definition 5.1 at z, using the purified sets (7.3). The terms present in z are $\{B, \ldots, G\}$ and the six sets $[N] \setminus \{i\}$, all of them six-element subsets of a seven-element set, so any two of them share at least five parties and (R1) holds. The six-element sets are distinct and of equal size, so none contains another, and $\{A\}$ is not contained in $\{B, \ldots, G\}$. The only inclusions among the LHS sets are therefore $\{A\} \subseteq [N] \setminus \{i\}$, and for these (R2) requires that a word containing the term A also contains iO. Since z does not contain the term A, (R2) holds at z, and $z \in \mathcal{A}(B_7^*)$. The occurrence words and 0 lie in $\mathcal{A}(B_7^*)$ by Chapter 5. This completes the proof of (iii). The inclusions $\{A\} \subseteq [N] \setminus \{i\}$ are visible only after purification, since the printed terms A and iO look disjoint, and the realizable domain is defined for the presentation with O as the purifier (Chapter 5). The statement that B_7^* has no integer map on $\mathcal{A}(B_7^*)$ refers to this presentation. By the proposition on changing the purifier in Chapter 5, which the paper does not state, the statement then holds for every choice of purifier, with the realizable domain computed in that presentation.

7.5. The real relaxation and the doubled inequality

Real values at z are not obstructed. The value $\theta_\epsilon = \frac{1}{2}$ on every edge and $\tau = 0$ satisfies (7.10): in (A) the sum is $6 \cdot \frac{1}{2} = 3 \leq 3$, in each (i) every edge contributes $\frac{1}{2}$, so the sum is $3 \leq 3$, and in (O) the sum is $3 + 14 = 17 \leq 19$. The obstruction is a failure of integrality.

The edge-cover language makes this precise. With $\tau = 0$, write $\theta(\mathcal{E}_i) = \sum_{\epsilon \in \mathcal{E}_i} \theta_\epsilon$ and $|\theta| = \sum_\epsilon \theta_\epsilon$. The term of condition (i) is $1 - \theta_\epsilon$ for the two edges at i and θ_ϵ for the other four, so

$$(A)\colon\ |\theta| \geq 3\ , \qquad (i)\colon\ 2 + |\theta| - 2\theta(\mathcal{E}_i) \leq 3\ , \quad \text{that is}\ \ \theta(\mathcal{E}_i) \geq \tfrac{1}{2}\big(|\theta| - 1\big)\ . \tag{7.15}$$

At total weight $|\theta| = 3$ the conditions (i) say $\theta(\mathcal{E}_i) \geq 1$ for every vertex: θ is a fractional edge cover of the two triangles. Summing $\theta(\mathcal{E}_i) \geq 1$ over the three vertices of one triangle counts every edge of that triangle twice, so a fractional edge cover of a triangle has weight at least $\frac{3}{2}$, and weight $\frac{1}{2}$ on each edge attains it. Two disjoint triangles therefore have fractional edge cover number 3, while an integral edge cover needs two edges per triangle, four in all. This integrality gap is what the integer case above runs into (Figure 1).

For $2B_7^*$ the ranges are $\{0,1,2\}$ on each $S(A\epsilon)$ and $\{0, \ldots, 28\}$ on $S(ABCDEFG)$, and the pins and the distances (7.7) are doubled: $n(x^{(A)}) = (2, \ldots, 2, 28)$, $n(x^{(i)}) = ((2[i \in \epsilon])_\epsilon, 28)$, $n(0) = 0$, and the distances are 6, 6 and 38. Twice the half-integral value above is the integer value $\theta'_\epsilon = 1$ on every edge, with $\tau' = 0$, that is $n(z) = (1,1,1,1,1,1,28)$. It satisfies the doubled conditions: against $x^{(A)}$ the sum is $6 \cdot 1 + 0 = 6 \leq 6$, against $x^{(i)}$ every edge contributes $|1 - 2[i \in \epsilon]| = 1$, so the sum is $6 \leq 6$, and against 0 the sum is $6 + 28 = 34 \leq 38$. So $2B_7^*$ is not obstructed at z. Feasibility at one point does not give a map on the whole cube, and that $2B_7^*$ has an integer map on all of $\{0,1\}^8$ is the certificate of Proposition 7.5(ii).

Physics aside 7.16. In the language of the Ryu–Takayanagi formula, a point u of the cube labels the bulk points that lie in the homology regions of exactly the LHS terms with $u_l = 1$. Read in this way, z labels bulk points that lie in the homology regions of the six large regions $[N] \setminus \{i\}$ and of $\{B, \dots, G\}$, but not in that of A: points deep in the bulk, on the side away from A. The map has to decide into which of the candidate regions $A\epsilon$ for the RHS to put these points. Putting them into $A\epsilon$ costs, against the pin of a vertex i, one unit for each chosen edge not at i and saves one unit for each chosen edge at i, and the budget of 3 forces a choice of three edges touching all six vertices. With $\theta = \frac{1}{2}$ the points are "half" in every $A\epsilon$, which a single copy of each candidate region cannot realize and two copies can.

7.6. What the obstruction uses

The point z lies coordinatewise between the occurrence words of every pair of parties: wherever $x^{(i)}$ and $x^{(j)}$ agree, z agrees with them. For $i, j \in \{B, \dots, G\}$ the two words agree in the positions of A (value 0), of AO (value 1) and of the four terms pO with p different from i and j (value 1), and z has the same values there. For A and $i \in \{B, \dots, G\}$ the words agree exactly in the positions jO with $j \neq i$, where both are 1, and so is z. This covers all 21 pairs of parties. For a point between $x^{(i)}$ and $x^{(j)}$ the weighted distances add up: in a position where the two words agree z agrees with both, and in a position where they differ z agrees with exactly one of them, so

$$d_\alpha\big(x^{(i)}, z\big) + d_\alpha\big(z, x^{(j)}\big) = d_\alpha\big(x^{(i)}, x^{(j)}\big) = d_\beta\big(y^{(i)}, y^{(j)}\big) + B_7^* \cdot r_{ij} \ , \tag{7.17}$$

where the second equality is the Bell-value formula (3.6).

Now suppose that the Bell value on the pair i, j were 0. The pinned values differ by $|n_r(x^{(i)}) - n_r(x^{(j)})| = \beta_r |y_r^{(i)} - y_r^{(j)}|$ in each term, so the triangle inequality in each coordinate and the two contraction conditions at z give

$$\begin{aligned} d_\beta\big(y^{(i)}, y^{(j)}\big) &\leq \sum_r \big|n_r(x^{(i)}) - n_r(z)\big| + \sum_r \big|n_r(z) - n_r(x^{(j)})\big| \\ &\leq d_\alpha\big(x^{(i)}, z\big) + d_\alpha\big(z, x^{(j)}\big) = d_\beta\big(y^{(i)}, y^{(j)}\big) \ . \end{aligned} \tag{7.18}$$

Both inequalities would be equalities. Then both contraction conditions at z would be tight, and in every coordinate $n_r(z)$ would lie between $n_r(x^{(i)})$ and $n_r(x^{(j)})$: the value at z would be squeezed onto an ℓ^1-geodesic between the two pinned values. This is the argument of a lemma of Bao and Naskar [**17**]: a contraction map that preserves the distance between two points sends every point between them (on a Hamming path between them, in their terms) to a point between their images. In B_7^* every pair of parties has Bell value 2 or 4, and the obstruction uses this room. At z real values are allowed, as Section 7.5 shows, and what fails is integrality. Superbalance sets every Bell value to 0 and removes exactly this room: for a superbalanced inequality, the value of a map at a point between two occurrence words is squeezed onto an ℓ^1-geodesic between the two pinned values. The squeeze applies only at points between two occurrence words. At a point that lies between no two pinned points, a superbalanced inequality can still fail by the same covering argument, and Section 8.1 gives a six-party example. Chapter 8 then follows the paper in a different construction, which starts from a balanced inequality and completes it to a superbalanced one, and there the failures are certified by computer on the subdomain $\mathcal{U}$ of the restriction lemma, of 128, 134 or 135 points.

7.7. A second example and the scope

Remark 7.19 (A second balanced example). The balanced inequality

$$\begin{aligned} B_7: \quad & 3\,(AO + BO + CO + DO + EO + FO + GO) + DEFG \\ & \geq \ ABC + ADE + AFG + BDEF + BDEG + CDFG + CEFG + 15\,O \end{aligned} \tag{7.20}$$

is a nonnegative $\frac{1}{2}\mathbb{Z}$-combination of 46 instances of subadditivity and MMI, and $k_{\min}(B_7) = 2$. Its obstruction is not visible at a single point: the contraction conditions against the pinned points force integer values at two points of $\{0,1\}^8$ at distance 1 whose values are at ℓ^1 distance 7.

Status: certified (identity with 46 instances; integer map for $2B_7$ on the full cube; DRAT, 2 encoders).

Both B_7^* and B_7 are positive combinations of instances of subadditivity and MMI. Every instance of subadditivity or MMI has at most seven terms, while B_7^* has 15 and B_7 has 16, so no nonzero instance is a multiple of either inequality, and by the positive-combination lemma (Lemma 2.13) neither is a facet of HEC_7, and they say nothing about whether facets have unscaled maps. In every balanced valid inequality without an unscaled map whose least scaling the paper determines, namely B_7^*, B_7, the balanced core Q_8^A of Chapter 8 and the superbalanced E_0, E_1, E_3 and E_4, the least scaling is 2. The six-party inequality U_6 of Section 8.1 is superbalanced, hence balanced, and has no unscaled map, so a balanced example exists with $N = 6$. The paper leaves open whether one exists with $N \leq 5$ (Chapter 12).

CHAPTER 8

Superbalanced inequalities without unscaled proofs

Where we are. Every facet of HEC_N other than subadditivity and its relabelings is superbalanced [**2**], so the question of Chapter 7 for facets is a question about superbalanced inequalities, and Section 7.6 showed that superbalance removes the room that the obstruction of B_7^* used at its point z. This chapter follows the paper's answer. We first give a six-party superbalanced inequality U_6 that fails at a single point, by the covering argument of Chapter 7 at a point that lies between no two pinned points, and prove the failure by hand. We then start from a balanced inequality Q_8^A on eight parties whose terms all contain the party A, complete it to a superbalanced inequality Q_8 by terms built from a Fano plane, split off a valid inequality, delete a party to reach seven parties, and split off instances of MMI in five exact steps. Every superbalanced inequality along the way is valid and has no unscaled map, on the full cube or on its realizable domain. These failures rest on computer certificates alone, DRAT proofs on the subdomain $\mathcal{U}$ of the restriction lemma, while those of B_7^* and of the six-party example also have proofs by hand.

8.1. A six-party example

Section 7.6 showed that superbalance squeezes the value of a map at a point between two occurrence words onto an ℓ^1-geodesic between the two pinned values. It says nothing about a point that lies between no two pinned points, and at such a point a superbalanced inequality can still fail by the covering argument of Chapter 7. The paper's smallest example has six parties, and its failure, like that of B_7^*, can be checked by hand.

The inequality. Let $N = 6$ with parties $A, \dots, F$, and let $\{A, B, C\}$ and $\{D, E, F\}$ be two triangles with the six edges $\mathcal{E} = \{AB, AC, BC, DE, DF, EF\}$. (In this section $\mathcal{E}$ denotes these edges, not those of Chapter 7.) The inequality is

$$\begin{aligned} U_6: \quad & AB + AC + AD + AE + AF + BC + BD + BE + BF + CD + CE + CF \\ & + DE + DF + EF + ABC + DEF + ABCDE + ABCDF + ABCEF \\ & + ABDEF + ACDEF + BCDEF \\ & \geq\ 4\,(A + B + C + D + E + F) + ABCD + ABCE + ABCF \\ & + ADEF + BDEF + CDEF + 3\,ABCDEF\ . \end{aligned} \tag{8.1}$$

Its left-hand side consists of all 15 pairs of parties, the two triangles ABC and DEF, and the six sets of five parties, each with coefficient 1, so $L = |\alpha| = 23$. Its right-hand side consists of the six single parties with coefficient 4, six sets of four parties with coefficient 1 and the set of all six parties with coefficient 3, so $R = 13$ and $|\beta| = 33$.

The four-party terms are clearer after purification. The sets $ABCD$ and EFO are complementary in $\{A, \dots, F, O\}$, so $S(ABCD) = S(EFO)$, and in the same way $ABCE$, $ABCF$, $ADEF$, $BDEF$, $CDEF$ stand for $S(DFO)$, $S(DEO)$, $S(BCO)$, $S(ACO)$, $S(ABO)$. So the six four-party terms are the terms $S(\epsilon O)$, one for each edge $\epsilon \in \mathcal{E}$, and $S(ABCDEF) = S(O)$. The right-hand side of B_7^* consists of the terms $S(A\epsilon)$ for the edges of its two triangles and of $14\,S(O)$. In U_6 the purifier takes the place of the hub A of B_7^*, and the triangles use the six remaining parties, so one

party fewer is needed. The relabelings that permute the vertices of each triangle and those that exchange the two triangles map U_6 to itself. There are $3! \cdot 3! \cdot 2 = 72$ of them, and the paper gives 72 as the order of the stabilizer of U_6, so there are no others.

Superbalance. By Lemma 3.5, the Bell value on a pair i, j of points of $\{A, \dots, F, O\}$ is the total coefficient of the LHS terms that separate i from j minus that of the RHS terms that separate them, where a term $S(I)$, $I \subseteq [N]$, separates i from j if it contains exactly one of them, and O lies in no I. The 72 symmetries map every one of the 21 pairs to one of the three pairs A, O (six pairs in the orbit), A, B (the six pairs inside a triangle) and A, D (the nine pairs across). For these three:

- A, O: on the left, the five pairs at A, the triangle ABC and the five five-party sets that contain A, total 11; on the right, $4A$, the four sets $ABCD$, $ABCE$, $ABCF$, $ADEF$ and $3ABCDEF$, total 11.
- A, B: on the left, the eight pairs $AC, AD, AE, AF, BC, BD, BE, BF$ and the two five-party sets $ACDEF$ and $BCDEF$, total 10; on the right, $4A$, $4B$, $ADEF$ and $BDEF$, total 10.
- A, D: on the left, the eight pairs $AB, AC, AE, AF, BD, CD, DE, DF$, the two triangles and the two five-party sets $ABCEF$ and $BCDEF$, total 12; on the right, $4A$, $4D$ and the four sets $ABCE$, $ABCF$, $BDEF$, $CDEF$, total 12.

All Bell values vanish, so U_6 is superbalanced, and in particular balanced.

Validity. For a vertex i let ϵ_i be the edge of its triangle that does not contain i, and for an edge ij let $\overline{ij}$ be the set of the other four parties. Consider the sums of instances of MMI

$$\begin{aligned} \psi_1 &= \sum_{ij \in \mathcal{E}} \mathrm{MMI}(i{:}j{:}\overline{ij}) \ , \qquad \psi_2 = \mathrm{MMI}(A{:}B{:}C) + \mathrm{MMI}(D{:}E{:}F) \ , \\ \psi_3 &= \sum_{ij \in \mathcal{E}} \sum_x \mathrm{MMI}(i{:}j{:}x) \ , \qquad \psi_4 = \sum_i \sum_x \mathrm{MMI}(i{:}x{:}\epsilon_i) \ , \end{aligned} \tag{8.2}$$

where x runs over the three vertices of the triangle that contains neither i nor j (in ψ_3) or not i (in ψ_4), and i runs over the six vertices in ψ_4. They contain 6, 2, 18 and 18 instances. Each sum is invariant under the 72 symmetries, and

$$6U_6 = 3\psi_1 + 3\psi_2 + \psi_3 + \psi_4 \tag{8.3}$$

as an identity of integer vectors. By symmetry it suffices to compare the coefficients of one term from each orbit of terms under the symmetries. Two of them, by hand, recalling that $\mathrm{MMI}(X{:}Y{:}Z)$ has $+1$ on XY, XZ, YZ and -1 on X, Y, Z, XYZ. The term AB occurs with $+1$ in $\mathrm{MMI}(A{:}B{:}CDEF)$ of ψ_1, in $\mathrm{MMI}(A{:}B{:}C)$ of ψ_2 and in the three $\mathrm{MMI}(A{:}B{:}x)$ of ψ_3, and with -1 as the set $Z = \epsilon_C$ in the three $\mathrm{MMI}(C{:}x{:}AB)$ of ψ_4, so its coefficient on the right of (8.3) is $3 + 3 + 3 - 3 = 6$, which is 6 times its coefficient 1 in U_6. The term A occurs only with -1: in $\mathrm{MMI}(A{:}B{:}CDEF)$ and $\mathrm{MMI}(A{:}C{:}BDEF)$ of ψ_1 (weight 3 each), in $\mathrm{MMI}(A{:}B{:}C)$ (weight 3), in the six instances $\mathrm{MMI}(A{:}j{:}x)$ and the three $\mathrm{MMI}(i{:}j{:}A)$ with $ij \in \{DE, DF, EF\}$ of ψ_3, and in the three $\mathrm{MMI}(A{:}x{:}BC)$ and the three $\mathrm{MMI}(i{:}A{:}\epsilon_i)$ with $i \in \{D, E, F\}$ of ψ_4. Its coefficient is $-(6 + 3 + 6 + 3 + 3 + 3) = -24$, which is 6 times -4. The full comparison of all 63 coefficients was done by two programs in exact integer arithmetic. Instances of MMI are valid, so U_6 is valid.

Proposition 8.4. *The inequality U_6 is valid and superbalanced, and*

(i) *$2U_6$ has an integer map on its realizable domain $\mathcal{A}(U_6)$, which has 580 points;*
(ii) *U_6 has no integer map on the full cube $\{0,1\}^{23}$, and none on $\mathcal{A}(U_6)$.*

Hence $k^{\mathcal{A}}_{\min}(U_6) = 2$.

Status: validity certified (identity (8.3), 2 programs); (i) certified (integer map on $\mathcal{A}$, all 167,910 pairs, 2 checkers); (ii) certified (DRAT proofs on $\mathcal{A}(U_6)$, 2 encoders) and argued (the hand proof

below); $k^{\mathcal{A}}_{\min}(U_6) = 2$ derived. Whether $2U_6$ has an integer map on the full cube was not decided, so on the full cube only $k_{\min}(U_6) \geq 2$ is known.

The point. Let $z \in \{0,1\}^{23}$ be the word that contains the six LHS terms with five parties and no other LHS term. It lies in $\mathcal{A}(U_6)$: any two sets of five of the six parties share four parties, so (R1) holds, and no LHS term other than a five-party set contains a five-party set, so (R2) asks nothing more of z. The occurrence word $x^{(A)}$ contains the five pairs $AB, \dots, AF$, the triangle ABC and the five five-party sets other than $BCDEF$. So z and $x^{(A)}$ differ in the five pairs, in ABC and in $BCDEF$, and the same count holds for every party:

$$d_\alpha\big(z, x^{(i)}\big) = 7 \quad (i \in \{A, \dots, F\}) \ , \qquad d_\alpha(z, 0) = 6 \ . \tag{8.5}$$

The pins. Order the RHS terms as $S(A), \dots, S(F)$, then the six terms $S(\epsilon O)$, then $S(ABCDEF)$. A party i lies in the set of parties that $S(\epsilon O)$ stands for exactly when $i \notin \epsilon$. By (4.13) the pinned value at $x^{(i)}$ is therefore 4 on $S(i)$, 0 on the other five singletons, 1 on the four terms $S(\epsilon O)$ with $i \notin \epsilon$, 0 on the two with $i \in \epsilon$, and 3 on $S(ABCDEF)$, in total $4 + 4 + 3 = 11$. For example, $n(x^{(A)}) = (4, 0, 0, 0, 0, 0;\ 0, 0, 1, 1, 1, 1;\ 3)$ with the edges in the order AB, AC, BC, DE, DF, EF. The pinned value at 0 is 0.

The conditions. Write a candidate value at z as

$$n(z) = (a, b, c) \ , \qquad a_i \in [0, 4] \ , \quad b_\epsilon \in [0, 1] \ , \quad c \in [0, 3] \ , \tag{8.6}$$

with a_i on $S(i)$, b_ϵ on $S(\epsilon O)$ and c on $S(ABCDEF)$, and let $|n| = \sum_i a_i + \sum_\epsilon b_\epsilon + c$. Every pinned coordinate is either 0 or the top β_r of its range, and every coordinate of $n(z)$ lies in the range. So the term of a coordinate in the contraction condition is $n_r(z)$ where the pin is 0 and $\beta_r - n_r(z)$ where the pin is β_r. Adding up,

$$\sum_r \big|n_r(z) - n_r\big(x^{(i)}\big)\big| = |n| + 11 - 2s_i \ , \qquad s_i = a_i + \sum_{\epsilon \not\ni i} b_\epsilon + c \ , \tag{8.7}$$

where s_i is the part of $n(z)$ on the coordinates pinned to their top at $x^{(i)}$. With (8.5), the contraction condition (4.14) between z and the seven pinned points reads

$$(O)\colon\ |n| \leq 6 \ , \qquad (i)\colon\ |n| + 11 - 2s_i \leq 7 \ , \ \text{ that is } \ 2s_i \geq |n| + 4 \quad (i \in \{A, \dots, F\}) \ . \tag{8.8}$$

Solving them. Add the six conditions (i). In $\sum_i s_i$ each a_i occurs once, each b_ϵ four times (once for each vertex not on ϵ) and c six times, so

$$2\Big(\sum_i a_i + 4\sum_\epsilon b_\epsilon + 6c\Big) \geq 6|n| + 24 \ , \qquad \text{that is} \qquad \sum_\epsilon b_\epsilon + 3c \geq 2\sum_i a_i + 12 \ .$$

Condition (O) gives $\sum_\epsilon b_\epsilon \leq 6 - \sum_i a_i - c$. Inserting this on the left,

$$6 - \sum_i a_i + 2c \geq 2\sum_i a_i + 12 \ , \qquad \text{that is} \qquad 2c \geq 3\sum_i a_i + 6 \ .$$

Since $c \leq 3$, the right-hand side is at most 6, so $\sum_i a_i = 0$ and $c = 3$. The two displays now give $\sum_\epsilon b_\epsilon \geq 3$ and $\sum_\epsilon b_\epsilon \leq 3$, so $\sum_\epsilon b_\epsilon = 3$ and $|n| = 6$, and condition (i) becomes $\sum_{\epsilon \not\ni i} b_\epsilon \geq 2$, or, subtracting from $\sum_\epsilon b_\epsilon = 3$,

$$\sum_{\epsilon \ni i} b_\epsilon \leq 1 \qquad \text{for every vertex } i \ . \tag{8.9}$$

Each edge has two endpoints, so the six sums in (8.9) add up to $2\sum_\epsilon b_\epsilon = 6$, and every one of them equals 1. In words: b puts weight on the edges so that every vertex receives exactly 1, a fractional perfect matching of the two triangles. In a triangle with vertices i, j, k this says $b_{ij} + b_{ik} = b_{ij} + b_{jk} = b_{ik} + b_{jk} = 1$, and subtracting the equations pairwise gives $b_{ij} = b_{ik} = b_{jk} = \frac{1}{2}$. So the conditions (8.8) admit exactly one real value,

$$a = 0 \ , \qquad b_\epsilon = \tfrac{1}{2} \ \ (\epsilon \in \mathcal{E}) \ , \qquad c = 3 \ , \tag{8.10}$$

which satisfies all seven conditions with equality ($|n| = 6$ and $s_i = 2 + 3 = 5$), and it is not an integer. Hence

$$\boxed{\begin{array}{l}\text{at } z\text{, an integer value would be a perfect matching}\\ \text{of two disjoint triangles, and none exists}\end{array}}\,, \tag{8.11}$$

because a triangle has three vertices and each edge covers two of them. No integer map exists on a domain that contains z, the occurrence words and 0, in particular on $\{0,1\}^{23}$ and on $\mathcal{A}(U_6)$. This proves (ii).

For B_7^* the integer case asked for three edges that touch all six vertices, which is the same as a perfect matching, since three edges have six endpoints. So the two failures are the same combinatorial fact. In both cases the conditions at the point admit exactly one real value, $\frac{1}{2}$ on every edge: for B_7^*, adding its six conditions (i) of (7.10) gives $|\theta| \le 3 - 3\tau$, which with (A) forces $\tau = 0$ and $|\theta| = 3$, so that every condition (i) holds with equality and $\theta(\mathcal{E}_i) = 1$ at every vertex.

Doubling. For $2U_6$ the ranges, the pins and the distances double, and twice the value (8.10), namely $a = 0$, $b_\epsilon = 1$ on every edge and $c = 6$, is an integer value that satisfies the doubled conditions with equality. So $2U_6$ is not obstructed at z. That it has an integer map on all 580 points of $\mathcal{A}(U_6)$ is the certificate of Proposition 8.4(i), checked on all pairs of points by two checkers.

Why superbalance does not prevent the failure. The squeeze (7.18) needs a point between two pinned points. The point z of B_7^* lies between the occurrence words of every pair of parties. The point z of U_6 lies between no two of the seven pinned points. The words $x^{(i)}$ and $x^{(j)}$ of two parties both contain the pair term ij, which z does not contain, and $x^{(i)}$ and 0 both miss the five-party set without i, which z contains. So z disagrees with both words of every pair in at least one position, and the distances from z to the pinned points are not tied to each other by the Bell values. The six-party example shows that a superbalanced inequality can fail at a single point, so for the superbalanced facets, which by Theorem 3.11 are all facets other than subadditivity and its relabelings, the argument of Section 7.6 alone leaves a failure possible. U_6 itself is not a facet: by (8.3) and the positive-combination lemma (Lemma 2.13) it is a positive combination of instances of MMI, none of which is a multiple of U_6. The inequality of Example 4.19, given by Grimaldi, Headrick, Hubeny and Shteyner as a valid inequality without a map in the presentation with merged right-hand side, is of the same kind [**5**].

Physics aside 8.12. Read through the Ryu–Takayanagi formula, z labels bulk points that lie in the homology region of every set of five parties, and in none of the homology regions of the pairs and the two triangles. On a pure state, when the RT surfaces are unique, the homology region of a five-party set is the complement of that of the remaining party together with the purifier, so these are the bulk points outside every region iO: deep in the bulk, away from each single boundary region. The map has to decide into which candidate regions for the right-hand side to put them. The conditions say: into no single-party region, into the region of all six parties, and "half" into each of the six regions ϵO, which a single copy of each region cannot realize and two copies can. The analogy stops at the graph models: the proofs use only those.

8.2. The balanced core

Let $N = 8$ with parties $A, \dots, H$, and let

$$\begin{aligned} Q_8^A: \quad & ADE + AFG + ABCH + ABDFH + ABEGH + ACDGH + ACEFH \\ & \ge\ 2A + ABDGH + ABEFH + ACDFH + ACEGH + ABCDEFGH\ . \end{aligned} \tag{8.13}$$

Every term contains A, and every coefficient is 1 except that of $2A$. So $L = 7$, $R = 6$ and $|\alpha| = |\beta| = 7$. Party A lies in all seven LHS terms and has total coefficient $2 + 5 = 7$ on the right. Party B lies in the three LHS terms $ABCH$, $ABDFH$, $ABEGH$ and in the three RHS terms

$ABDGH$, $ABEFH$, $ABCDEFGH$, and the two sides agree in the same way for $C, \dots, H$ (three terms on each side for $C, \dots, G$ and five for H), so Q_8^A is balanced. Its Bell values on pairs of parties, computed exactly from (3.6), are 2 on sixteen of the 28 pairs and 0 on the other twelve, which are the seven pairs containing A and the pairs BC, BH, CH, DE, FG. So Q_8^A is not superbalanced.

Since every LHS term contains A, the occurrence word $x^{(A)}$ is the all-ones word, the interval $[0, x^{(A)}]$ is the whole cube $\{0,1\}^7$, and the subdomain $\mathcal{U}(Q_8^A)$ of (5.18) is the full cube with 128 points.

PROPOSITION 8.14. *Q_8^A is valid and has no integer map on $\{0,1\}^7$, and $2Q_8^A$ has an integer map on $\{0,1\}^7$. Hence $k_{\min}(Q_8^A) = 2$.*

Status: no integer map: certified (DRAT, 3 encoders); map for $2Q_8^A$: certified (integer map, all pairs checked); validity and $k_{\min}(Q_8^A) = 2$: derived.

Validity follows from the map for $2Q_8^A$ by the contraction theorem (Theorem 4.1), applied as in Section 6.5. The proposition is the balanced starting point of the construction, and it is one of the balanced failures with least scaling 2 mentioned in Section 7.7.

8.3. The Fano plane and the superbalanced completion

A *Fano plane* is a set of seven points together with seven three-element subsets, the *lines*, such that every pair of points lies on exactly one line. Seven lines contain $7 \cdot 3 = 21$ pairs of points, which is the number $\binom{7}{2}$ of all pairs, so the condition says exactly that no pair occurs on two lines. On the points $B, \dots, H$ the seven sets

$$BCH\ ,\quad BDF\ ,\quad BEG\ ,\quad CDG\ ,\quad CEF\ ,\quad DEH\ ,\quad FGH \tag{8.15}$$

are the lines of a Fano plane: their pairs are BC, BH, CH; BD, BF, DF; BE, BG, EG; CD, CG, DG; CE, CF, EF; DE, DH, EH; FG, FH, GH, and these 21 pairs are distinct. Every point lies on three lines, since it has six partners and each line through it supplies two. The complements of the lines in $\{B, \dots, H\}$ are the four-element sets

$$DEFG\ ,\quad CEGH\ ,\quad CDFH\ ,\quad BEFH\ ,\quad BDGH\ ,\quad BCFG\ ,\quad BCDE\ , \tag{8.16}$$

in the order of (8.15), and every point lies in the four complements of the lines that miss it.

The superbalanced completion of Q_8^A is

(8.17)

$$\begin{aligned} Q_8 :\quad & ADE + AFG + ABCH + ABDFH + ABEGH + ACDGH + ACEFH \\ & + 2\,(BCDE + BCFG + BDGH + BEFH + CDFH + CEGH + DEFG) \\ & \geq\ 2A + 4\,(B + C + D + E + F + G + H) + DE + FG + BCH \\ & + ABDGH + ABEFH + ACDFH + ACEGH + 3\,BCDEFGH + ABCDEFGH\ . \end{aligned}$$

Its terms that contain A are exactly those of Q_8^A. Its other LHS terms are the seven complements (8.16), each with coefficient 2. It has $L = 14$, $R = 17$, $|\alpha| = 7{+}14 = 21$ and $|\beta| = 2{+}28{+}3{+}4{+}3{+}1 = 41$. All 36 Bell values, on the pairs of $A, \dots, H, O$, vanish, so Q_8 is superbalanced, and its stabilizer has order $|\operatorname{Stab}(Q_8)| = 8$. Both statements are exact computations from (8.17). Two of the Bell values show how the computation goes. On the pair B, O the Bell value is the total coefficient of the terms containing B, which is $3 \cdot 1 + 4 \cdot 2 = 11$ on the left (the terms $ABCH$, $ABDFH$, $ABEGH$ and $BCDE$, $BCFG$, $BDGH$, $BEFH$) and $4 + 1 + 1 + 1 + 3 + 1 = 11$ on the right. On the pair B, C the LHS terms that contain exactly one of B and C are $ABDFH$, $ABEGH$, $ACDGH$, $ACEFH$ (weight 1 each) and $BDGH$, $BEFH$, $CDFH$, $CEGH$ (weight 2 each), so $d_\alpha(x^{(B)}, x^{(C)}) = 12$, and the RHS terms that do so are $4B$, $4C$, $ABDGH$, $ABEFH$, $ACDFH$, $ACEGH$, so $d_\beta(y^{(B)}, y^{(C)}) = 12$.

A reading of the construction. Bell values are linear in the inequality, so the Bell values of Q_8 are those of Q_8^A plus those of $Q_8 - Q_8^A$, the Fano part. Since all Bell values of Q_8 vanish, $Q_8 - Q_8^A$ has Bell value -2 on exactly the sixteen pairs on which Q_8^A has $+2$, and 0 on all other pairs. So the Fano part cancels the slack of the core, while the coordinates of the LHS terms containing A, which carry the interval $[0, x^{(A)}]$ of the restriction lemma, stay those of Q_8^A. This explains how the construction is put together. It does not prove that the obstruction of Q_8^A survives the completion, because the occurrence words of the other parties gain the new coordinates, and with them their pins and distances change. That Q_8 has no unscaled map is a separate certificate (Theorem 8.36).

8.4. The step to Q'_8 and Q''_8

Let R_8 be the inequality whose LHS is the sum of the seven complements (8.16) with unit coefficients,

$$\begin{aligned} R_8: \quad & BCDE + BCFG + BDGH + BEFH + CDFH + CEGH + DEFG \\ & \geq\ 2\,(B + C + D + E + F + G + H) + 2\,BCDEFGH\ . \end{aligned} \tag{8.18}$$

The vector $42R_8$ is a positive integer combination of 147 instances of MMI, with coefficients 1, 3 and 18 on three orbits of instances, so R_8 is valid. Let

$$Q'_8 = 3Q_8 - 2R_8\ , \qquad \text{so that} \qquad Q_8 = \tfrac{1}{3}Q'_8 + \tfrac{2}{3}R_8\ . \tag{8.19}$$

Written out, multiplying (8.17) by 3 and subtracting twice (8.18) lowers the coefficients of the Fano complements from 6 to 4, those of the single parties $B, \dots, H$ from 12 to 8 and that of $BCDEFGH$ from 9 to 5:

$$\begin{aligned} Q'_8: \quad & 3\,(ADE + AFG + ABCH + ABDFH + ABEGH + ACDGH + ACEFH) \\ & + 4\,(BCDE + BCFG + BDGH + BEFH + CDFH + CEGH + DEFG) \\ & \geq\ 6A + 8\,(B + C + D + E + F + G + H) + 3\,(DE + FG + BCH) \\ & + 3\,(ABDGH + ABEFH + ACDFH + ACEGH) \\ & + 5\,BCDEFGH + 3\,ABCDEFGH\ , \end{aligned} \tag{8.20}$$

with $L = 14$, $R = 17$, $|\alpha| = 49$ and $|\beta| = 91$. A second step subtracts three times the instance MMI(A:H:DE), which is $AH + ADE + DEH \geq A + H + DE + ADEH$, and defines $Q''_8 = Q'_8 - 3\,\mathrm{MMI}(A{:}H{:}DE)$. Subtracting the instance three times moves ADE off the left and DE off the right, adds $3ADEH$ on the left and $3AH$, $3DEH$ on the right, and lowers $6A$ to $3A$ and $8H$ to $5H$:

$$\begin{aligned} Q''_8: \quad & 3\,(AFG + ABCH + ADEH + ABDFH + ABEGH + ACDGH + ACEFH) \\ & + 4\,(BCDE + BCFG + BDGH + BEFH + CDFH + CEGH + DEFG) \\ & \geq\ 3A + 8\,(B + C + D + E + F + G) + 5H + 3\,(AH + FG + BCH + DEH) \\ & + 3\,(ABDGH + ABEFH + ACDFH + ACEGH) \\ & + 5\,BCDEFGH + 3\,ABCDEFGH\ , \end{aligned} \tag{8.21}$$

with $L = 14$, $R = 18$, $|\alpha| = 49$ and $|\beta| = 88$. Both Q'_8 and Q''_8 are superbalanced, by exact computation, and both are valid (Theorem 8.36).

Proposition 8.22. *Q_8 is not a facet of* HEC_8*, and* $\sup\{\lambda\colon Q_8 - \lambda R_8 \text{ valid}\} = \frac{2}{3}$.

Status: derived (validity certificates; 37 graph models, entropy vectors by 2 max-flow codes).

Proof. By (8.19), Q_8 is a positive combination of Q'_8 and R_8, both valid. R_8 is nonzero and has no term containing A, while Q_8 has, so R_8 is not a positive multiple of Q_8, and by the positive-combination lemma (Lemma 2.13) Q_8 is not a facet. For the step, (8.19) gives

$$Q_8 - \lambda R_8 = \tfrac{1}{3}Q'_8 + \big(\tfrac{2}{3} - \lambda\big)R_8\ . \tag{8.23}$$

For $\lambda \le \frac{2}{3}$ the right side is a nonnegative combination of valid inequalities, so $Q_8 - \lambda R_8$ is valid. For the upper bound, the paper gives graph models with $Q'_8 \cdot S_\Gamma = 0$ and $R_8 \cdot S_\Gamma > 0$. On such a graph model (8.23) evaluates to $(\frac{2}{3} - \lambda) R_8 \cdot S_\Gamma$, which is negative for every $\lambda > \frac{2}{3}$ (Section 6.6). □

For the second step only the lower bound is known: since Q''_8 is valid, $Q'_8 - \lambda\,\mathrm{MMI}(A{:}H{:}DE) = Q''_8 + (3-\lambda)\,\mathrm{MMI}(A{:}H{:}DE)$ is valid for $\lambda \le 3$, and whether a larger multiple can be split off is not determined.

8.5. Deleting a party

The *deletion* of the party H replaces every term $S(I)$ by $S(I \setminus \{H\})$, with $S(\emptyset) = 0$, and collects equal terms. By the isolated-party lemma (Lemma 2.16), the deletion of a valid inequality is valid [**1**], and by Chapter 3 the deletion of a superbalanced inequality is superbalanced. Deletion acts term by term, so it is linear. The deletion of $\mathrm{MMI}(A{:}H{:}DE)$ is

$$S(A) + S(ADE) + S(DE) - S(A) - S(\emptyset) - S(DE) - S(ADE) = 0\ , \tag{8.24}$$

so Q'_8 and Q''_8 have the same deletion. In Q'_8 the deletion turns $ABCH$, $ABDFH$, $ABEGH$, $ACDGH$, $ACEFH$ into ABC, $ABDF$, $ABEG$, $ACDG$, $ACEF$, the complements $BDGH$, $BEFH$, $CDFH$, $CEGH$ into BDG, BEF, CDF, CEG, the RHS terms BCH, $ABDGH$, $ABEFH$, $ACDFH$, $ACEGH$, $BCDEFGH$, $ABCDEFGH$ into BC, $ABDG$, $ABEF$, $ACDF$, $ACEG$, $BCDEFG$, $ABCDEFG$, and the term $8H$ into $8S(\emptyset) = 0$. No two terms become equal, and the result is the seven-party inequality

$$\begin{aligned} E_0:\quad & 3\,(ABC + ADE + AFG) + 4\,(BDG + BEF + CDF + CEG) \\ & + 3\,(ABDF + ABEG + ACDG + ACEF) + 4\,(BCDE + BCFG + DEFG) \\ & \ge\ 6A + 8\,(B + C + D + E + F + G) + 3\,(BC + DE + FG) \\ & + 3\,(ABDG + ABEF + ACDF + ACEG) + 5\,BCDEFG + 3\,ABCDEFG\ , \end{aligned} \tag{8.25}$$

with $L = 14$, $R = 16$, $|\alpha| = 49$ and $|\beta| = 83$. Its realizable domain has $|\mathcal{A}(E_0)| = 2187$ points of the 2^{14} points of $\{0,1\}^{14}$. So E_0 is valid and superbalanced, because Q'_8 is valid and superbalanced, which Theorem 8.36 proves from certificates that do not involve E_0.

8.6. The Fano structure of E_0

The seven parties of E_0 carry a Fano plane with the lines

$$ABC\ ,\quad ADE\ ,\quad AFG\ ,\quad BDG\ ,\quad BEF\ ,\quad CDF\ ,\quad CEG\ , \tag{8.26}$$

whose 21 pairs are again distinct (Figure 1). Three lines pass through A and four miss it. In these terms the LHS of E_0 consists of the seven lines, with weight 3 for the lines through A and 4 for the lines missing A, the complements $ACEF$, $ACDG$, $ABEG$, $ABDF$ of the four lines missing A, with weight 3, and the complements $DEFG$, $BCFG$, $BCDE$ of the three lines through A, with weight 4. On the right, BC, DE, FG are the lines through A with A removed, and $ABDG$, $ABEF$, $ACDF$, $ACEG$ are the lines missing A with A added.

The stabilizer. A *collineation* of the Fano plane is a permutation of its points that maps lines to lines. A collineation that fixes A maps the lines through A among themselves and the lines missing A among themselves, and so it maps each of the classes of terms just listed onto itself, with the same coefficients. Extended by $O \mapsto O$, it therefore lies in $\mathrm{Stab}(E_0)$. These collineations form a group of order 24, which we now count. Label the points by the nonzero vectors of $\mathbb{F}_2^3$, the three-dimensional vector space over the field with two elements, by

$$A = 100\ ,\quad B = 010\ ,\quad C = 110\ ,\quad D = 001\ ,\quad E = 101\ ,\quad F = 111\ ,\quad G = 011\ . \tag{8.27}$$

Then each line (8.26) is a triple $\{a, b, a+b\}$, as one checks line by line ($A+B=C$, $A+D=E$, $A+F=G$, $B+D=G$, $B+E=F$, $C+D=F$, $C+E=G$), and since there are seven

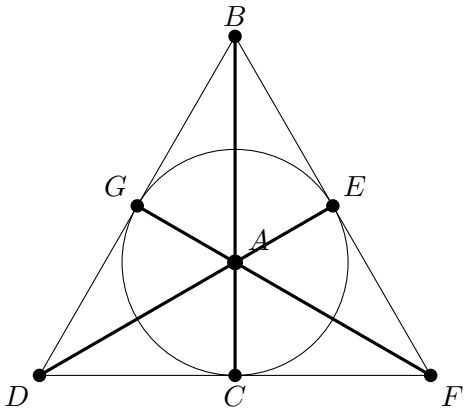


FIGURE 1. The Fano plane on the parties of $E_0, \dots, E_5$. Its seven lines are the three sides of the triangle (BDG, BEF, CDF), the three medians (ABC, ADE, AFG) and the circle (CEG). The thick lines pass through A.

such triples, these are all of them. A collineation φ, extended by $\varphi(0) = 0$, therefore satisfies $\varphi(a+b) = \varphi(a)+\varphi(b)$, because $a+b$ is the third point on the line through a and b, and $\varphi(a)+\varphi(b)$ is the third point on the image line. An additive map of $\mathbb{F}_2^3$ is linear, so the collineations are the invertible 3×3 matrices over $\mathbb{F}_2$, and conversely every such matrix maps triples $\{a, b, a+b\}$ to triples. An invertible matrix is fixed by the images of the three basis vectors: the first any of the 7 nonzero vectors, the second any of the 6 vectors outside the span of the first, the third any of the 4 vectors outside the span of the first two. So there are $7 \cdot 6 \cdot 4 = 168$ collineations. They act transitively on the seven points, since every nonzero vector is the first column of some invertible matrix, so by the orbit-stabilizer theorem the collineations fixing A form a group of order $168/7 = 24$. An exact computation over all 8! relabelings of $A, \dots, G, O$ shows that $\mathrm{Stab}(E_0)$ has order 24 and fixes A and O, so it is exactly the group of collineations of the plane (8.26) that fix A.

The antichain. The LHS terms of E_0 that contain A are the three lines ABC, ADE, AFG and the four sets $ABDF$, $ABEG$, $ACDG$, $ACEF$, which are A together with the complements BDF, BEG, CDG, CEF in $\{B, \dots, G\}$ of the four lines missing A. Two distinct sets of the same size are never nested. A line $A \cup \{p, q\}$ through A, with $\{p, q\}$ one of BC, DE, FG, lies in one of the four-element sets only if $\{p, q\}$ lies in one of the triples BDF, BEG, CDG, CEF, and none of them contains BC, DE or FG. So the seven terms containing A form an antichain, and by the restriction lemma (Lemma 5.19) the subdomain $\mathcal{U}(E_0)$ lies in $\mathcal{A}(E_0)$.

8.7. The descent

We split E_0 along the instances and orbit sums of MMI

$$\begin{aligned} \mu_0 &= \mathrm{MMI}(BC{:}DE{:}FG)\ , \qquad \mu_1 = \mathrm{MMI}(B{:}D{:}F)\ , \\ \mu_2 &= \mathrm{MMI}(A{:}C{:}DF) + \mathrm{MMI}(A{:}E{:}BF) + \mathrm{MMI}(A{:}G{:}BD)\ , \\ \mu_3 &= \mathrm{MMI}(A{:}B{:}D) + \mathrm{MMI}(A{:}B{:}F) + \mathrm{MMI}(A{:}D{:}F)\ , \\ \mu_4 &= \mathrm{MMI}(B{:}E{:}G) + \mathrm{MMI}(C{:}D{:}G) + \mathrm{MMI}(C{:}E{:}F)\ , \end{aligned} \tag{8.28}$$

and define

(8.29)

$$E_1 = E_0 - 2\mu_0\ , \qquad E_2 = E_1 - 2\mu_1\ , \qquad E_3 = E_2 - \mu_2\ , \qquad E_4 = E_3 - \mu_3\ , \qquad E_5 = E_4 - 2\mu_4\ .$$

Here s indexes the steps of the chain. Adding the five definitions, all intermediate inequalities cancel, and

$$E_0 = E_5 + 2\mu_0 + 2\mu_1 + \mu_2 + \mu_3 + 2\mu_4 \ . \tag{8.30}$$

The first step uses $\mu_0 = BCDE+BCFG+DEFG-BC-DE-FG-BCDEFG$. Subtracting it twice lowers the weight of the three complements of the lines through A from 4 to 2, lowers $3\,(BC+DE+FG)$ to $BC+DE+FG$ and $5\,BCDEFG$ to $3\,BCDEFG$:

$$\begin{aligned} E_1: \quad & 3\,(ABC+ADE+AFG) + 4\,(BDG+BEF+CDF+CEG) \\ & + 3\,(ABDF+ABEG+ACDG+ACEF) + 2\,(BCDE+BCFG+DEFG) \\ & \geq\ 6A + 8\,(B+C+D+E+F+G) + BC+DE+FG \\ & + 3\,(ABDG+ABEF+ACDF+ACEG) + 3\,BCDEFG + 3\,ABCDEFG \ . \end{aligned} \tag{8.31}$$

Its LHS consists of the seven lines, with weight 3 for the lines through A and 4 for the others, the complements of the four lines missing A, with weight 3, and the complements of the three lines through A, with weight 2. An exact computation shows that E_1 has the same stabilizer as E_0. The other members of the chain, computed from (8.25), (8.28) and (8.29) and written with the terms of each side ordered by size and then alphabetically, are

(8.32)

$$\begin{aligned} E_2: \quad & 3ABC+3ADE+3AFG+2BDF+4BDG+4BEF+4CDF+4CEG+3ABDF \\ & + 3ABEG+3ACDG+3ACEF+2BCDE+2BCFG+2DEFG \\ & \geq\ 6A+6B+8C+6D+8E+6F+8G+BC+2BD+2BF+DE \\ & + 2DF+FG+3ABDG+3ABEF+3ACDF+3ACEG \\ & + 3BCDEFG+3ABCDEFG \ , \end{aligned}$$

(8.33)

$$\begin{aligned} E_3: \quad & 3ABC+3ADE+3AFG+2BDF+3BDG+3BEF+3CDF+4CEG+3ABDF \\ & + 3ABEG+3ACDG+3ACEF+2BCDE+2BCFG+2DEFG \\ & \geq\ 3A+6B+7C+6D+7E+6F+7G+AC+AE+AG+BC+BD \\ & + BF+DE+DF+FG+ABD+ABF+ADF+2ABDG+2ABEF \\ & + 2ACDF+3ACEG+3BCDEFG+3ABCDEFG \ , \end{aligned}$$

(8.34)

$$\begin{aligned} E_4: \quad & 3ABC+3ADE+3AFG+2BDF+3BDG+3BEF+3CDF+4CEG+3ABDF \\ & + 3ABEG+3ACDG+3ACEF+2BCDE+2BCFG+2DEFG \\ & \geq\ 4B+7C+4D+7E+4F+7G+2AB+AC+2AD+AE+2AF \\ & + AG+BC+2BD+2BF+DE+2DF+FG+2ABDG+2ABEF \\ & + 2ACDF+3ACEG+3BCDEFG+3ABCDEFG \ , \end{aligned}$$

$$\begin{aligned} E_5: \quad & 3ABC+3ADE+3AFG+2BDF+3BDG+3BEF+2BEG+3CDF+2CDG \\ & + 2CEF+4CEG+3ABDF+3ABEG+3ACDG+3ACEF \\ & + 2BCDE+2BCFG+2DEFG \\ & \geq\ 2B+3C+2D+3E+2F+3G+2AB+AC+2AD+AE+2AF+AG \\ & + BC+2BD+2BE+2BF+2BG+2CD+2CE+2CF+2CG+DE \\ & + 2DF+2DG+2EF+2EG+FG+2ABDG+2ABEF+2ACDF \\ & + 3ACEG+3BCDEFG+3ABCDEFG \ . \end{aligned} \tag{8.35}$$

	N	L	R	$\lvert\alpha\rvert$	$\lvert\beta\rvert$	$\lvert\mathcal{A}\rvert$	validity	next step, $\sup\lambda$
Q_8	8	14	17	21	41		map on $\{0,1\}^{14}$	$-\lambda R_8$, $\frac{2}{3}$
Q'_8	8	14	17	49	91		map on $\{0,1\}^{14}$	$-\lambda\,\mathrm{MMI}(A{:}H{:}DE)$, ≥ 3
Q''_8	8	14	18	49	88		map on $\{0,1\}^{14}$	delete H
E_0	7	14	16	49	83	2187	map for $2Q$ on $\{0,1\}^{14}$	$-\lambda\mu_0$, 2
E_1	7	14	16	43	75	2187	map for $2Q$ on $\{0,1\}^{14}$	$-\lambda\mu_1$, 2
E_2	7	15	19	45	75	2916	map on $\mathcal{A}$	$-\lambda\mu_2$, ≥ 1
E_3	7	15	25	42	69	2916	map for $2Q$ on $\{0,1\}^{15}$	$-\lambda\mu_3$, ≥ 1
E_4	7	15	24	42	66	2916	map for $2Q$ on $\{0,1\}^{15}$	$-\lambda\mu_4$, 2
E_5	7	18	33	48	66	6912	map on $\mathcal{A}$	

TABLE 1. The superbalanced inequalities of Theorem 8.36. L and R count the distinct LHS and RHS terms, and $|\mathcal{A}|$ is the number of points of the realizable domain, given for $N = 7$. "Validity" names the certificate: an exact rational map on the full cube, an integer map for $2Q$ on the full cube (Proposition 8.42), or an exact rational map on the realizable domain together with the realizable-domain criterion. The last column gives the next step of the chain and $\sup\{\lambda\colon Q - \lambda X \text{ valid}\}$ for the subtracted vector X. A lower bound $\geq c$ comes from the validity of the next inequality, and an exact value is attained by an explicit graph model on which the next inequality vanishes and X does not.

In all of them the seven LHS terms that contain A are those of E_0, with coefficient 3, as one reads off the displays, so the antichain argument above applies to every E_s. Table 1 collects the data of the chain.

8.8. The theorem

THEOREM 8.36. *The inequalities Q_8, Q'_8, Q''_8 ($N = 8$) and $E_0, \dots, E_5$ ($N = 7$) are superbalanced and valid. None of them has an integer map, either on the full cube or on its realizable domain.*

Status: superbalance by an exact computation of all Bell values; validity of Q_8, Q'_8, Q''_8 certified (rational maps on $\{0,1\}^{14}$; 2, 3 and 2 checkers), of E_0, E_1, E_3, E_4 certified (integer maps for $2E_s$ on the full cube, 2 checkers each), of E_2, E_5 derived (rational maps on $\mathcal{A}$, 2 checkers each); no integer map certified (DRAT; encoders: Q_8 3, Q'_8 2, Q''_8 3, E_0 3, E_1 3, E_2 2, E_3 2, E_4 2, E_5 1).

PROOF. *Superbalance* is an exact integer computation of all Bell values from the coefficient vectors, 36 values for $N = 8$ and 28 for $N = 7$.

Validity. For Q_8, Q'_8 and Q''_8 there are exact rational real maps on the full cube $\{0,1\}^{14}$, of 16,384 points, each checked in exact arithmetic on all edges of the cube, which suffice by the irreducible-pairs lemma (Lemma 5.15). By the realizable-domain criterion (Lemma 5.9), or by the contraction theorem applied to MQ_8, MQ'_8, MQ''_8 with M a common denominator of the map (Section 6.5), these three inequalities are valid. The inequality E_0 is the deletion of H from Q'_8 (Section 8.5) and is valid by the isolated-party lemma. For $E_1, \dots, E_5$ there are exact rational real maps on the realizable domains, of 2187, 2916, 2916, 2916 and 6912 points, checked in exact arithmetic on all pairs of points, and the realizable-domain criterion gives validity. For E_0, E_1, E_3 and E_4 there is also a route that uses neither the realizable-domain criterion nor the deletion: the integer maps for $2E_0$, $2E_1$, $2E_3$ and $2E_4$ on the full cube of Proposition 8.42 show by the contraction theorem that these doubled inequalities are valid, and hence so are E_0, E_1, E_3 and E_4. This is the certificate behind the status line for these four.

Non-existence. In each of the nine inequalities the seven LHS terms that contain A form an antichain: for $E_0, \dots, E_5$ by Section 8.6 and Section 8.7, and for Q_8, Q'_8, Q''_8 by the same kind of check, since their terms containing A are those of Q_8^A, with ADE replaced by $ADEH$ in Q''_8. So the subdomain $\mathcal{U}$ of (5.18), consisting of the occurrence words, the word 0 and the $2^7 = 128$ points of the interval $[0, x^{(A)}]$, lies in the realizable domain. The interval contains 0 and $x^{(A)}$, and no other occurrence word, because every party other than A lies in an LHS term without A (a Fano complement for $N = 8$, a line missing A for $N = 7$). Hence

$$|\mathcal{U}| = 128 + (N-1) = \begin{cases} 135 & (N=8)\ , \\ 134 & (N=7)\ . \end{cases} \tag{8.37}$$

For every one of the nine inequalities, the map problem with $k = 1$ on $\mathcal{U}$ was written as a CNF and shown unsatisfiable by a DRAT proof checked with `drat-trim`. By the chain (6.7) of Chapter 6, which uses the soundness of the encoding (Lemma 6.5) and the restriction lemma, no integer map exists on the full cube or on the realizable domain. □

By Chapter 4, the theorem also excludes unscaled contraction maps on the full cube in the fully split presentation and in the presentations with merged right-hand side.

Two points of logic in this proof are spelled out here, because the telescoped identity (8.30) may suggest shortcuts. First, validity passes only up the chain. The identity $E_s = E_{s+1} + c_s\mu_s$, with $c_s = 2, 2, 1, 1, 2$, shows that E_s is valid if E_{s+1} is, but not the converse: subtracting a valid inequality from a valid one can produce an invalid one, as the bound $\frac{2}{3}$ of Proposition 8.22 illustrates. So every E_{s+1} needs its own validity certificate, and has one. Second, the failure has to be shown for each member separately. Whether E_{s+1} has an unscaled map is a question about a different cube, with different pins and distances, and a map for a sum is not a sum of maps when terms cancel between the summands. So the failure of every member of the chain is certified on its own.

8.9. Maximal steps and the end of the chain

PROPOSITION 8.38 (The descent). *$E_0 = E_5 + 2\mu_0 + 2\mu_1 + \mu_2 + \mu_3 + 2\mu_4$, and the steps along μ_0, μ_1 and μ_4 are maximal:* $\sup\{\lambda\colon E_s - \lambda\mu_s \text{ valid}\} = 2$ *for* $s = 0, 1, 4$.

Status: identity exact (it follows from the definitions (8.29)); maximal steps derived (validity certificates; graph models, entropy vectors by at least 3 max-flow codes).

PROOF. The identity is (8.30). For the lower bound, $E_s - \lambda\mu_s = E_{s+1} + (2-\lambda)\mu_s$ for $s = 0, 1, 4$, which is valid for $\lambda \le 2$ since E_{s+1} is valid by Theorem 8.36 and μ_s is a sum of instances of MMI. For the upper bound, the paper gives for each of $s = 0, 1, 4$ a graph model with $E_{s+1} \cdot S_\Gamma = 0$ and $\mu_s \cdot S_\Gamma > 0$, with 138, 168 and 10 vertices. On it, $(E_s - \lambda\mu_s) \cdot S_\Gamma = (2-\lambda)\mu_s \cdot S_\Gamma < 0$ for every $\lambda > 2$. □

At a maximal step the next inequality vanishes on a graph model on which the subtracted MMI sum does not, so no larger multiple of the same sum can be split off while keeping validity. For $s = 2, 3$ only the lower bound $\sup\lambda \ge 1$ from the validity of the next inequality is known.

EXAMPLE 8.39 (A graph for $s = 4$). The following graph, with ten vertices, does this for $s = 4$, and its required properties can be checked by hand. Its vertices are the boundary vertices $A, \dots, G, O$, of which A, B and O have no edges, and two bulk vertices w_0, w_1, with the edges

(8.40)

C–w_1 (2) , D–w_0 , D–w_1 , E–w_0 , E–w_1 , F–w_0 , F–w_1 (1 each) , G–w_0 (4) , w_0–w_1 (1) ,

weights in parentheses. We verify by hand that $\mu_4 \cdot S_\Gamma = 4$. An I-cut contains the boundary vertices of I and is fixed by which of the two bulk vertices it contains, so $S_\Gamma(I)$ is the least of four cut weights. Table 2 lists them for the sets that μ_4 needs, excluding B.

I	$W \cap \{w_0, w_1\} = \emptyset$	$\{w_0\}$	$\{w_1\}$	$\{w_0, w_1\}$	$S_\Gamma(I)$
C	2	10	4	10	2
D, E, F	2	8	6	10	2
G	4	4	10	8	4
CD, CE, CF	4	10	4	8	4
CG	6	6	8	6	6
DG	6	4	10	6	4
EF	4	8	6	8	4
CDG	8	6	8	4	4
CEF	6	10	4	6	4

TABLE 2. Cut weights of the graph (8.40) for the sets I that enter μ_4, by the bulk vertices the cut contains, and their minimum $S_\Gamma(I)$.

Since B has no edges, adding it to a cut changes nothing, so $S_\Gamma(BX) = S_\Gamma(X)$ for every X and $S_\Gamma(B) = 0$. In $\mathrm{MMI}(B{:}E{:}G) = S(BE) + S(BG) + S(EG) - S(B) - S(E) - S(G) - S(BEG)$ the terms then cancel in pairs, leaving $-S_\Gamma(B) = 0$. With Table 2,

$$\begin{aligned} \mathrm{MMI}(C{:}D{:}G) \cdot S_\Gamma &= 4 + 6 + 4 - 2 - 2 - 4 - 4 = 2 \ , \\ \mathrm{MMI}(C{:}E{:}F) \cdot S_\Gamma &= 4 + 4 + 4 - 2 - 2 - 2 - 4 = 2 \ , \\ \mu_4 \cdot S_\Gamma &= 0 + 2 + 2 = 4 \ . \end{aligned} \tag{8.41}$$

For $E_5 \cdot S_\Gamma$ one needs all 51 terms of (8.35). Since A, B and O have no edges, $S_\Gamma(I)$ depends only on $I \cap \{C, D, E, F, G\}$ and does not change when this intersection is replaced by its complement in $\{C, D, E, F, G\}$. The values are 2 on C, D, E, F and 4 on G, 6 on CG and 4 on the other nine pairs, the complements give the sets of three and four parties, and $S_\Gamma(CDEFG) = 0$. Evaluating (8.35) with these values gives $E_5 \cdot S_\Gamma = 0$, and then $E_4 \cdot S_\Gamma = E_5 \cdot S_\Gamma + 2\mu_4 \cdot S_\Gamma = 8$, which a direct evaluation of (8.34) confirms. So this graph shows that $E_4 - \lambda\mu_4$ is invalid for every $\lambda > 2$.

None of the chain is a facet. Each of Q_8, Q_8', $E_0, \ldots, E_4$ is a positive combination of the next inequality, which is valid, and a nonzero sum of instances of MMI or the inequality R_8, which is not a multiple of it: R_8 has no term containing A, $\mathrm{MMI}(A{:}H{:}DE)$ has 7 terms against $L + R = 31$ for Q_8', and μ_s has at most 21 terms against at least 30 for E_s. By the positive-combination lemma, none of them is a facet. For the last one, $6E_5 = 2X_7 + Y_7 + 3\mu_5$, where X_7 and Y_7 are valid seven-party inequalities and μ_5 is a sum of three instances of MMI, all introduced in Chapter 9. Since μ_5 has at most 21 terms and E_5 has 51, E_5 is not a facet either. The paper does not decide whether Q_8'' is a facet of HEC_8.

After each split the remaining inequality still has no unscaled map, while the subtracted instances of MMI have unscaled maps, given by the map of Bao et al. for $\mathrm{MMI}(A{:}B{:}C)$ [**1**] (Example 4.9). This is an observation read off the certificates step by step. Chapter 9 follows the chain one step further, past E_5, to a facet of HEC_7.

8.10. Scaled maps for the chain

Since none of these inequalities has an unscaled map, one can ask how much scaling they need. The following scaled cases are decided.

PROPOSITION 8.42. (i) *$3Q_8$ has no integer map, either on the full cube or on its realizable domain.*

(ii) *$2E_0$, $2E_1$, $2E_3$ and $2E_4$ have integer maps on the full cube. Hence $k_{\min}(E_s) = k_{\min}^{\mathcal{A}}(E_s) = 2$ for $s = 0, 1, 3, 4$.*

Status: (i) certified (DRAT, 1 encoder); (ii) certified (integer maps, 2 checkers each); $k_{\min}(E_s) = k^{\mathcal{A}}_{\min}(E_s) = 2$ derived.

Part (i) is certified like Theorem 8.36, by a DRAT proof for the problem for $3Q_8$ on $\mathcal{U}(Q_8)$, which by the antichain property and the restriction lemma excludes integer maps for $3Q_8$ on the full cube and on $\mathcal{A}(Q_8)$. For part (ii), the four maps are tables of integers on the full cubes, $\{0,1\}^{14}$ with 16,384 points for E_0 and E_1 and $\{0,1\}^{15}$ with 32,768 points for E_3 and E_4. Each was checked in exact integer arithmetic by two independent checkers on all edges of its cube, which suffice by the irreducible-pairs lemma, and by at least one of them also on all $\binom{16384}{2} = 134{,}209{,}536$ or $\binom{32768}{2} = 536{,}854{,}528$ pairs of points. The maps restrict to the realizable domains, so $k^{\mathcal{A}}_{\min}(E_s) \le k_{\min}(E_s) \le 2$, and Theorem 8.36 excludes $k = 1$ on both domains. By the contraction theorem the maps also prove that $2E_s$, and hence E_s, is valid for $s = 0, 1, 3, 4$.

The map for $2E_0$ is built from the one for $2E_1$, and this step can be followed by hand. By (8.29), $E_0 = E_1 + 2\mu_0$ with $\mu_0 = BCDE + BCFG + DEFG - BC - DE - FG - BCDEFG$, and comparing (8.25) with (8.31) shows that every term of μ_0 is a term of E_1 on the same side. So E_0 and E_1 have the same 14 LHS terms, the same 16 RHS terms, the same cube $\{0,1\}^{14}$ and the same occurrence words, and their coefficients satisfy $\alpha(E_0) = \alpha(E_1) + 2\alpha(\mu_0)$ and $\beta(E_0) = \beta(E_1) + 2\beta(\mu_0)$, where $\alpha(\mu_0)$ and $\beta(\mu_0)$ are 1 on the terms of μ_0 and 0 on the other terms. Let n be an integer map for $2E_1$, and let m be an unscaled map for μ_0, which exists because the map of Bao et al. for MMI(A:B:C) [**1**] proves every instance (Example 4.9). Read m as a function on $\{0,1\}^{14}$ that depends only on the three coordinates of $BCDE$, $BCFG$ and $DEFG$, with $m_r = 0$ for the RHS terms r outside μ_0. The pins of n and of $4m$ add up to the pins of $2E_0$, their ranges add up to the ranges of $2E_0$, and for all $u, v \in \{0,1\}^{14}$ the triangle inequality and the contraction conditions of n and m give

$$\begin{aligned} \sum_r \big|(n+4m)_r(u) - (n+4m)_r(v)\big| &\le \sum_r \big|n_r(u) - n_r(v)\big| + 4\sum_r \big|m_r(u) - m_r(v)\big| \\ &\le 2d_{\alpha(E_1)}(u,v) + 4d_{\alpha(\mu_0)}(u,v) = 2d_{\alpha(E_0)}(u,v)\ . \end{aligned} \tag{8.43}$$

So $n + 4m$ is an integer map for $2E_0$. Section 8.8 noted that a map for a sum is not a sum of maps when terms cancel between the summands. Here no term cancels, and the sum of the maps is a map.

On the realizable domain, $2E_4$ also has an integer map equivariant under $\mathrm{Stab}(E_4)$, which has order 6 and fixes the purifier, checked in exact arithmetic on all $4{,}250{,}070 = \binom{2916}{2}$ pairs of points of $\mathcal{A}(E_4)$. The least scaling of E_2, E_5, Q_8, Q_8' and Q_8'' is not known; see Chapter 12.

Part III

A facet, the least scaling, and what is open

CHAPTER 9

A superbalanced facet F_7 of HEC_7

Where we are. The inequalities of Chapter 8 are valid and superbalanced and have no unscaled map, but none of Q_8, Q_8', $E_0, \dots, E_5$ is a facet of the cone, and the paper does not decide whether Q_8'' is one. A reader may therefore suspect that the failure is an artifact of inequalities that are sums of simpler ones, and that it disappears at the facets. This chapter follows the chain one step further. The last inequality E_5 splits as $6E_5 = 2X_7 + Y_7 + 3\mu_5$, and splitting Y_7 once more produces a facet F_7 of HEC_7. We write F_7 out and list its properties (Section 9.1), prove that it is a facet (Section 9.2), compare it with the seven-party inequalities in the literature (Section 9.3), place it in an exact decomposition of $24E_0$ (Section 9.4), and show that it has an unscaled map on its realizable domain, while no such map respects its symmetry (Section 9.5). Section 9.6 records what is known about the other pieces of the decomposition.

9.1. The inequality F_7

We use seven parties $A, \dots, G$ and the purifier O, and the notation of Chapter 1: a term ABC stands for $S(ABC)$, and every term is a subset of the parties. The inequality is

$$\begin{aligned} F_7 : \quad & 2ABC + 2ADE + 2AFG + BDF + 4BDG + 4BEF + BEG + 4CDF \\ & + CDG + CEF + 2ABDF + 2ABEG + 2ACDG + 2ACEF + BCDE \\ & + BCFG + DEFG + 2BCDEG + 2BCEFG + 2CDEFG \\ \geq \ & 3B + 2C + 3D + 2E + 3F + 2G + 2AB + 2AD + 2AF + 3BE + 3BG \\ & + 3CD + 3CF + 3DG + 3EF + ABDG + ABEF + ACDF + 3ACEG \\ & + BCDEF + BCDFG + BDEFG + 4BCDEFG + 2ABCDEFG \ . \end{aligned} \tag{9.1}$$

Counting terms and adding coefficients on each side gives

$$L = 20 \ , \qquad R = 24 \ , \qquad |\alpha| = 39 \ , \qquad |\beta| = 54 \ . \tag{9.2}$$

The coefficients take the values $1, 2, 3, 4$, and since some of them equal 1 they have no common factor. So F_7 is not a multiple of a smaller integer inequality.

Superbalance. By the Bell-value formula (3.6), the Bell value of F_7 on a pair i, j is the total coefficient of the LHS terms that contain exactly one of i, j, minus the same total on the RHS. For a pair i, O this is the total coefficient of the terms containing i, LHS minus RHS, since no term contains O. Two examples show how the count goes. For the pair A, O, the LHS terms containing A are the seven terms with coefficient 2 beginning with A, total 14, and the RHS terms containing A are $2AB$, $2AD$, $2AF$, $ABDG$, $ABEF$, $ACDF$, $3ACEG$ and $2ABCDEFG$, total $2+2+2+1+1+1+3+2 = 14$. For the pair B, C, the LHS terms containing B but not C have total $1+4+4+1+2+2 = 14$ (from BDF, $4BDG$, $4BEF$, BEG, $2ABDF$, $2ABEG$), those containing C but not B have total $4+1+1+2+2+2 = 12$ (from $4CDF$, CDG, CEF, $2ACDG$, $2ACEF$, $2CDEFG$), and on the RHS the corresponding totals are $3+2+3+3+1+1+1 = 14$ (from $3B$, $2AB$, $3BE$, $3BG$, $ABDG$, $ABEF$, $BDEFG$) and $2+3+3+1+3 = 12$ (from $2C$, $3CD$, $3CF$, $ACDF$, $3ACEG$). So

$$F_7 \cdot r_{AO} = 14 - 14 = 0 \ , \qquad F_7 \cdot r_{BC} = (14+12) - (14+12) = 0 \ . \tag{9.3}$$

The same count for all $\binom{8}{2} = 28$ pairs of $A, \dots, G, O$ gives 0 every time, so F_7 is superbalanced in the sense of Chapter 3.

The terms that contain A. The Fano plane on $A, \dots, G$ of Figure 1 has the lines ABC, ADE, AFG through A and the four lines BDG, BEF, CDF, CEG that miss A. Their complements in $\{A, \dots, G\}$ are $ACEF$, $ACDG$, $ABEG$, $ABDF$. The seven LHS terms of F_7 that contain A are

$$2ABC + 2ADE + 2AFG + 2ABDF + 2ABEG + 2ACDG + 2ACEF \ , \tag{9.4}$$

the three lines through A and the complements of the four lines that miss A, all with weight 2. These are the same seven sets as the terms of E_0 that contain A (in E_0 they have weight 3), and they form an antichain. To see this, note that two distinct sets of the same size are never nested, so a nested pair would consist of a line through A inside the complement of a line ℓ that misses A. Any two lines of the Fano plane meet, so ℓ meets the line through A in a point other than A, and that point is missing from the complement of ℓ. The antichain property is what the restriction lemma (Lemma 5.19(b)) asks for.

The realizable domain. Enumerating the $2^{20} = 1{,}048{,}576$ points of $\{0,1\}^{20}$ and testing the conditions (R1) and (R2) of Definition 5.1 for every pair of LHS terms shows that $\mathcal{A}(F_7)$ has 11,046 points.

The stabilizer. Testing all $8! = 40{,}320$ relabelings of $A, \dots, G, O$, with purification, shows that $\mathrm{Stab}(F_7)$ has six elements,

$$\mathrm{Stab}(F_7) = \{\mathrm{id},\ (BD)(CE),\ (BF)(CG),\ (DF)(EG),\ (BDF)(CEG),\ (BFD)(CGE)\} \ . \tag{9.5}$$

All six fix A and O. They permute the three ordered pairs (B,C), (D,E), (F,G) as blocks: $(BD)(CE)$ exchanges the first two blocks, $(BF)(CG)$ the first and the third, $(DF)(EG)$ the last two, and the two remaining elements permute the blocks cyclically. Every permutation of the three blocks occurs exactly once, so $\mathrm{Stab}(F_7)$ is isomorphic to $\mathfrak{S}_3$, generated by $(BD)(CE)$ and $(BF)(CG)$. Each element is also a collineation of the Fano plane fixing A (for instance $(BD)(CE)$ exchanges the lines ABC and ADE, fixes AFG, and maps BDG, BEF, CDF, CEG to BDG, CDF, BEF, CEG), and the same exhaustive test shows that $\mathrm{Stab}(F_7)$ is a subgroup of the stabilizer of E_0, the group of order 24 of the collineations that fix A (Chapter 8). By the orbit-stabilizer theorem the orbit of F_7 under the 8! relabelings has $40{,}320/6 = 6{,}720$ members.

Not a lift from fewer parties. An inequality for six parties can be turned into one for seven by letting one of its parties stand for the union of two parties i, j. Every term of the result then contains both of i, j or neither. If one of the two is the purifier, the other occurs in no term. Inspecting the 44 terms of (9.1) shows that no pair of $A, \dots, G, O$ has this property (for example BDF contains B and not C), so F_7 does not arise in this way, that is, it is not a zero-lift in the sense of [**4**].

All statements of this section are exact integer computations from (9.1).

9.2. The facet

Recall from Chapter 2 that a valid $Q \neq 0$ defines a facet of HEC_N when the face $\mathrm{HEC}_N \cap Q^{\perp}$ has dimension $2^N - 2$, and that, because HEC_N is full-dimensional [**4**], this holds exactly when there are $2^N - 2$ graph models with linearly independent entropy vectors on which Q vanishes. For $N = 7$ the number is 126.

THEOREM 9.6. *The inequality F_7 is valid on* HEC_7 *and defines a facet of* HEC_7.

Status: derived (validity from exact rational maps on $\mathcal{A}(F_7)$ through the realizable-domain criterion; facet from 126 tight graph models with rank certified modulo primes).

PROOF. *Validity.* There is a map $n\colon \mathcal{A}(F_7) \to \prod_{r=1}^{24}[0,\beta_r]$ with rational values that satisfies the pins (4.13) and the contraction condition (4.14) on all

$$\binom{11{,}046}{2} = \frac{11{,}046 \cdot 11{,}045}{2} = 61{,}001{,}535 \tag{9.7}$$

pairs of points of $\mathcal{A}(F_7)$. The certificate is the table of its values, and a checker rebuilds $\mathcal{A}(F_7)$ from the list of terms, confirms that the table has exactly these points, and checks the pins, the ranges $0 \le n_r(u) \le \beta_r$ and the contraction condition in exact integer arithmetic after multiplying by the common denominator (Chapter 6). There are two such maps, one for F_7 and one for $2F_7$, and each was checked by four checkers. By the realizable-domain criterion (Lemma 5.9) the existence of one of them proves that $F_7 \cdot S \ge 0$ for every graph model.

Facet. There are 126 explicit graph models $\Gamma_1, \dots, \Gamma_{126}$ with integer edge weights. Their entropy vectors $S_{\Gamma_a} \in \mathbb{Z}^{127}$ are recomputed by a checker with its own exact max-flow computation, one minimum cut for each of the 127 nonempty subsets of $\{A, \dots, G\}$, and $F_7 \cdot S_{\Gamma_a} = 0$ for every a. Let $\mathbf{S}$ be the 126×127 integer matrix with rows S_{Γ_a}. Its rank modulo a prime p is 126, and since reduction modulo p does not increase the rank of an integer matrix (Section 6.6), its rank over $\mathbb{Q}$ is at least 126. All rows lie in the hyperplane $F_7^{\perp}$, which has dimension 126, so

$$\operatorname{rk}_{\mathbb{Q}} \mathbf{S} = 126 = 2^7 - 2\ . \tag{9.8}$$

The rows are entropy vectors of graph models, so they lie in HEC_7, and since F_7 vanishes on them they lie in the face $\mathrm{HEC}_7 \cap F_7^{\perp}$. That face therefore contains 126 linearly independent vectors and has dimension at least 126. It lies in the hyperplane $F_7^{\perp}$, so its dimension is exactly 126. With the validity proved above,

$$\boxed{F_7 \text{ valid, and 126 independent tight entropy vectors } \Longrightarrow\ F_7 \text{ defines a facet of } \mathrm{HEC}_7\ .} \tag{9.9}$$

Two lists of 126 graph models were built independently, and for each list the entropy vectors were recomputed by two max-flow implementations that share no code with the program that built the graphs, and the rank was computed modulo three different primes. □

Full-dimensionality enters at the last step: without it, a face of dimension 126 could be the whole cone. The realizable-domain criterion is what keeps the validity certificate small, since the map is needed on 11,046 points instead of the 2^{20} points of the full cube.

Physics aside 9.10. A tight graph model is a discretized time-symmetric slice on which the seven RT entropies saturate F_7. The facet property says that the entropy vectors of these saturating configurations span a hyperplane, the largest span that the saturating configurations of a nonzero valid inequality can have. By the positive-combination lemma (Lemma 2.13), a facet is not a positive combination of valid inequalities that are not multiples of it, so F_7 is one of the irreducible linear constraints that the RT formula imposes on seven regions.

9.3. Comparison with the literature

The paper reports that F_7 does not appear among the seven-party inequalities that its search of the literature found. We describe what was compared and how.

REMARK 9.11 (Comparison with the literature). The facet F_7 does not appear among the 43 seven-party inequalities of the works listed in Table 1. For the comparison every inequality is divided by the greatest common divisor of its coefficients and brought to a canonical form under the 8! relabelings of $A, \dots, G, O$ with purification, and no canonical form in the table agrees with that of F_7 (an exact computation). A second argument needs no canonical forms. A relabeling only permutes the coefficients of an inequality, and F_7 has a coefficient 4, while the largest coefficient in the table, after division by the greatest common divisor, is 3.

The three infinite families in the table also have members with more than seven parties, and merging or deleting parties turns such a member into a seven-party inequality. Both operations keep every coefficient on its side and can only combine, cancel or remove terms, so they do not increase $|\alpha|$ or $|\beta|$. An integer multiple of F_7 has $|\alpha| \geq 39$ and $|\beta| \geq 54$, while every member of the three families with at most 14 parties has $|\beta| \leq 50$ or $|\alpha| < 39$. So none of these members gives a multiple of F_7 in this way.

The table contains every seven-party inequality that the paper found in the works on holographic entropy inequalities it cites and in the papers that cite [**15, 11, 31, 32**], screened by their abstracts. The paper does not compare F_7 with the conjectured generalizations of the toric inequalities in [**31**], whose seven-party instances are not written out there, with the averaged inequalities of the symmetrized cones [**33, 34**], which constrain only sums over relabelings, with family members with more than 14 parties, or with works outside this search, and it found no published list of facets of HEC_7. The facet property of the seven-party members of the toric and projective-plane families was shown in [**12**].

Status: the paper assigns no status word to this comparison. It is an exact computation, and it covers only the literature searched, as described above.

source	inequalities	number	max. coefficient
Czech, Wang [**15**]	the seven-region inequality, both printed forms	2	1
Akers, Hernández-Cuenca, Rath [**19**]	eq. (1.5)	1	1
Bao, Naskar [**17**]	eqs. (A.11), (A.12)	2	1
Bao, Furuya, Naskar [**21**]	eq. (3.2)	1	1
Chou, Lao, Yang [**35**]	eq. (5.15)	1	1
toric family [**11, 36**]	the seven-party members	4	1
projective-plane family [**11, 36**]	the seven-party member	1	3
cyclic family [**1**]	the nonzero seven-party members	31	≤ 2

TABLE 1. The 43 seven-party inequalities with which F_7 was compared. Inequalities that agree up to relabeling are counted separately, so several rows contain the same orbit. The largest coefficient is taken after division by the greatest common divisor of the coefficients. The largest coefficient of F_7 is 4.

9.4. Where F_7 comes from

The facet arises from the chain $E_0, \ldots, E_5$ of Chapter 8 by exact splittings. We first write out the inequalities involved. Let

$$g = (DF)(EG) \ , \tag{9.12}$$

the element of $\mathrm{Stab}(F_7)$ in (9.5) that exchanges the blocks (D, E) and (F, G). In terms of the generators, $g = (BD)(CE)(BF)(CG)(BD)(CE)$. The letter g is used for this relabeling only. Besides the sums $\mu_0, \ldots, \mu_4$ of instances of MMI in (8.28), we need the orbit sum

$$\mu_5 = \mathrm{MMI}(C{:}DF{:}EG) + \mathrm{MMI}(E{:}BF{:}CG) + \mathrm{MMI}(G{:}BD{:}CE) \ , \tag{9.13}$$

and three further seven-party inequalities,

(9.14)

$$\begin{aligned} X_7: \quad & 7ABC + 7ADE + 7AFG + 5BDF + 5BDG + 5BEF + 5BEG + 5CDF + 5CDG \\ & + 5CEF + 6CEG + 7ABDF + 7ABEG + 7ACDG + 7ACEF \\ & + 4BCDE + 4BCFG + 4DEFG \\ & \geq\ 4B + 5C + 4D + 5E + 4F + 5G + 4AB + 3AC + 4AD \\ & + 3AE + 4AF + 3AG + 3BC + 4BD + 4BE + 4BF + 4BG + 4CD \\ & + 4CE + 4CF + 4CG + 3DE + 4DF + 4DG + 4EF + 4EG \\ & + 3FG + 5ABDG + 5ABEF + 5ACDF + 6ACEG + 5BCDEFG + 7ABCDEFG\ , \end{aligned}$$

(9.15)

$$\begin{aligned} V_7: \quad & 2ABC + 2ADE + 2AFG + BDF + 2BDG + 2BEF + BEG \\ & + 2CDF + CDG + CEF + 2CEG + 2ABDF + 2ABEG + 2ACDG \\ & + 2ACEF + 3BCFG + BCDEF + 2BCDEG + BDEFG + 2CDEFG \\ & \geq\ B + 2C + 3D + 4E + F + 2G + 2AB + 2AD \\ & + 2AF + BE + 2BF + 3BG + CD + 3CF + 2CG + DG \\ & + EF + ABDG + ABEF + ACDF + 3ACEG + BCDE + DEFG + BCDFG \\ & + 4BCDEFG + 2ABCDEFG\ , \end{aligned}$$

and

(9.16)

$$\begin{aligned} V_7': \quad & 2ABC + 2ADE + 2AFG + BDF + BDG + BEF + BEG \\ & + CDF + CDG + CEF + 3CEG + 2ABDF + 2ABEG + 2ACDG \\ & + 2ACEF + 2DEFG + BCDEF + BCDEG + 2BCDFG + 2BCEFG \\ & \geq\ 2B + 4C + 2E + F + 3G + 2AB + 2AD + 2AF \\ & + BD + BE + CD + CE + 2DF + 2DG + 2EF + 2EG \\ & + ABDG + ABEF + ACDF + 3ACEG + 2BCFG + 4BCDEFG + 2ABCDEFG\ . \end{aligned}$$

Finally, Y_7 is the integer vector defined by

$$Y_7 = 6E_5 - 2X_7 - 3\mu_5\ , \qquad \text{that is,} \qquad 6E_5 = 2X_7 + Y_7 + 3\mu_5\ , \tag{9.17}$$

with E_5 given by (8.25), (8.28) and (8.29). The relabeling g lies in the stabilizers of F_7, X_7 and Y_7, which are the same group (9.5), but not in that of V_7, so $g(V_7)$ is a different inequality from V_7.

Proposition 9.18. *The identities*

$$4Y_7 = 3F_7 + 2V_7 + g(V_7) + 2V_7' \tag{9.19}$$

and

$$24E_0 = 3F_7 + 2V_7 + g(V_7) + 2V_7' + 8X_7 + 12\mu_5 + 48\mu_4 + 24\mu_3 + 24\mu_2 + 48\mu_1 + 48\mu_0 \tag{9.20}$$

hold exactly in $\mathbb{Z}^{127}$, and every inequality on their right-hand sides is valid. Consequently every graph model on which E_0 vanishes has $F_7 \cdot S = 0$, that is, the face $\mathrm{HEC}_7 \cap E_0^\perp$ lies in the facet $\mathrm{HEC}_7 \cap F_7^\perp$. The same holds with E_0 replaced by any of $E_1, \ldots, E_5$ or Y_7.

Status: certified (integer identities, checked from the coefficient vectors); the containment of faces is derived.

Proof. *The identities.* The identity (9.19) is an identity between integer vectors in $\mathbb{Z}^{127}$, one coordinate for each nonempty subset of $\{A, \ldots, G\}$, and it is checked coordinate by coordinate

from (9.1), (9.15), (9.16), the relabeling (9.12) and the definition (9.17). Multiplying (9.17) by 4 and inserting (9.19) gives

$$24E_5 = 8X_7 + 4Y_7 + 12\mu_5 = 3F_7 + 2V_7 + g(V_7) + 2V_7' + 8X_7 + 12\mu_5 \ . \tag{9.21}$$

The telescoped descent of Proposition 8.38, $E_0 = E_5 + 2\mu_0 + 2\mu_1 + \mu_2 + \mu_3 + 2\mu_4$, multiplied by 24, reads

$$24E_0 = 24E_5 + 48\mu_0 + 48\mu_1 + 24\mu_2 + 24\mu_3 + 48\mu_4 \ , \tag{9.22}$$

and inserting (9.21) gives (9.20). Two programs check the decomposition (9.20) exactly.

Validity of the summands. F_7 is valid by Theorem 9.6. V_7, V_7' and X_7 are valid by the realizable-domain criterion, each with an exact rational map on its realizable domain, which has 10,032, 10,032 and 6,912 points respectively, checked like the map of Theorem 9.6 by two checkers each. $g(V_7)$ is valid because HEC_7 is invariant under relabelings (Chapter 2). The μ_s are sums of instances of MMI, which is valid [**9**]. Finally Y_7, which does not appear in (9.20), is valid by (9.19), since $4Y_7$ is a sum of valid inequalities.

Containment of faces. Let S be the entropy vector of a graph model with $E_0 \cdot S = 0$. Every term on the right of (9.20), evaluated on S, is nonnegative, and they add up to $24E_0 \cdot S = 0$, so each of them vanishes, in particular $3F_7 \cdot S = 0$. This is the positive-combination lemma (Lemma 2.13) applied to (9.20). For E_s with $1 \le s \le 5$ the descent (8.29) gives $24E_s = 24E_5 +$ (a nonnegative combination of $\mu_s, \dots, \mu_4$), and with (9.21) the same argument applies. For Y_7 it applies to (9.19). □

The coefficient 3 of F_7 in (9.19) can be read as follows. Since $g \in \mathrm{Stab}(F_7)$, we have $g(F_7) = F_7$, and (9.19) can be written

$$4Y_7 = 2(F_7 + V_7) + g(F_7 + V_7) + 2V_7' \ . \tag{9.23}$$

The coefficient of F_7 is $2 + 1$, from $F_7 + V_7$ and from its image under g. Among the summands of (9.20), F_7 is the only one that the paper shows to be a facet. Table 2 collects what is known about the others.

	L	R	$\lvert\alpha\rvert$	$\lvert\beta\rvert$	$\lvert\,\mathrm{Stab}\,\rvert$	$\lvert\mathcal{A}\rvert$	integer map on $\mathcal{A}$	tight rank
F_7	20	24	39	54	6	11,046	yes, none equivariant	126, facet
V_7	20	26	35	48	2	10,032	yes, none equivariant	125, open
V_7'	20	23	32	44	1	10,032	yes	125, open
X_7	18	33	102	138	6	6,912	open	not a facet
Y_7	21	27	66	93	6	12,774	open	not a facet

TABLE 2. The inequalities in (9.19) and (9.20) other than the sums of instances of MMI. All five are valid and superbalanced. "Integer map on $\mathcal{A}$" says whether an unscaled map on the realizable domain is known, and whether one equivariant under Stab exists. "Tight rank" is the number of linearly independent entropy vectors of explicit tight graph models, and 126 is the dimension of a facet of HEC_7. Y_7 is not a facet by (9.19), and X_7 by (9.33) of Section 9.7, each with the positive-combination lemma.

9.5. Unscaled maps of F_7

The statement that F_7 has an unscaled map but none that respects its symmetry needs the action of a relabeling on maps, which we set up first. As in Chapter 2, σ denotes a generic relabeling, and g stays the fixed relabeling (9.12).

The action on terms and points. Let Q be an inequality as in Chapter 1, with LHS terms $I_1, \dots, I_L$ and RHS terms $J_1, \dots, J_R$, and let $\sigma \in \mathrm{Stab}(Q)$ fix the purifier. For a term $I \subseteq [N]$ the image $\sigma(I)$ is again a subset of $[N]$, because σ fixes O and no purification is needed. Since σ fixes the coefficient vector of Q, it permutes the LHS terms among themselves with their coefficients, and likewise the RHS terms. We write $\sigma(I_l) = I_{\sigma(l)}$ with $\alpha_{\sigma(l)} = \alpha_l$, and $\sigma(J_r) = J_{\sigma(r)}$ with $\beta_{\sigma(r)} = \beta_r$. On points $u \in \{0,1\}^L$ the relabeling acts by moving coordinates along with their terms,

$$(\sigma \cdot u)_{\sigma(l)} = u_l \qquad (l = 1, \dots, L) \ . \tag{9.24}$$

This action has four properties.

(a) It maps occurrence words to occurrence words, $\sigma \cdot x^{(i)} = x^{(\sigma(i))}$, because $(\sigma \cdot x^{(i)})_{\sigma(l)} = [i \in I_l] = [\sigma(i) \in \sigma(I_l)] = [\sigma(i) \in I_{\sigma(l)}] = x^{(\sigma(i))}_{\sigma(l)}$. It fixes the word 0.

(b) It preserves the weighted distance, $d_\alpha(\sigma \cdot u, \sigma \cdot v) = d_\alpha(u, v)$, because the coordinates are permuted together with their weights.

(c) It preserves the realizable domain, $\sigma \cdot \mathcal{A}(Q) = \mathcal{A}(Q)$, because $\sigma(I) \cap \sigma(I') = \sigma(I \cap I')$ is empty exactly when $I \cap I'$ is, and $\sigma(I) \subseteq \sigma(I')$ exactly when $I \subseteq I'$, so the conditions (R1) and (R2) of Definition 5.1 hold at $\sigma \cdot u$ exactly when they hold at u.

(d) It composes, $\sigma \cdot (\tau \cdot u) = (\sigma\tau) \cdot u$, since $(\sigma \cdot (\tau \cdot u))_{\sigma\tau(l)} = (\tau \cdot u)_{\tau(l)} = u_l$.

Conjugate maps. For a real map n for Q on $\mathcal{A}(Q)$ (Definition 4.12) and σ as above put

$$n^\sigma_r(u) = n_{\sigma(r)}(\sigma \cdot u) \qquad (u \in \mathcal{A}(Q) \ , \ r = 1, \dots, R) \ . \tag{9.25}$$

This is again a map for Q on $\mathcal{A}(Q)$, integer if n is. The values lie in the right ranges because $\beta_{\sigma(r)} = \beta_r$. The pins hold because, by (a),

$$n^\sigma_r(x^{(i)}) = n_{\sigma(r)}(x^{(\sigma(i))}) = \beta_{\sigma(r)}[\sigma(i) \in J_{\sigma(r)}] = \beta_r[i \in J_r] \ , \tag{9.26}$$

and $n^\sigma(0) = n(0) = 0$. The contraction condition holds because, by (b),

$$\sum_r \left| n^\sigma_r(u) - n^\sigma_r(v) \right| = \sum_{r'} \left| n_{r'}(\sigma \cdot u) - n_{r'}(\sigma \cdot v) \right| \le d_\alpha(\sigma \cdot u, \sigma \cdot v) = d_\alpha(u, v) \ , \tag{9.27}$$

where we substituted $r' = \sigma(r)$, which runs over all RHS indices. By (d), $(n^\sigma)^\tau = n^{\sigma\tau}$. A map is *equivariant* under a group H of such relabelings if $n^\sigma = n$ for all $\sigma \in H$, that is, $n_{\sigma(r)}(\sigma \cdot u) = n_r(u)$ for all σ, r and u.

THEOREM 9.28. (i) *F_7 has an integer map on its realizable domain $\mathcal{A}(F_7)$. Hence F_7 has an unscaled contraction proof on its realizable domain, and $k^{\mathcal{A}}_{\min}(F_7) = 1$.*

(ii) *No integer map for F_7 on $\mathcal{A}(F_7)$ is equivariant under* $\mathrm{Stab}(F_7)$.

Status: (i) certified (integer map on $\mathcal{A}(F_7)$), and $k^{\mathcal{A}}_{\min}(F_7) = 1$ derived; (ii) certified (DRAT, 2 encoders).

PROOF. (i) The certificate is a table of integers $n_r(u) \in \{0, \dots, \beta_r\}$ for the 11,046 points $u \in \mathcal{A}(F_7)$ and the 24 RHS terms. A checker rebuilds $\mathcal{A}(F_7)$ from Definition 5.1, verifies that it is the set of points of the table, and verifies the pins, the ranges and the contraction condition on all pairs of points in exact arithmetic. Two checkers did this. An integer map is in particular a real map, so the realizable-domain criterion applies to it and proves validity once more, and by Definition 4.23 the least k for which kF_7 has an integer map on $\mathcal{A}(F_7)$ is 1.

(ii) All six elements of $\mathrm{Stab}(F_7)$ fix the purifier, so the action above applies. The equivariant encoding of Chapter 6 is the order encoding of the map problem for F_7 on $\mathcal{A}(F_7)$ with the variables of (u, r) and $(\sigma \cdot u, \sigma(r))$ identified for all $\sigma \in \mathrm{Stab}(F_7)$, and with the contraction condition imposed on one pair from each orbit of pairs. An equivariant integer map gives an assignment that satisfies the plain order encoding, by the soundness of the encoding (Lemma 6.5). By equivariance this assignment gives the same value to any two identified variables, so it is an assignment of the

equivariant formula, and it satisfies it, because that formula keeps only some of the contraction constraints (after the identification, the constraints of the pairs of one orbit coincide). The formula is unsatisfiable, certified by DRAT proofs from two independent encoders, each accepted by `drat-trim` (Section 6.3). So no equivariant integer map exists. □

Avis and Hernández-Cuenca ask in their Problem 3 whether their contraction criterion becomes a necessary condition for validity once every RHS term $\beta_r S(J_r)$ with $\beta_r \geq 2$ is replaced by β_r unit terms [**4**], and in the formulation of Chapter 5 this is the question whether $k^{\mathcal{A}}_{\min}(Q) = 1$ for every valid Q. The inequalities of Theorem 8.36 show that the answer in this formulation is negative, while the one facet treated here has $k^{\mathcal{A}}_{\min} = 1$.

The obstruction in (ii) is one of integrality. Let n be an integer map of (i) and average its conjugates over the group,

$$\overline{n} = \frac{1}{6} \sum_{\sigma \in \mathrm{Stab}(F_7)} n^\sigma \ . \tag{9.29}$$

Each n^σ is a map for F_7 on $\mathcal{A}(F_7)$, and an average of maps is a map: the pins are linear, the ranges are intervals, and by the triangle inequality $\sum_r |\overline{n}_r(u) - \overline{n}_r(v)| \leq \frac{1}{6} \sum_\sigma \sum_r |n^\sigma_r(u) - n^\sigma_r(v)| \leq d_\alpha(u, v)$. It is equivariant, because $(\overline{n})^\tau = \frac{1}{6} \sum_\sigma n^{\sigma\tau}$ by (d), and $\sigma\tau$ runs over the whole group when σ does. Its values lie in $\frac{1}{6}\mathbb{Z}$. So $6\overline{n} = \sum_\sigma n^\sigma$ is an equivariant integer map for $6F_7$, by the rule that maps for $k_1 Q$ and $k_2 Q$ add to a map for $(k_1 + k_2)Q$ (Chapter 4). Together with (ii) this gives the following.

Corollary 9.30. *The least k for which kF_7 has an integer map on $\mathcal{A}(F_7)$ that is equivariant under* $\mathrm{Stab}(F_7)$ *satisfies* $2 \leq k \leq 6$.

Status: derived.

Since F_7 is invariant under $\mathrm{Stab}(F_7)$, one might look for an unscaled proof with the same symmetry. Theorem 9.28(ii) excludes one on $\mathcal{A}(F_7)$, and the corollary shows that a symmetric proof exists after scaling by at most 6.

The full cube. The paper did not attempt to construct an integer map for F_7 on the full cube $\{0,1\}^{20}$, so it leaves open whether $k_{\min}(F_7) = 1$ (Chapter 12). The restriction lemma cannot exclude it, because the map of (i) restricts to a map on $\mathcal{U}(F_7)$, which lies in $\mathcal{A}(F_7)$ by the antichain property of (9.4). So the map problem on $\mathcal{U}(F_7)$ has a solution, and no non-existence certificate on $\mathcal{U}(F_7)$ exists.

9.6. The other summands

Remark 9.31 (The other summands). The inequalities V_7 and V_7' also have integer maps on their realizable domains, each certified like the map of Theorem 9.28(i), by two checkers. As for F_7, no integer map of V_7 on $\mathcal{A}(V_7)$ is equivariant under its stabilizer, which has order 2 and is generated by $(BF)(CG)$. This is certified by a DRAT proof. The stabilizer of V_7' is trivial. For each of V_7 and V_7' there are 125 explicit tight graph models with linearly independent entropy vectors, one fewer than a facet needs, so the paper does not decide whether they are facets. For V_7 the 125 vectors also lie in $F_7^\perp$ (an exact computation). They are linearly independent and lie in the subspace $V_7^\perp \cap F_7^\perp$, which has dimension 125 because V_7 and F_7 are not proportional, so they span it. Hence V_7 defines a facet if and only if some graph model has $V_7 \cdot S = 0$ and $F_7 \cdot S > 0$. For X_7 and Y_7 validity is known, but not whether they have unscaled maps on their realizable domains.

Consequently (9.20) writes $24E_0$ as a positive combination of X_7 and of inequalities known to have unscaled contraction proofs on their realizable domains: F_7, V_7, $g(V_7)$ (relabeling the terms and the points carries a map of V_7 to one of $g(V_7)$), V_7', and the instances of MMI, which have unscaled maps even on the full cube, given by the map of Bao et al. for MMI(A:B:C) [**1**] (Example 4.9). Section 9.7 writes E_0 as a positive combination of inequalities known to have them, without X_7.

9.7. A half-sum behind the failures

The decomposition (9.20) has denominator 24. A shorter one, with denominator 2, writes every member of the chain through the same two inequalities. It uses a further seven-party inequality,

$$\begin{aligned} P_7: \quad & 4ABC + 4ADE + 4AFG + 3BDF + 3BDG + 3BEF + 3BEG + 3CDF + 3CDG \\ & + 3CEF + 3CEG + 4ABDF + 4ABEG + 4ACDG + 4ACEF \\ & + 4BCDE + 2BCFG + 2DEFG \\ & \geq\ 2B + 2C + 2D + 2E + 3F + 3G + 2AB + 2AC + 2AD \\ & + 2AE + 2AF + 2AG + 2BC + 3BD + 3BE + 2BF + 2BG + 3CD \\ & + 3CE + 2CF + 2CG + 2DE + 2DF + 2DG + 2EF + 2EG \\ & + 2FG + 3ABDG + 3ABEF + 3ACDF + 3ACEG + BCDEF + BCDEG \\ & + 2BCDEFG + 4ABCDEFG\ , \end{aligned} \tag{9.32}$$

with $L = 18$, $R = 35$, $|\alpha| = 60$, $|\beta| = 80$, all Bell values 0 and a stabilizer of order 8. Its LHS terms are those of X_7, so $\mathcal{A}(P_7) = \mathcal{A}(X_7)$, with 6,912 points. There is an integer map for P_7 on $\mathcal{A}(P_7)$, which was checked on all 23,884,416 pairs of points by three checkers, and by the realizable-domain criterion (Lemma 5.9) P_7 is valid.

Two exact identities connect P_7 with the chain. With $g = (DF)(EG)$,

$$2X_7 = V_7' + P_7 + 2g(P_7)\ , \qquad 2E_5 = V_7' + P_7 + 2\,\mathrm{MMI}(D{:}BG{:}CF) + 2\,\mathrm{MMI}(E{:}BF{:}CG)\ , \tag{9.33}$$

both checked in exact integer arithmetic, by three programs, from the printed coefficients. So E_5, the last member of the chain, is half the sum of V_7' and P_7 plus two instances of MMI. Every earlier member is E_5 plus a nonnegative integer combination of the μ_t by the descent identities of Chapter 8, so

$$2E_s = V_7' + P_7 + 2M_s \qquad (s = 0, \ldots, 5)\ , \tag{9.34}$$

with $M_s = \mathrm{MMI}(D{:}BG{:}CF) + \mathrm{MMI}(E{:}BF{:}CG)$ plus a nonnegative integer combination of $\mu_s, \ldots, \mu_4$. Counted with multiplicity, $2M_s$ consists of 36, 32, 28, 22, 16 and 4 instances of MMI for $s = 0, \ldots, 5$.

Read the identities as statements about maps. Every summand on their right-hand sides has an unscaled map on its realizable domain: V_7' by Remark 9.31, P_7 as just said, $g(P_7)$ by relabeling the map of P_7, and the instances of MMI even on the full cube. So X_7 and every E_s are positive combinations of valid inequalities with unscaled maps on their realizable domains, and E_0 has such a decomposition without the summand X_7, for which no unscaled map is known (Remark 9.31). The coefficients $\frac{1}{2}$ in (9.34) fall on exactly two summands, and their half-sum $\frac{1}{2}(V_7' + P_7)$ is again an integer vector, valid and superbalanced. Its seven LHS terms that contain A are those of E_0 and form an antichain, so its subdomain $\mathcal{U}$, of 134 points, lies in its realizable domain, and DRAT proofs on $\mathcal{U}$ from two encoders show that $\frac{1}{2}(V_7' + P_7)$ has no integer map, on the full cube or on its realizable domain.

So two inequalities with unscaled maps have a half-sum without one. The two statements are compatible. When no term of one inequality cancels against a term of the other, the two maps combine into a map of the sum, as in the product construction of Grimaldi et al. [**5**], by applying them to the restrictions of a point to the two sets of LHS terms and adding the values. Here $V_7' + P_7$ cancels LHS terms of one summand against RHS terms of the other, $|\alpha|$ dropping from $32 + 60 = 92$ to 88, so the cube and the distances of $V_7' + P_7$ differ from those of the two summands, and the construction above is unavailable for $V_7' + P_7$ and its half. Whether V_7' and P_7 are facets, and whether they have unscaled maps on the full cube, are open questions. By (9.33) and the positive-combination lemma (Lemma 2.13), X_7 is not a facet: it is a positive combination of valid inequalities that are not multiples of it.

In the one case where a superbalanced inequality without unscaled maps was followed down to a facet, the facet has an unscaled contraction proof on its realizable domain, and every such proof breaks its symmetry. If V_7, V_7' or P_7 is not a facet, the facets into which it decomposes are not known, and any of them could lack an unscaled proof.

CHAPTER 10

Star inequalities and their scaling spectrum

Where we are. Chapters 7–9 show that the least scaling $k_{\min}$ of Definition 4.23 can exceed 1. They decide its value only in a few cases: $k_{\min}(B_7^*) = 2$, $k_{\min}(B_7) = 2$, $k_{\min}(Q_8^A) = 2$, and for the superbalanced failures the scaled facts of Proposition 8.42, among them $k_{\min}(E_s) = 2$ for $s = 0, 1, 3, 4$. The question of this chapter and the next is how large $k_{\min}$ can be for a valid N-party inequality. For a general inequality, deciding whether kQ has an integer map is a finite problem on 2^L points, which the paper decides by SAT solvers case by case, and nothing forces the answer to have a closed form. Star inequalities are a class in which it has one: the whole map problem is decided at a single point of the cube, and the answer is a statement about lattice points of a polytope. This chapter develops that description. Chapter 11 reads it as a coin-weighing problem, uses it to build star inequalities, which are implied by subadditivity and are not balanced, whose least scaling grows faster than exponentially with N, and bounds the least scaling of star inequalities from above.

10.1. Star inequalities

The left-hand side of a star inequality consists of single-party entropies $S(i)$ and of $S([N])$. Since $S([N]) = S(O)$ by purification, it is a weighted sum of the single-party entropies of all $N + 1$ parties, the purifier included.

Definition 10.1 (Star inequalities). Let $m \geq 1$, let $J_1, \ldots, J_m \subseteq [N]$ be distinct sets with $2 \leq |J_K| \leq N - 1$, let $\beta_1, \ldots, \beta_m$ be positive integers, and let ρ_i, $i \in [N] \cup \{O\}$, be nonnegative integers, the *radii*. The *star inequality* with these data is

$$Q: \qquad \sum_{i\in[N]} \rho_i S(i) + \rho_O S([N]) \;\geq\; \sum_{K=1}^{m} \beta_K S(J_K)\ . \tag{10.2}$$

The *word* of a party $i \in [N]$ is $\eta^{(i)} = (\beta_K[i \in J_K])_{K=1}^m$, and $\eta^{(O)} = 0$. The *star polytope* of Q is

$$\Pi(Q) = \Big\{ c \in \prod_{K=1}^{m} [0, \beta_K] \colon \|c - \eta^{(i)}\|_1 \leq \rho_i \text{ for all } i \in [N] \cup \{O\} \Big\}\ . \tag{10.3}$$

The size conditions on the J_K make (10.2) an inequality of the form used throughout the companion: $|J_K| \geq 2$ and $J_K \neq [N]$ keep every RHS term different from every LHS term, which are singletons and $[N]$, and distinctness keeps the RHS terms different from each other. The index K runs over the RHS terms and is never a party. In Chapter 11 we call the values of K *coins*.

The words have a direct meaning for maps. The RHS of Q has $R = m$ terms, and the pin (4.13) of a party i in a map for kQ is $n_K(x^{(i)}) = k\beta_K[i \in J_K] = k\eta_K^{(i)}$. So the words are the pins of a map for Q, and $\Pi(Q)$ is the set of points of the box $\prod_K [0, \beta_K]$ that lie in the ℓ_1-ball of radius ρ_i around the pin of i for every $i \in [N] \cup \{O\}$.

The cube of Q explains the name. Give the LHS term $\{i\}$ the bit i and the term $[N]$ the bit $\star$. A party i lies in the LHS terms $\{i\}$ and $[N]$ and in no other, so its occurrence word has exactly the bits i and $\star$ set, and the purifier's occurrence word is 0. The point z with only the bit $\star$ set differs

from each of these $N+1$ words in exactly one bit. In the graph of the cube the occurrence words form a star around z, and Theorem 10.13 below shows that the map problem is decided at z.

10.2. Linearity on the box

The words are corners of the box $\prod_K [0,\beta_K]$, and on the box the ℓ_1-distance to a corner is an affine function.

LEMMA 10.4 (Linearity on the box). *Let $\eta \in \prod_K \{0,\beta_K\}$ and $c \in \prod_K [0,\beta_K]$. Then*

$$\|c-\eta\|_1 = \sum_{K\,:\,\eta_K=0} c_K + \sum_{K\,:\,\eta_K=\beta_K} (\beta_K - c_K)\ , \tag{10.5}$$

and $\|c-\eta\|_1 + \|c-\overline{\eta}\|_1 = |\beta|$ for the complementary word $\overline{\eta} = \beta - \eta$. Hence $\Pi(Q)$ is cut out of the box by $N+1$ linear inequalities with coefficients ± 1 and integer right-hand sides, and if two words of Q are complementary and their radii add up to $|\beta|$, both of their constraints hold with equality on $\Pi(Q)$.

PROOF. For $c_K \in [0,\beta_K]$ we have $|c_K - 0| = c_K$ and $|c_K - \beta_K| = \beta_K - c_K$, which gives (10.5). Moving the constants to the right, the constraint $\|c-\eta^{(i)}\|_1 \le \rho_i$ reads

$$\sum_{K\,:\,\eta^{(i)}_K=0} c_K - \sum_{K\,:\,\eta^{(i)}_K=\beta_K} c_K \ \le\ \rho_i - \sum_{K\,:\,\eta^{(i)}_K=\beta_K} \beta_K\ , \tag{10.6}$$

with coefficients ± 1 and an integer right-hand side. For complementary words η and $\overline{\eta}$, each coordinate K has $\eta_K = 0$ in one of them and $\eta_K = \beta_K$ in the other, so by (10.5) it contributes c_K to one of the two distances and $\beta_K - c_K$ to the other, together β_K. Summing over K gives $|\beta|$. Finally, let $d_1 = \|c-\eta\|_1 \le \rho_1$ and $d_2 = \|c-\overline{\eta}\|_1 \le \rho_2$ with $\rho_1 + \rho_2 = |\beta| = d_1 + d_2$. Then $\rho_1 - d_1 = d_2 - \rho_2$, where the left side is ≥ 0 and the right side is ≤ 0, so both vanish. □

10.3. Validity from a real point

Validity of a star inequality follows from subadditivity as soon as its polytope has a point. For each K consider the two inequalities

$$\mathrm{SA}^1_K :\ \sum_{i\in J_K} S(i) \ \ge\ S(J_K)\ , \qquad \mathrm{SA}^2_K :\ \sum_{i\in [N]\setminus J_K} S(i) + S([N]) \ \ge\ S(J_K)\ . \tag{10.7}$$

The first is subadditivity applied repeatedly to the disjoint regions $\{i\}$, $i \in J_K$. For the second, apply subadditivity repeatedly to the disjoint regions $\{i\}$, $i \in [N]\setminus J_K$, and $\{O\}$, which gives

$$\sum_{i\in [N]\setminus J_K} S(i) + S(O) \ \ge\ S\big(([N]\setminus J_K)\cup\{O\}\big)\ , \tag{10.8}$$

and use purification, $S(O) = S([N])$ and $S(([N]\setminus J_K)\cup\{O\}) = S(J_K)$. Both are sums of instances of subadditivity, so they hold on HEC_N and for all quantum states [**13**]. We call them the *subadditivity chains* of Q and use the same symbols for their coefficient vectors. We write P_I for the coefficient vector of $S(I) \ge 0$, which holds because cut weights are nonnegative.

LEMMA 10.9 (Validity from a real point). *If $c \in \Pi(Q)$, then*

$$Q = \sum_{K=1}^{m} \big[(\beta_K - c_K)\,\mathrm{SA}^1_K + c_K\,\mathrm{SA}^2_K\big] + \sum_{i\in[N]} \big(\rho_i - \|c-\eta^{(i)}\|_1\big)\mathrm{P}_{\{i\}} + \big(\rho_O - \|c\|_1\big)\mathrm{P}_{[N]} \tag{10.10}$$

with all coefficients nonnegative. In particular Q is valid, and if $kc \in \mathbb{Z}^m$, then kQ is a nonnegative integer combination of instances of subadditivity and positivity.

PROOF. Write e_I for the unit vector of the term $S(I)$, so that

(10.11)
$$Q = \sum_{i\in[N]} \rho_i e_{\{i\}} + \rho_O e_{[N]} - \sum_K \beta_K e_{J_K} , \qquad \mathrm{SA}^1_K = \sum_{i\in J_K} e_{\{i\}} - e_{J_K} , \qquad \mathrm{SA}^2_K = \sum_{i\notin J_K} e_{\{i\}} + e_{[N]} - e_{J_K} ,$$

and $\mathrm{P}_I = e_I$. We compare the coefficients of each unit vector on the two sides of (10.10). The term $S(J_K)$ occurs only in the two subadditivity chains of index K, since the J_K are distinct and are neither singletons nor $[N]$. Its coefficient on the right is $-(\beta_K - c_K) - c_K = -\beta_K$, as in Q. The singleton $S(i)$ occurs in SA^1_K for each K with $i \in J_K$, where $\eta^{(i)}_K = \beta_K$, and in SA^2_K for each K with $i \notin J_K$, where $\eta^{(i)}_K = 0$. So the subadditivity chains give it the coefficient

$$\sum_{K\colon i\in J_K} (\beta_K - c_K) + \sum_{K\colon i\notin J_K} c_K = \sum_K \left|\eta^{(i)}_K - c_K\right| = \|c - \eta^{(i)}\|_1 , \tag{10.12}$$

and the positivity term adds $\rho_i - \|c - \eta^{(i)}\|_1$, for a total of ρ_i. The term $S([N])$ occurs only in the SA^2_K, with total coefficient $\sum_K c_K = \|c\|_1 = \|c - \eta^{(O)}\|_1$, and the last term brings it to ρ_O. No other term occurs. The coefficients $\beta_K - c_K$ and c_K are nonnegative because c lies in the box, and the positivity coefficients are nonnegative because $c \in \Pi(Q)$. A nonnegative combination of valid inequalities is valid. If $kc \in \mathbb{Z}^m$, then k times (10.10) has nonnegative integer coefficients, because $k\beta_K$, $k\rho_i$ and $k\|c - \eta^{(i)}\|_1 = \sum_K |k\eta^{(i)}_K - kc_K|$ are integers, and each subadditivity chain is a sum of instances of subadditivity. □

10.4. The scaling spectrum

Recall from Definition 4.12 that an integer map for kQ on the full cube $\{0,1\}^L$ is a map $n = (n_K)_{K=1}^m$ with $n_K(u) \in \{0, 1, \ldots, k\beta_K\}$, with the pins $n_K(x^{(i)}) = k\beta_K[i \in J_K]$ and $n(0) = 0$, and with $\sum_K |n_K(u) - n_K(v)| \le k d_\alpha(u, v)$ for all $u, v \in \{0,1\}^L$.

THEOREM 10.13 (Scaling spectrum of star inequalities). *Let Q be a star inequality and $k \ge 1$. Then kQ has an integer map on the full cube if and only if $\Pi(Q) \cap \frac{1}{k}\mathbb{Z}^m \ne \emptyset$. Consequently, if $\Pi(Q) \ne \emptyset$, then Q is valid and*

$$k_{\min}(Q) = \min\left\{k \ge 1\colon \Pi(Q) \cap \tfrac{1}{k}\mathbb{Z}^m \ne \emptyset\right\} < \infty , \tag{10.14}$$

and if $\Pi(Q) = \emptyset$, then no kQ has an integer map on the full cube.

Status: argued.

PROOF. *Step 1: one cube for all radii.* The LHS terms of Q are the singletons $\{i\}$ with $\rho_i > 0$ and the set $[N]$ if $\rho_O > 0$, since a term with coefficient 0 is absent. It is convenient to keep one bit for every singleton and one bit $\star$ for $[N]$ also when the corresponding radius is 0. So let

$$\mathcal{B} = \{0,1\}^{[N]\cup\{\star\}} , \qquad d(x, x') = \sum_{i\in[N]} \rho_i |x_i - x'_i| + \rho_O |x_\star - x'_\star| , \tag{10.15}$$

and write e_i, $e_\star$ for the unit vectors of $\mathcal{B}$. A party i lies in $\{i\}$ and in $[N]$, so its occurrence word in $\mathcal{B}$ is $e_i + e_\star$, and the purifier's is 0. We claim that kQ has an integer map on its full cube if and only if there is a map $n\colon \mathcal{B} \to \prod_{K=1}^m \{0, 1, \ldots, k\beta_K\}$ with

$$n(e_i + e_\star) = k\eta^{(i)} \ \ (i \in [N]) , \qquad n(0) = 0 , \qquad \|n(x) - n(x')\|_1 \le k d(x, x') \ \ (x, x' \in \mathcal{B}) . \tag{10.16}$$

If all radii are positive, $\mathcal{B}$ is the full cube of Q with $d = d_\alpha$, and (10.16) is the definition of an integer map for kQ. Otherwise let π delete the bits of weight 0. It maps $\mathcal{B}$ onto the full cube of Q, maps $e_i + e_\star$ to the occurrence word $x^{(i)}$ (whose bits are exactly the LHS terms that contain i) and 0 to 0, and satisfies $d(x, x') = d_\alpha(\pi x, \pi x')$. So an integer map n' for kQ gives the map $n = n' \circ \pi$ satisfying (10.16). Conversely, let n satisfy (10.16). If bit l has weight 0, then $\|n(x) - n(x \oplus e_l)\|_1 \le k \cdot 0 = 0$,

where $x \oplus e_l$ is x with bit l flipped. So n does not depend on the bits of weight 0, and $n = n' \circ \pi$ for a well-defined map n' on the full cube of Q. It has the pins of an integer map for kQ, and the contraction condition, because $u = \pi x$ and $v = \pi x'$ give $d_\alpha(u, v) = d(x, x')$.

Step 2: edges suffice. Bao et al. observe that the contraction condition needs to be checked only on the edges of the cube $[\mathbf{1}]$, and the same holds for the weights d. If x and x' differ in the bits $l_1, \dots, l_s$, the walk $x = x_0, x_1, \dots, x_s = x'$ that flips these bits one at a time has $d(x, x') = \sum_{t=1}^{s} d(x_{t-1}, x_t)$, because each step changes one bit and the weights add. By the triangle inequality for $\|\cdot\|_1$, $\|n(x) - n(x')\|_1 \le \sum_t \|n(x_{t-1}) - n(x_t)\|_1$. So the contraction condition in (10.16) holds for all pairs once it holds on every edge of $\mathcal{B}$, that is, once

$$\|n(x) - n(x \oplus e_l)\|_1 \le k\rho_l \qquad \text{for all } x \in \mathcal{B} \text{ and } l \in [N] \cup \{\star\} \ , \tag{10.17}$$

where $\rho_\star = \rho_O$.

Step 3: a map gives a grid point. Let n satisfy (10.16), and let $z = e_\star$. The point z differs from $e_i + e_\star$ only in bit i, and from 0 only in bit $\star$, so $d(z, e_i + e_\star) = \rho_i$ for $i \in [N]$ and $d(z, 0) = \rho_O$. The contraction condition at these $N + 1$ pairs, with the pins, gives

$$\|n(z) - k\eta^{(i)}\|_1 \le k\rho_i \quad (i \in [N]) \ , \qquad \|n(z)\|_1 \le k\rho_O \ . \tag{10.18}$$

Put $c = n(z)/k$. Dividing (10.18) by k shows that c satisfies all constraints of (10.3). Since $n(z) \in \prod_K \{0, \dots, k\beta_K\}$, the point c lies in the box and $kc = n(z) \in \mathbb{Z}^m$. So $c \in \Pi(Q) \cap \frac{1}{k}\mathbb{Z}^m$.

Step 4: a grid point gives a map. Let $c \in \Pi(Q)$ with $kc \in \mathbb{Z}^m$. Then

$$a_K = k(\beta_K - c_K) \ , \qquad b_K = kc_K \tag{10.19}$$

are nonnegative integers with $a_K + b_K = k\beta_K$. For each K define two functions $\mathcal{B} \to \{0, 1\}$,

$$\phi^1_K(x) = [x_i = 1 \text{ for some } i \in J_K] \ , \qquad \phi^2_K(x) = [x_\star = 1 \text{ and } x_i = 0 \text{ for all } i \in [N] \setminus J_K] \ , \tag{10.20}$$

and put

$$n_K(x) = a_K \phi^1_K(x) + b_K \phi^2_K(x) \ . \tag{10.21}$$

Then $n_K(x) \in \{0, a_K, b_K, k\beta_K\} \subseteq \{0, \dots, k\beta_K\}$. For the pins, at $x = e_i + e_\star$ the only singleton bit that is set is i, so $\phi^1_K = [i \in J_K]$, and $\phi^2_K = [i \notin [N] \setminus J_K] = [i \in J_K]$. Hence $n_K(e_i + e_\star) = (a_K + b_K)[i \in J_K] = k\eta^{(i)}_K$. At $x = 0$ both functions vanish, so $n(0) = 0$. For the edges, ϕ^1_K depends only on the bits $i \in J_K$, and ϕ^2_K only on the bits $i \in [N] \setminus J_K$ and $\star$, and flipping one bit changes each function by at most 1. So flipping a bit $i \in [N]$ changes n_K by at most a_K if $i \in J_K$, where $\eta^{(i)}_K = \beta_K$, and by at most b_K if $i \notin J_K$, where $\eta^{(i)}_K = 0$. Summing over K,

$$\|n(x) - n(x \oplus e_i)\|_1 \le \sum_{K\colon i \in J_K} k(\beta_K - c_K) + \sum_{K\colon i \notin J_K} kc_K = k \sum_K \left|\eta^{(i)}_K - c_K\right| = k\|c - \eta^{(i)}\|_1 \le k\rho_i \ . \tag{10.22}$$

Flipping $\star$ changes only the ϕ^2_K, so

$$\|n(x) - n(x \oplus e_\star)\|_1 \le \sum_K kc_K = k\|c - \eta^{(O)}\|_1 \le k\rho_O \ . \tag{10.23}$$

This is (10.17), so by Steps 2 and 1, kQ has an integer map on the full cube.

Step 5: consequences. If $\Pi(Q) \neq \emptyset$, then Q is valid by Lemma 10.9. By Lemma 10.4, $\Pi(Q)$ is defined by finitely many linear inequalities with integer coefficients and integer right-hand sides, namely the box constraints $0 \le c_K \le \beta_K$ and the rows (10.6), and it is bounded. A nonempty bounded polyhedron has a vertex, and a vertex is the unique solution of m linearly independent constraints taken with equality. By Cramer's rule this solution is rational, so it lies in $\frac{1}{k}\mathbb{Z}^m$ for k a common denominator of its coordinates. Hence the minimum in (10.14) exists, and by Steps 3 and 4 it equals $k_{\min}(Q)$. If $\Pi(Q) = \emptyset$, Step 3 shows that no kQ has an integer map. □

For a star inequality with an empty polytope, the theorem states only that no multiple of it has an integer map on the full cube, and it leaves the validity of such an inequality undecided.

When the polytope is a single point the theorem takes a simple form. If $\Pi(Q) = \{c\}$, the set of k for which kQ has an integer map is the set of k with $kc \in \mathbb{Z}^m$, which is the set of multiples of the least common denominator D of the coordinates of c (written in lowest terms). So kQ has an integer map exactly when D divides k, and

$$k_{\min}(Q) = D \ . \tag{10.24}$$

All the examples below are of this kind.

Remark 10.25 (The maps of the subadditivity chains). With a single copy of $S(J_K)$, that is, $\beta_K = 1$, $a_K = 1$ and $b_K = 0$, the function ϕ^1_K is the standard contraction map that proves SA^1_K: the output bit is 1 as soon as one of the regions $i \in J_K$ is inside. Likewise ϕ^2_K proves SA^2_K in the purified picture, where $S(J_K)$ is the entropy of $([N] \setminus J_K) \cup \{O\}$ and the output bit is 1 exactly when the point lies in the region $[N]$ and in none of the regions $i \notin J_K$. In the language of the sorting lemma (Lemma 4.15), the count (10.21) uses a_K of the $k\beta_K$ unit copies of $S(J_K)$ for the first map and b_K for the second, which are exactly the multiplicities of the subadditivity chains in k times the decomposition (10.10). So the map of Step 4 is the contraction-map form of the identity of Lemma 10.9.

Physics aside 10.26. A graph vertex whose word is $z = e_\star$ lies in the minimal min cut of $[N]$ and in none of the minimal min cuts of the single regions. In the bulk picture it sits in the entanglement wedge of the union of all regions, outside the wedge of every single region. The value $n(z)/k = c$ says which fraction of the candidate homology regions for each $S(J_K)$ contains such a point, and the theorem says that a consistent choice of these fractions is all that a contraction proof of a star inequality needs.

10.5. The smallest example

Example 10.27. The three-party star inequality

$$Q' : \qquad A + B + C + ABC \ \geq \ AB + AC \tag{10.28}$$

has $m = 2$, $J_1 = AB$, $J_2 = AC$, $\beta = (1, 1)$ and $|\beta| = 2$, all radii equal to 1 (the coefficients of $S(A)$, $S(B)$, $S(C)$ and of $S(ABC) = S([N])$), and the words $\eta^{(A)} = (1, 1)$, $\eta^{(B)} = (1, 0)$, $\eta^{(C)} = (0, 1)$ and $\eta^{(O)} = (0, 0)$ on (AB, AC). For $c = (c_1, c_2) \in [0, 1]^2$ the four constraints of (10.3) read, by (10.5),

$$(1 - c_1) + (1 - c_2) \leq 1 \ , \qquad (1 - c_1) + c_2 \leq 1 \ , \qquad c_1 + (1 - c_2) \leq 1 \ , \qquad c_1 + c_2 \leq 1 \ . \tag{10.29}$$

The first and the last give $c_1 + c_2 = 1$, and the middle two give $c_1 = c_2$. So

$$\Pi(Q') = \left\{ \left(\tfrac{1}{2}, \tfrac{1}{2}\right) \right\} \ . \tag{10.30}$$

This is the mechanism of Lemma 10.4: the pairs A, O and B, C have complementary words with radii adding up to $|\beta| = 2$. By (10.24), kQ' has an integer map exactly for even k, and $k_{\min}(Q') = 2$.

In this case the theorem can be checked directly at the point z. With the LHS terms in the order A, B, C, ABC, the occurrence words are $x^{(A)} = 1001$, $x^{(B)} = 0101$, $x^{(C)} = 0011$ and $x^{(O)} = 0000$, and $z = 0001$ differs from each of them in one bit of weight 1. For an integer map n of kQ', the contraction condition at these four pairs says that $n(z) \in \{0, \ldots, k\}^2$ lies within ℓ_1-distance k of all four corners (k, k), $(k, 0)$, $(0, k)$, $(0, 0)$ of $[0, k]^2$, which forces $n(z) = (\frac{k}{2}, \frac{k}{2})$. For $k = 1$ this is visible at once: $n(z) \in \{0, 1\}^2$ is one of the four corners, and the opposite corner is at distance $2 > 1$.

For $k = 2$ the recipe of Step 4 with $c = (\frac{1}{2}, \frac{1}{2})$ uses $a_K = b_K = 1$. Here $[N] \setminus J_1 = \{C\}$ and $[N] \setminus J_2 = \{B\}$, so the map is

$$\begin{aligned} n_1(x) &= [x_A = 1 \text{ or } x_B = 1] + [x_\star = 1 \text{ and } x_C = 0] \ , \\ n_2(x) &= [x_A = 1 \text{ or } x_C = 1] + [x_\star = 1 \text{ and } x_B = 0] \ . \end{aligned} \tag{10.31}$$

Its sixteen values, written $x_A x_B x_C x_\star \mapsto n_1 n_2$, are

$0000 \mapsto 00$	$0001 \mapsto 11$	$0010 \mapsto 01$	$0011 \mapsto 02$
$0100 \mapsto 10$	$0101 \mapsto 20$	$0110 \mapsto 11$	$0111 \mapsto 11$
$1000 \mapsto 11$	$1001 \mapsto 22$	$1010 \mapsto 11$	$1011 \mapsto 12$
$1100 \mapsto 11$	$1101 \mapsto 21$	$1110 \mapsto 11$	$1111 \mapsto 11$

The pins are $n(1001) = (2,2) = 2\eta^{(A)}$, $n(0101) = (2,0) = 2\eta^{(B)}$, $n(0011) = (0,2) = 2\eta^{(C)}$ and $n(0000) = (0,0)$, and the value at the center is $n(0001) = (1,1) = 2c$. Each n_K is a sum of two brackets, so its values lie in $\{0,1,2\}$. Flipping x_A can change only the two brackets with x_A, flipping x_B only $[x_A = 1 \text{ or } x_B = 1]$ and $[x_\star = 1 \text{ and } x_B = 0]$, flipping x_C only the two brackets with x_C, and flipping $x_\star$ only the two brackets with $x_\star$. Each bracket changes by at most 1, so every one of the 32 edges of $\{0,1\}^4$ has $\|n(x) - n(x')\|_1 \le 2 = 2\alpha_l$, and by Step 2 the map is an integer map for $2Q'$. An edge is tight, with $\|n(x) - n(x')\|_1 = 2$, exactly when both brackets that contain the flipped bit change. For a flip of x_A or of $x_\star$ this happens when $x_B = x_C = 0$, and for a flip of x_B or of x_C when $x_A = 0$ and $x_\star = 1$. Each condition fixes two of the three other bits, so 8 of the 32 edges are tight.

The same fact as an identity: the four subadditivity chains are SA_1^1: $A + B \ge AB$, SA_1^2: $C + ABC \ge AB$ (which is $S(C) + S(O) \ge S(CO)$ with $S(CO) = S(AB)$), SA_2^1: $A + C \ge AC$ and SA_2^2: $B + ABC \ge AC$. Their sum is $2A + 2B + 2C + 2ABC \ge 2AB + 2AC$, which is $2Q'$. So $2Q'$ is a sum of four instances of subadditivity, and the map (10.31) is the sum of their four maps, one copy each.

We do not present Q' as new. It is the smallest case of the mechanism of Chapter 11.

Status: certified (DRAT, 3 encoders, for $k = 1, 3$; integer maps for $k = 2, 4$), and also proved by hand above.

CHAPTER 11

Coin weighing and large least scaling

Where we are. By (10.24), a star inequality whose polytope is a single point c has $k_{\min}$ equal to the least common denominator of c. To make $k_{\min}$ large we therefore need star polytopes that are single points with large denominators. This chapter reads the constraints of a star inequality as weighings of coins on a balance and proves a general theorem: when a list of weighings determines the weights of the coins up to a common factor, the corresponding star inequality has $k_{\min}$ equal to the total weight of the coins, written as coprime integers. It then builds an explicit family D_j on $N = 4j + 7$ parties with $k_{\min}(D_j) = 2^{j+2} = 2^{(N+1)/4}$, a seven-party example Z_7 with $k_{\min}(Z_7) = 8$, and further examples on 9 to 17 parties. With the weighing matrices of Alon and Vũ it shows that $k_{\min}$ of an N-party star inequality, a class implied by subadditivity and not balanced, can be as large as $N^{(1/4-o(1))N}$, and it proves that it is never larger than $N^{(1/2+o(1))N}$ when it is finite. All the inequalities of this chapter are implied by subadditivity and are not balanced (Section 11.12).

11.1. Coin weighing

A classical kind of puzzle gives a set of coins and a balance with two pans and asks what a few weighings reveal about the weights of the coins. A weighing puts some of the coins on the left pan and the others on the right pan and reports whether the pans balance. The constraints of a star inequality can be read in this language.

Take all RHS coefficients equal to 1, so that the box is $[0,1]^m$ and every word is the indicator vector $\mathbf{1}_X \in \{0,1\}^m$ of a set $X \subseteq [m]$ of coordinates. We call the coordinates $K \in [m]$ *coins*, write $\overline{X} = [m] \setminus X$, $\mathbf{1} = \mathbf{1}_{[m]}$ and $\xi(X) = \sum_{K \in X} \xi_K$ for $\xi \in \mathbb{R}^m$, and read ξ as an assignment of weights to the coins. A set $T \subseteq [m]$ stands for the weighing with the coins of T on one pan and those of $\overline{T}$ on the other, and ξ *balances* T if $\xi(T) = \xi(\overline{T})$. This use of the word balance for weighings is unrelated to balanced inequalities in the sense of Chapter 3. For $\xi \in [0,1]^m$ and $X \subseteq [m]$, Lemma 10.4 gives

$$\|\xi - \mathbf{1}_X\|_1 = \sum_{K \notin X} \xi_K + \sum_{K \in X} (1 - \xi_K) = \|\xi\|_1 + |X| - 2\xi(X) \ . \tag{11.1}$$

On the box $\|\xi\|_1 = \xi(T) + \xi(\overline{T})$, so the constraint of a party with word $\mathbf{1}_T$ and radius $|T|$ reads

$$\|\xi - \mathbf{1}_T\|_1 \le |T| \iff \xi(T) + \xi(\overline{T}) - 2\xi(T) \le 0 \iff \xi(\overline{T}) \le \xi(T) \ : \tag{11.2}$$

the pan with the coins of T is at least as heavy as the other.

Two mechanisms turn these one-sided constraints into equalities. The first is a complementary pair. A party with word $\mathbf{1}_T$ and radius $|T|$ and a party with word $\mathbf{1}_{\overline{T}}$ and radius $|\overline{T}|$ demand $\xi(\overline{T}) \le \xi(T)$ and $\xi(T) \le \xi(\overline{T})$, so every point of the polytope balances T. This is the equality statement of Lemma 10.4, read as a weighing. In the same way the purifier, with word 0 and radius 1, and a party with word $\mathbf{1}$ and radius $m-1$ fix the total weight, $\|\xi\|_1 = 1$. The second mechanism needs no pairs. Suppose the weighings $T_1, T_2, \dots$ of the parties can be given positive weights $\gamma_1, \gamma_2, \dots$ such that, for every coin, the weights of the weighings that put it on one pan add up to the same total as the weights of those that put it on the other pan. Then the weighted sum of the differences $\xi(T_i) - \xi(\overline{T_i})$ vanishes for every ξ, because each coin enters it with coefficient

zero, and a vanishing sum of nonnegative terms with positive weights has every term equal to zero. So again every point of the polytope balances every weighing.

Either way, the points of the polytope are weight distributions of total weight 1 that balance every listed weighing. If the weighings determine the weights up to a common factor, the polytope is a single point, and its least common denominator is the total weight of the coins when the weights are written as coprime integers. In the reading of Kozlov and Vũ [**37**] and of Alon and Vũ [**3**], where the kernel entries of a matrix of weighings count coins of each kind rather than weigh them, the same total, the least ℓ_1 norm of a nonzero integer kernel vector, is the number of coins whose equality the weighings decide. Section 11.2 makes this precise. A doubling recursion then gives the family D_j, in which the lightest coin carries the fraction $2^{-(j+2)}$ of the total while the number of parties grows linearly in j (Sections 11.4 and 11.5). The ill-conditioned weighing matrices of Alon and Vũ [**3**] do much better (Section 11.9).

11.2. Coin-weighing stars

The following theorem allows any number of weighings, lets a set occur several times in the list, and needs no complementary pairs of weighings. Only the purifier and one party with the word $\mathbf{1}$ form a pair, which fixes the total weight.

THEOREM 11.3 (Coin-weighing stars). *Let $m \geq 2$ and $N \geq 3$, and let $\mathcal{T} = (T_1, \dots, T_{N-1})$ be a list of subsets of $[m]$, in which a set may occur more than once, such that*

(W1) *the vectors $\xi \in \mathbb{R}^m$ that balance every T_i form a line $\mathbb{R}\omega$, where $\omega \in \mathbb{Z}^m$ has positive entries without a common factor, and*

(W2) *there are $\gamma_1, \dots, \gamma_{N-1} > 0$ with $\sum_{i=1}^{N-1} \gamma_i(\mathbf{1} - 2 \cdot \mathbf{1}_{T_i}) = 0$.*

Put $q = \sum_K \omega_K$, and let $Q_{\mathcal{T}}$ be the inequality (10.2) with $\beta = \mathbf{1}$ in which the party $i \leq N-1$ has the word $\mathbf{1}_{T_i}$ and the radius $|T_i|$, the party N has the word $\mathbf{1}$ and the radius $m-1$, and the purifier has the radius 1, so that J_K is the set of parties whose word has a 1 at coin K. Then:

(a) $Q_{\mathcal{T}}$ is a star inequality on N parties with m RHS terms, and $\Pi(Q_{\mathcal{T}}) = \{\omega/q\}$;

(b) $qQ_{\mathcal{T}} = \sum_{K=1}^{m} \big[(q - \omega_K)\,\mathrm{SA}_K^1 + \omega_K\,\mathrm{SA}_K^2\big]$, so $Q_{\mathcal{T}}$ is valid, indeed for all quantum states;

(c) $kQ_{\mathcal{T}}$ has an integer map on the full cube if and only if q divides k, and $k_{\min}(Q_{\mathcal{T}}) = q$.

Status: argued.

Before the proof, two remarks on the conditions. Coordinate K of the identity in (W2) reads $\sum_{i:\,K \notin T_i} \gamma_i = \sum_{i:\,K \in T_i} \gamma_i$, so (W2) is the condition of the second mechanism above: with the weights γ_i, every coin is on each pan equally often. Condition (W1) is a statement of linear algebra. Since $(\mathbf{1} - 2 \cdot \mathbf{1}_T) \cdot \xi = \xi(\overline{T}) - \xi(T)$, a vector balances T_i exactly when it is orthogonal to the vector $\mathbf{1} - 2 \cdot \mathbf{1}_{T_i}$, which has entry -1 on the coins of T_i and $+1$ on the others. So (W1) says that the $(N-1) \times m$ matrix with these rows has rank $m-1$ and a kernel spanned by a vector with positive entries. Two kinds of lists satisfy (W2) with all $\gamma_i = 1$. A *paired* list consists of sets $T_1, \dots, T_n$ and their complements $\overline{T_1}, \dots, \overline{T_n}$, in some order, and (W2) holds because $(\mathbf{1} - 2 \cdot \mathbf{1}_T) + (\mathbf{1} - 2 \cdot \mathbf{1}_{\overline{T}}) = 2 \cdot \mathbf{1} - 2 \cdot \mathbf{1} = 0$. In a *half-regular* list every coin lies in exactly half of the sets.

PROOF. *Step 1: the radii are positive.* The vector ω balances every T_i by (W1), and its entries are positive, so $\omega(T_i) = \omega(\overline{T_i}) = q/2 > 0$. Hence neither T_i nor $\overline{T_i}$ is empty, the radii $|T_i|$ are positive integers, and so is $m - 1 \geq 1$.

Step 2: the sizes of the J_K. The set J_K consists of the party N and the parties $i \leq N-1$ with $K \in T_i$. If K lay in no T_i, coordinate K of (W2) would read $\sum_i \gamma_i = 0$, which is false because all γ_i are positive. So K lies in some T_i, and $|J_K| \geq 2$. If K lay in every T_i, the same coordinate would read $0 = \sum_i \gamma_i$, again false. So some party $i \leq N-1$ is not in J_K, and $|J_K| \leq N-1$.

Step 3: the J_K are distinct. If $J_K = J_{K'}$ for two coins $K \neq K'$, then K and K' lie in exactly the same sets T_i. The vector $\mathbf{1}_{\{K\}} - \mathbf{1}_{\{K'\}}$, which puts weight 1 on K and -1 on K', then balances every T_i: both coins are on the same pan in every weighing, and their weights cancel. By (W1) it would be a multiple of ω, which is impossible because ω has positive entries. Distinctness is needed. If two coins had $J_K = J_{K'}$, the inequality written down would have the single term $2S(J_K)$ in place of two terms $S(J_K)$, it would be a different star inequality, with one coordinate in $[0, 2]$ in place of two coordinates in $[0, 1]$, and the computation of the polytope below would not apply to it. With Steps 1 to 3, $Q_{\mathcal{T}}$ satisfies Definition 10.1: it is a star inequality on N parties with m RHS terms.

Step 4: the point $c = \omega/q$ attains every radius. Since $m \geq 2$ and the entries of ω are positive, every entry of c lies strictly between 0 and 1, so c lies in the box. Also $\|c\|_1 = q/q = 1$, and $c(T_i) = c(\overline{T_i}) = \frac{1}{2}$ because ω balances T_i. By (11.1),

$$\|c - \mathbf{1}_{T_i}\|_1 = 1 + |T_i| - 1 = |T_i| \ , \qquad \|c - \mathbf{1}\|_1 = 1 + m - 2 = m - 1 \ , \qquad \|c - 0\|_1 = 1 \ . \tag{11.4}$$

So $c \in \Pi(Q_{\mathcal{T}})$, and every constraint holds with equality at c.

Step 5: the polytope is the point c. Let $\xi \in \Pi(Q_{\mathcal{T}})$. The words 0 of the purifier and $\mathbf{1}$ of the party N are complementary, with radii adding up to $1 + (m-1) = m = |\beta|$, so by Lemma 10.4 both constraints hold with equality, and $\|\xi\|_1 = 1$. By (11.2) the constraint of a party $i \leq N-1$ says $\xi(\overline{T_i}) - \xi(T_i) \leq 0$. Taking the scalar product of the identity in (W2) with ξ gives

$$\sum_{i=1}^{N-1} \gamma_i \big(\xi(\overline{T_i}) - \xi(T_i)\big) = 0 \ . \tag{11.5}$$

Every term is a positive number γ_i times a number ≤ 0, and the sum vanishes, so every term vanishes: ξ balances every T_i. By (W1), $\xi = \kappa\omega$ for some $\kappa \in \mathbb{R}$, and $\|\xi\|_1 = \kappa q = 1$ (the entries of ξ are nonnegative on the box) gives $\kappa = 1/q$. So $\xi = c$.

Step 6: the identity. Lemma 10.9 at c gives (10.10) with all positivity coefficients equal to 0, because every radius is attained at c by (11.4). Multiplying by q and using $qc_K = \omega_K$ and $\beta_K = 1$ gives

$$qQ_{\mathcal{T}} = \sum_{K=1}^{m} \left[(q - \omega_K)\, \mathrm{SA}_K^1 + \omega_K\, \mathrm{SA}_K^2 \right] \ , \tag{11.6}$$

a combination with nonnegative integer coefficients of the subadditivity chains (10.7), which hold for all quantum states. This is (b).

Step 7: the least scaling. By (a) and (10.24), $kQ_{\mathcal{T}}$ has an integer map on the full cube exactly when $kc = k\omega/q \in \mathbb{Z}^m$, that is, when q divides $k\omega_K$ for every K. If q divides k, this holds. Conversely, since the ω_K have no common factor, some integer combination of them equals 1 (Bézout's identity). If q divides every $k\omega_K$, it divides k times that combination, which is k. So $kQ_{\mathcal{T}}$ has an integer map exactly when q divides k, and $k_{\min}(Q_{\mathcal{T}}) = q$. □

Remark 11.7 (What just happened). The pair formed by the purifier and the party N fixes the total weight. Condition (W2) makes every one-sided pan constraint an equality, condition (W1) then fixes the weights up to scale, and the total weight fixes the scale. The least scaling is the least common denominator of the weights divided by their total, ω/q, which is q itself because the weights have no common factor. The same two conditions also give the size and distinctness conditions of a star inequality.

11.3. A worked example: H_9

Take $m = 8$ coins and eight weighings, named after the parties $A, \dots, H$ that will carry them, and the weight vector

$$\omega = (1,1,2,2,2,3,3,4)\ , \qquad q = \sum_K \omega_K = 18\ . \tag{11.8}$$

The weighings, their sizes and their weights under ω are

party	coins of T	$\lvert T\rvert$	$\omega(T)$
A	$\{5,7,8\}$	3	$2+3+4=9$
B	$\{1,2,3,5,7\}$	5	$1+1+2+2+3=9$
C	$\{2,4,5,8\}$	4	$1+2+2+4=9$
D	$\{1,4,6,7\}$	4	$1+2+3+3=9$
E	$\{1,3,4,8\}$	4	$1+2+2+4=9$
F	$\{2,3,6,7\}$	4	$1+2+3+3=9$
G	$\{1,2,6,8\}$	4	$1+1+3+4=9$
H	$\{3,4,5,6\}$	4	$2+2+2+3=9$

Every set carries $9 = q/2$, so ω balances all eight weighings.

Condition (W2). Coin 1 lies in the sets of B, D, E, G, coin 2 in those of B, C, F, G, coin 3 in B, E, F, H, coin 4 in C, D, E, H, coin 5 in A, B, C, H, coin 6 in D, F, G, H, coin 7 in A, B, D, F, and coin 8 in A, C, E, G. Every coin lies in exactly four of the eight sets, so the list is half-regular, and (W2) holds with all $\gamma_i = 1$: coordinate K of $\sum_i (\mathbf{1} - 2\cdot\mathbf{1}_{T_i})$ is $4-4=0$.

Condition (W1) *by hand.* Let $\xi \in \mathbb{R}^8$ balance all eight sets. Then each of the eight sets and each of their complements carries $\frac{1}{2}\xi([8])$, so any two of these sixteen sets carry the same weight. We use the complement $\overline{T_H} = \{1,2,7,8\}$ of the set of H and compare pairs. The sets of G and $\overline{T_H}$ give $\xi_1+\xi_2+\xi_6+\xi_8 = \xi_1+\xi_2+\xi_7+\xi_8$, so $\xi_6 = \xi_7$. The sets of A and $\overline{T_H}$ give $\xi_5+\xi_7+\xi_8 = \xi_1+\xi_2+\xi_7+\xi_8$, so $\xi_5 = \xi_1+\xi_2$. The sets of C and E give $\xi_2+\xi_5 = \xi_1+\xi_3$, and with the previous relation $\xi_3 = 2\xi_2$. The sets of D and F give $\xi_1+\xi_4 = \xi_2+\xi_3 = 3\xi_2$, so $\xi_4 = 3\xi_2 - \xi_1$. The sets of E and $\overline{T_H}$ give $\xi_3+\xi_4 = \xi_2+\xi_7$, so

$$\xi_7 = \xi_3+\xi_4-\xi_2 = 4\xi_2-\xi_1 = \xi_6\ . \tag{11.9}$$

The sets of B and A give $\xi_1+\xi_2+\xi_3 = \xi_8$, so $\xi_8 = \xi_1+3\xi_2$. Finally the set of A and its complement $\{1,2,3,4,6\}$ give

$$(\xi_1+\xi_2)+(4\xi_2-\xi_1)+(\xi_1+3\xi_2) = \xi_1+\xi_2+2\xi_2+(3\xi_2-\xi_1)+(4\xi_2-\xi_1)\ , \tag{11.10}$$

that is, $\xi_1+8\xi_2 = -\xi_1+10\xi_2$, so $\xi_1 = \xi_2$. Substituting, $\xi = \xi_1(1,1,2,2,2,3,3,4) = \xi_1\omega$. Every vector that balances the eight weighings is a multiple of ω, ω balances them, and $\omega_1 = 1$, so (W1) holds.

The inequality. By the theorem, the parties $A, \dots, H$ get the words $\mathbf{1}_T$ of the table with the radii $\lvert T\rvert$, the party I gets the word $\mathbf{1}$ and the radius 7, and the purifier gets the radius 1. The RHS term of coin K is the set of parties whose set contains K, together with I. With the RHS terms in the order of the coins,

$$\begin{aligned} H_9:\quad & 3A+5B+4C+4D+4E+4F+4G+4H+7I+ABCDEFGHI \\ &\geq\ BDEGI+BCFGI+BEFHI+CDEHI \\ &\quad +ABCHI+DFGHI+ABDFI+ACEGI\ . \end{aligned} \tag{11.11}$$

Each RHS term has five parties, and the eight terms are distinct, as Step 3 of the proof guarantees.

Consequences. By the theorem, $\Pi(H_9) = \{\frac{1}{18}(1,1,2,2,2,3,3,4)\}$, kH_9 has an integer map on the full cube exactly when 18 divides k, and $k_{\min}(H_9) = 18$, while $2^{(N+1)/4} = 2^{5/2} \approx 5.7$

at $N = 9$. The identity (11.6) reads $18H_9 = \sum_K [(18-\omega_K)\,\mathrm{SA}^1_K + \omega_K\,\mathrm{SA}^2_K]$ with $18-\omega = (17,17,16,16,16,15,15,14)$. For instance, $S(A)$ gets $16+15+14 = 45$ from SA^1_5, SA^1_7, SA^1_8, since A lies in J_5, J_7, J_8, and $1+1+2+2+3 = 9$ from SA^2_K for $K = 1,2,3,4,6$, in total $54 = 18\cdot 3$. The term $S(I)$ gets $\sum_K (18-\omega_K) = 144-18 = 126 = 18\cdot 7$, and $S(ABCDEFGHI)$ gets $\sum_K \omega_K = 18$.

Certificates. Independently of the theorems, DRAT proofs from two encoders show that kH_9 has no integer map for $k \le 17$ on the 11 points 0, z and the occurrence words, to which every map on the full cube restricts, and an integer map for $18H_9$ on the full cube of $2^{10} = 1{,}024$ points is checked exactly by three checkers.

Status of $k_{\min}(H_9) = 18$: certified (DRAT, 2 encoders, for $k \le 17$; integer map for $k = 18$), and also a consequence of the argued Theorem 11.3.

11.4. Coins and halves

Put $\omega^{(0)} = (1,1,1,1)$ and, recursively,

$$\omega^{(j+1)} = \big(\omega^{(j)}, \tfrac{q_j}{2}, \tfrac{q_j}{2}\big)\ , \qquad q_j = \sum_K \omega^{(j)}_K\ . \tag{11.12}$$

We call the coordinates *coins* and $\omega^{(j)}_K$ the weight of coin K. Level $j+1$ keeps the old coins and adds the two coins $u_j = 2j+5$ and $v_j = 2j+6$, each weighing half the total of the old coins. (The letters u_j, v_j are coin labels and are used only in this section.) The *halves* form the list $\mathcal{T}_0 = (\{1,2\},\{1,3\},\{1,4\})$ and, recursively, the list $\mathcal{T}_{j+1}$ of the sets

$$T \cup \{u_j\}\ \ (T \in \mathcal{T}_j, \text{ in order})\ , \qquad T^{(j)}_1 \cup \{v_j\}\ , \qquad \{u_j, v_j\}\ , \tag{11.13}$$

where $T^{(j)}_1$ is the first member of $\mathcal{T}_j$. At level j the coins are $[m_j]$, and $\overline{T}$, $\mathbf{1}_T$ and $\mathbf{1}$ are as in Section 11.1 with $m = m_j$. For a weight vector ω we write $\omega(T) = \sum_{K\in T}\omega_K$, as for ξ.

The first three levels are as follows. At level 0 there are four coins of weight 1, $q_0 = 4$, and the three halves $\{1,2\}$, $\{1,3\}$, $\{1,4\}$ weigh 2 each. At level 1 the coins $u_0 = 5$ and $v_0 = 6$ are added with weight $q_0/2 = 2$, so

$$\omega^{(1)} = (1,1,1,1,2,2)\ , \qquad q_1 = 8\ , \qquad \mathcal{T}_1 = \big(\{1,2,5\},\{1,3,5\},\{1,4,5\},\{1,2,6\},\{5,6\}\big)\ , \tag{11.14}$$

and each half weighs 4. At level 2 the coins $u_1 = 7$ and $v_1 = 8$ are added with weight $q_1/2 = 4$, so

$$\begin{aligned} &\omega^{(2)} = (1,1,1,1,2,2,4,4)\ , \qquad q_2 = 16\ , \\ &\mathcal{T}_2 = \big(\{1,2,5,7\},\{1,3,5,7\},\{1,4,5,7\},\{1,2,6,7\},\{5,6,7\},\{1,2,5,8\},\{7,8\}\big)\ , \end{aligned} \tag{11.15}$$

and each of the seven halves weighs 8.

Lemma 11.16. *For every $j \ge 0$, $\omega^{(j)}$ has $m_j = 2j+4$ entries, $q_j = 2^{j+2}$, $\omega^{(j)}_1 = 1$, the list $\mathcal{T}_j$ has $m_j - 1$ members, and every $T \in \mathcal{T}_j$ has $\omega^{(j)}(T) = q_j/2$.*

Proof. By induction on j. For $j = 0$ we have $m_0 = 4$, $q_0 = 4$, $\omega^{(0)}_1 = 1$, three halves, and each half has two coins of weight 1. Assume the statement for j. Two coins are added, so $m_{j+1} = m_j + 2 = 2(j+1)+4$. The total weight is $q_{j+1} = q_j + \frac{q_j}{2} + \frac{q_j}{2} = 2q_j = 2^{j+3}$, and coin 1 is old and keeps the weight 1. The list $\mathcal{T}_{j+1}$ has the $m_j - 1$ lifted halves and two new ones, so $(m_j - 1) + 2 = m_{j+1} - 1$ members. A lifted half $T \cup \{u_j\}$ weighs $\frac{q_j}{2} + \frac{q_j}{2} = q_j$, the half $T^{(j)}_1 \cup \{v_j\}$ weighs $\frac{q_j}{2} + \frac{q_j}{2} = q_j$, and $\{u_j, v_j\}$ weighs q_j. Since $q_j = q_{j+1}/2$, every member of $\mathcal{T}_{j+1}$ weighs $q_{j+1}/2$. □

11.5. The inequality D_j

Fix j and write $m = m_j$, $q = q_j$ and $\omega = \omega^{(j)}$. The inequality D_j is the coin-weighing star $Q_{\mathcal{T}}$ of Theorem 11.3 for the paired list $\mathcal{T}$ in which each half $T \in \mathcal{T}_j$, in order, is followed by its complement $\overline{T}$. Written out, it is the star inequality with $\beta = \mathbf{1}$ whose parties are given through their words. For each half $T \in \mathcal{T}_j$, in order, there is one party with word $\mathbf{1}_T$ and radius $|T|$ and one party with word $\mathbf{1}_{\overline{T}}$ and radius $|\overline{T}|$. A last party has the word $\mathbf{1}$ and the radius $m-1$. The purifier has the radius 1. This gives

$$N_j = 2(m_j - 1) + 1 = 4j + 7 \tag{11.17}$$

parties. The RHS term J_K of coin K is the set of parties whose word has a 1 at coin K, so that the word of a party i is indeed $\eta_K^{(i)} = [i \in J_K]$, as Definition 10.1 requires. The inequality reads $\sum_i \rho_i S(i) + S([N]) \geq \sum_{K=1}^{m} S(J_K)$.

THEOREM 11.18 (An explicit family). *For every $j \geq 0$:*

(a) D_j is a star inequality on $N_j = 4j + 7$ parties with $m_j = 2j + 4$ RHS terms, and $\Pi(D_j) = \{c^{(j)}\}$ with $c^{(j)} = \omega^{(j)}/q_j$;

(b) $q_j D_j = \sum_{K=1}^{m_j} \left[(q_j - \omega_K^{(j)})\, \mathrm{SA}_K^1 + \omega_K^{(j)}\, \mathrm{SA}_K^2 \right]$, so D_j is valid, indeed for all quantum states;

(c) kD_j has an integer map on the full cube if and only if 2^{j+2} divides k, and $k_{\min}(D_j) = 2^{j+2} = 2^{(N_j+1)/4}$.

Status: argued; (a) and (b) also verified in exact rational arithmetic for $j = 0, 1, 2$, and the values of k listed in Table 1 certified for D_0 and D_1.

PROOF. *Reduction to Theorem 11.3.* The list $\mathcal{T}$ has $2(m-1) = N_j - 1$ members, and $m \geq 4$ and $N_j \geq 7$. Condition (W2) holds with all $\gamma_i = 1$, because the list is paired. A vector balances $\overline{T}$ exactly when it balances T, since the two balance conditions are the same equation, so the vectors that balance every member of $\mathcal{T}$ are the solutions of the *balance equations*

$$\xi(T) = \xi(\overline{T}) \qquad \text{for every } T \in \mathcal{T}_j\ . \tag{11.19}$$

Let $\Lambda_j \subseteq \mathbb{R}^{m_j}$ be the space of these solutions. The entries of $\omega^{(j)}$ are positive, and $\omega_1^{(j)} = 1$ by Lemma 11.16, so they have no common factor, and (W1) is the statement

$$\Lambda_j = \mathbb{R}\omega^{(j)}\ . \tag{11.20}$$

The induction. We prove (11.20) by induction on j. The vector $\omega^{(j)}$ lies in Λ_j, since $\omega^{(j)}(T) = q_j/2 = \omega^{(j)}(\overline{T})$ by Lemma 11.16. For $j = 0$ the equations are

$$\xi_1 + \xi_2 = \xi_3 + \xi_4\ , \qquad \xi_1 + \xi_3 = \xi_2 + \xi_4\ , \qquad \xi_1 + \xi_4 = \xi_2 + \xi_3\ . \tag{11.21}$$

Adding the first two gives $2\xi_1 + \xi_2 + \xi_3 = \xi_2 + \xi_3 + 2\xi_4$, so $\xi_1 = \xi_4$. Adding the first and the third gives $\xi_1 = \xi_3$, and adding the last two gives $\xi_1 = \xi_2$. So $\xi = \xi_1(1, 1, 1, 1)$, and (11.20) holds for $j = 0$.

Now assume (11.20) for j and let $\xi = (\xi', \xi_u, \xi_v) \in \Lambda_{j+1}$, where $\xi' \in \mathbb{R}^{m_j}$ collects the old coordinates and ξ_u, ξ_v are those of u_j, v_j. For a set T of old coins let $\overline{T}' = [m_j] \setminus T$ be its complement among the old coins, and put $T_1 = T_1^{(j)}$. In $[m_{j+1}]$ the complement of $T \cup \{u_j\}$ is $\overline{T}' \cup \{v_j\}$, and that of $T_1 \cup \{v_j\}$ is $\overline{T_1}' \cup \{u_j\}$. So the balance equations of the halves $T_1 \cup \{u_j\}$ and $T_1 \cup \{v_j\}$ read

$$\xi'(T_1) + \xi_u = \xi'(\overline{T_1}') + \xi_v\ , \qquad \xi'(T_1) + \xi_v = \xi'(\overline{T_1}') + \xi_u\ . \tag{11.22}$$

Subtracting the second from the first gives $2\xi_u = 2\xi_v$, so $\xi_u = \xi_v$. With this, the equation of each lifted half $T \cup \{u_j\}$, $\xi'(T) + \xi_u = \xi'(\overline{T}') + \xi_v$, becomes

$$\xi'(T) = \xi'(\overline{T}') \qquad \text{for every } T \in \mathcal{T}_j\ , \tag{11.23}$$

so $\xi' \in \Lambda_j$, and by the induction hypothesis $\xi' = \kappa\omega^{(j)}$ for some $\kappa \in \mathbb{R}$. (The paper writes t for this scalar.) Finally, the complement of $\{u_j, v_j\}$ is the set $[m_j]$ of all old coins, and its equation reads

$$\xi_u + \xi_v = \xi'([m_j]) = \kappa q_j \ , \tag{11.24}$$

so $\xi_u = \xi_v = \kappa q_j/2$ and $\xi = \kappa(\omega^{(j)}, q_j/2, q_j/2) = \kappa\omega^{(j+1)}$. Hence $\Lambda_{j+1} = \mathbb{R}\omega^{(j+1)}$.

Conclusion. With (11.20), Theorem 11.3 applies with $q = q_j$. Its part (a) is part (a) here, with $c^{(j)} = \omega/q$. Its part (b) is

$$qD_j = \sum_{K=1}^{m} \left[(q - \omega_K)\,\mathrm{SA}^1_K + \omega_K\,\mathrm{SA}^2_K \right] \ , \tag{11.25}$$

which is (b). Its part (c) says that kD_j has an integer map on the full cube exactly when q_j divides k, and $q_j = 2^{j+2}$ by Lemma 11.16. So $k_{\min}(D_j) = 2^{j+2}$, and since $N_j + 1 = 4j + 8$, this is $2^{(N_j+1)/4}$. □

The only work specific to D_j is the induction (11.20), which establishes (W1). Everything else is Theorem 11.3: its Steps 1 to 3 show that the radii are positive and the J_K distinct, Step 4 that all radii are attained at $c^{(j)}$, and Step 7 obtains the least scaling from the fact that the weights have no common factor, here because $\omega_1^{(j)} = 1$.

11.6. The first member D_0

For $j = 0$ the halves are $\{1,2\}$, $\{1,3\}$, $\{1,4\}$, with complements $\{3,4\}$, $\{2,4\}$, $\{2,3\}$. Name the parties with the words $\mathbf{1}_{12}$, $\mathbf{1}_{34}$, $\mathbf{1}_{13}$, $\mathbf{1}_{24}$, $\mathbf{1}_{14}$, $\mathbf{1}_{23}$, $\mathbf{1}$ (where $\mathbf{1}_{12} = \mathbf{1}_{\{1,2\}}$) as $A, \dots, G$. In coin order their words are $A = 1100$, $B = 0011$, $C = 1010$, $D = 0101$, $E = 1001$, $F = 0110$ and $G = 1111$, with radii $2, 2, 2, 2, 2, 2, 3$, and $\rho_O = 1$. Coin 1 lies in the words of A, C, E, G, coin 2 in those of A, D, F, G, coin 3 in those of B, C, F, G, and coin 4 in those of B, D, E, G. So, with the RHS terms in the order of the coins,

$$D_0: \qquad 2A{+}2B{+}2C{+}2D{+}2E{+}2F{+}3G{+}ABCDEFG \ \geq\ ACEG{+}ADFG{+}BCFG{+}BDEG\,. \tag{11.26}$$

The polytope by hand. For D_0 the single-point property can be seen without the induction. Let $\xi \in \Pi(D_0) \subseteq [0,1]^4$. The purifier gives $\|\xi\|_1 \leq 1$, and G gives $\|\xi - \mathbf{1}\|_1 = 4 - \|\xi\|_1 \leq 3$, so $\|\xi\|_1 = 1$. For a pair X of coins, the party with word $\mathbf{1}_X$ gives, by (11.1), $1 + 2 - 2\xi(X) \leq 2$, that is, $\xi(X) \geq \frac{1}{2}$. The party with the complementary word gives $\xi(\overline{X}) \geq \frac{1}{2}$, and since $\xi(X) + \xi(\overline{X}) = 1$, both equal $\frac{1}{2}$. These are the three equations (11.21), so $\xi = \xi_1(1,1,1,1)$, and $\|\xi\|_1 = 1$ gives

$$\Pi(D_0) = \left\{ \tfrac{1}{4}(1,1,1,1) \right\} \ . \tag{11.27}$$

So every map of kD_0 takes the value $\frac{k}{4}(1,1,1,1)$ at z, and $k_{\min}(D_0) = 4$.

The identity. Part (b) of Theorem 11.18 reads $4D_0 = \sum_{K=1}^{4}(3\,\mathrm{SA}^1_K + \mathrm{SA}^2_K)$. To see it term by term, recall that SA^1_K contains $S(i)$ for $i \in J_K$ and SA^2_K contains $S(i)$ for $i \notin J_K$ together with $S(ABCDEFG)$. The term $S(A)$ lies in SA^1_1 and SA^1_2, since $A \in J_1, J_2$, with coefficient 3 each, and in SA^2_3 and SA^2_4 with coefficient 1 each, for a total of $8 = 4 \cdot 2$. The parties $B, \dots, F$ also lie in exactly two of the J_K and get 8 in the same way. The term $S(G)$ lies in all four SA^1_K, for a total of $4 \cdot 3 = 12 = 4 \cdot 3$. The term $S(ABCDEFG)$ lies once in each SA^2_K, for a total of $4 = 4 \cdot 1$, and each $S(J_K)$ gets $-(3+1) = -4$. This is $4D_0$.

Certificates. Independently of the theorem, DRAT proofs from three encoders show that kD_0 has no integer map on the full cube of $2^8 = 256$ points for $k = 1, 2, 3$, and an integer map for $4D_0$ is checked exactly by two checkers. The map (10.21) of Step 4 with $a_K = 3$ and $b_K = 1$ is an integer map for $4D_0$ by the proof of Theorem 10.13.

Status of $k_{\min}(D_0) = 4$: certified (DRAT, 3 encoders, for $k \leq 3$; integer map for $k = 4$).

For $j = 1$ the polytope is the single point $\frac{1}{8}(1,1,1,1,2,2)$, and $k_{\min}(D_1) = 8$ on $N_1 = 11$ parties. Here the certificates cover $k \le 2$ (no map, DRAT proofs from three encoders) and $k = 8$ (an integer map on the full cube of $2^{12} = 4{,}096$ points, two checkers), while the cases $3 \le k \le 7$ rest on the theorem. For $j = 2$ the point is $\frac{1}{16}(1,1,1,1,2,2,4,4)$ on 15 parties, and there are no certificates beyond the theorem.

11.7. A seven-party star with $k_{\min} = 8$

Single-point polytopes also arise without complementary pairs, and then the denominators can be larger at the same N. At $N = 7$, D_0 has $k_{\min} = 4$, and the following example has $k_{\min} = 8$.

Example 11.28. The star inequality

$$\begin{aligned} Z_7: \quad & 4A + 3B + 4C + 4D + 4E + 3F + 4G + 2ABCDEFG \\ & \ge\ ABCG + ABDE + ACEF + BCEF + BEFG + CDEG + CDFG \end{aligned} \tag{11.29}$$

has $m = 7$, $\beta = \mathbf{1}$ and $\rho_O = 2$. Number the coins $K = 1, \dots, 7$ in the order of the RHS terms. The words $\eta_K^{(i)} = [i \in J_K]$ and the radii are

party	word	coins in the word	radius
A	1110000	$\{1,2,3\}$	4
B	1101100	$\{1,2,4,5\}$	3
C	1011011	$\{1,3,4,6,7\}$	4
D	0100011	$\{2,6,7\}$	4
E	0111110	$\{2,3,4,5,6\}$	4
F	0011101	$\{3,4,5,7\}$	3
G	1000111	$\{1,5,6,7\}$	4
O	0000000	$\emptyset$	2

No two of these words are complementary. Two complementary words have sizes adding up to 7, so a pair would consist of a 3-coin word (A or D) and a 4-coin word (B, F or G), and each of these six pairs shares a coin (A, B and A, G share coin 1, A, F share coin 3, D, B share coin 2, D, F share coin 7, D, G share coin 6). No party has the all-ones word, which would be complementary to the purifier's. So the mechanism of Lemma 10.4 does not apply directly.

The point $c = \frac{1}{8}(2,1,1,6,3,1,2)$ *attains all eight radii.* We have $\|c\|_1 = \frac{16}{8} = 2 = \rho_O$, and by (11.1), $\|c - \mathbf{1}_X\|_1 = 2 + |X| - 2c(X)$ for the coin set X of each word:

party	$\lvert X\rvert$	$c(X)$	$\lVert c - \mathbf{1}_X\rVert_1$	radius
A	3	$\frac{2+1+1}{8} = \frac{1}{2}$	$2 + 3 - 1 = 4$	4
B	4	$\frac{2+1+6+3}{8} = \frac{3}{2}$	$2 + 4 - 3 = 3$	3
C	5	$\frac{2+1+6+1+2}{8} = \frac{3}{2}$	$2 + 5 - 3 = 4$	4
D	3	$\frac{1+1+2}{8} = \frac{1}{2}$	$2 + 3 - 1 = 4$	4
E	5	$\frac{1+1+6+3+1}{8} = \frac{3}{2}$	$2 + 5 - 3 = 4$	4
F	4	$\frac{1+6+3+2}{8} = \frac{3}{2}$	$2 + 4 - 3 = 3$	3
G	4	$\frac{2+3+1+2}{8} = 1$	$2 + 4 - 2 = 4$	4

Every distance equals the radius, and $0 \le c_K \le 1$, so $c \in \Pi(Z_7)$ with all eight constraints tight.

Every point of $\Pi(Z_7)$ *has all eight constraints tight.* Every RHS term of Z_7 has exactly four parties, so every coin K lies in exactly four of the seven party words and has $\eta_K = 0$ in the other three words and in the purifier's. For $\xi \in [0,1]^7$, coin K therefore contributes $4(1 - \xi_K) + 4\xi_K = 4$

to the sum of the eight distances (10.5), and

$$\sum_{i\in[N]\cup\{O\}} \|\xi - \eta^{(i)}\|_1 = 4\cdot 7 = 28 \qquad \text{for every } \xi \in [0,1]^7 \ . \tag{11.30}$$

The radii add up to $4+3+4+4+4+3+4+2 = 28$ as well. At a point of $\Pi(Z_7)$ each distance is at most its radius and the distances add up to the sum of the radii, so each distance equals its radius. In terms of the rows (10.6), the sum of all eight rows has zero coefficients, a linear dependence with all weights equal to 1.

The eight equalities have the unique solution c. Let $\xi \in \Pi(Z_7)$. Then $\|\xi\|_1 = 2$, and $\|\xi - \mathbf{1}_X\|_1 = \rho$ for each party, which by (11.1) reads $\xi(X) = \frac{1}{2}(2 + |X| - \rho)$. Write $\xi_{abc} = \xi_a + \xi_b + \xi_c$, and so on. The seven parties give

$$\begin{aligned} &A\colon\ \xi_{123} = \tfrac{1}{2}\ , \qquad B\colon\ \xi_{1245} = \tfrac{3}{2}\ , \qquad C\colon\ \xi_{13467} = \tfrac{3}{2}\ , \qquad D\colon\ \xi_{267} = \tfrac{1}{2}\ , \\ &E\colon\ \xi_{23456} = \tfrac{3}{2}\ , \qquad F\colon\ \xi_{3457} = \tfrac{3}{2}\ , \qquad G\colon\ \xi_{1567} = 1\ . \end{aligned} \tag{11.31}$$

Subtracting the equations of C, E and F from $\|\xi\|_1 = 2$ gives the complementary sums

$$\xi_{25} = \tfrac{1}{2}\ , \qquad \xi_{17} = \tfrac{1}{2}\ , \qquad \xi_{126} = \tfrac{1}{2}\ . \tag{11.32}$$

The equation of D and the third of (11.32) give $\xi_2 + \xi_6 + \xi_7 = \xi_1 + \xi_2 + \xi_6$, so $\xi_7 = \xi_1$, and with $\xi_{17} = \frac{1}{2}$ we get $\xi_1 = \xi_7 = \frac{1}{4}$. From $\xi_{25} = \frac{1}{2}$ we get $\xi_5 = \frac{1}{2} - \xi_2$, and from D, $\xi_6 = \frac{1}{2} - \xi_2 - \xi_7 = \frac{1}{4} - \xi_2$. The equation of G becomes

$$\tfrac{1}{4} + \big(\tfrac{1}{2} - \xi_2\big) + \big(\tfrac{1}{4} - \xi_2\big) + \tfrac{1}{4} = 1\ , \tag{11.33}$$

so $\frac{5}{4} - 2\xi_2 = 1$ and $\xi_2 = \frac{1}{8}$, hence $\xi_5 = \frac{3}{8}$ and $\xi_6 = \frac{1}{8}$. The equation of A gives $\xi_3 = \frac{1}{2} - \frac{1}{4} - \frac{1}{8} = \frac{1}{8}$, and that of B gives $\xi_4 = \frac{3}{2} - \frac{1}{4} - \frac{1}{8} - \frac{3}{8} = \frac{3}{4}$. So

$$\xi = \big(\tfrac{1}{4}, \tfrac{1}{8}, \tfrac{1}{8}, \tfrac{3}{4}, \tfrac{3}{8}, \tfrac{1}{8}, \tfrac{1}{4}\big) = \tfrac{1}{8}(2,1,1,6,3,1,2) = c\ , \tag{11.34}$$

and the total $\frac{16}{8} = 2$ checks. Hence $\Pi(Z_7) = \{c\}$. Since the eight tight constraints have a unique solution, their rows have rank 7.

Consequences. The least common denominator of c is 8, because $8c \in \mathbb{Z}^7$ and $c_2 = \frac{1}{8}$. By (10.24), kZ_7 has an integer map on the full cube exactly when 8 divides k, and $k_{\min}(Z_7) = 8$. By Lemma 10.9, with zero positivity coefficients because every radius is attained,

(11.35)

$$8Z_7 = \sum_{K=1}^{7} \big[(8-8c_K)\,\mathrm{SA}^1_K + 8c_K\,\mathrm{SA}^2_K\big]\ , \qquad 8-8c = (6,7,7,2,5,7,6)\ , \qquad 8c = (2,1,1,6,3,1,2)\ ,$$

a sum of instances of subadditivity. For instance, $S(A)$ gets $6+7+7 = 20$ from $\mathrm{SA}^1_1, \mathrm{SA}^1_2, \mathrm{SA}^1_3$ and $6+3+1+2 = 12$ from $\mathrm{SA}^2_4, \ldots, \mathrm{SA}^2_7$, in total $32 = 8{\cdot}4$, and $S(ABCDEFG)$ gets $2+1+1+6+3+1+2 = 16 = 8\cdot 2$.

The value $k_{\min}(Z_7) = 8$ is also certified independently of Theorem 10.13: for each $k = 1, \ldots, 7$, DRAT proofs from three encoders show that kZ_7 has no integer map on the full cube of 256 points, and an integer map for $8Z_7$ is checked exactly by two checkers. That no point of $\frac{1}{k}\mathbb{Z}^7$ lies in $\Pi(Z_7)$ for $k \le 7$ follows by hand from $\Pi(Z_7) = \{c\}$ and $c_2 = \frac{1}{8}$.

Status: certified (DRAT, 3 encoders, for $k \le 7$; integer map for $k = 8$), and also a consequence of the argued Theorem 10.13.

11.8. Larger certified examples

The paper certifies four more star inequalities with large least scaling, on 9 to 17 parties. For each of them, DRAT proofs show that kQ has no integer map for every $k < k_{\min}(Q)$, on the $N+2$ points 0, z and the occurrence words, to which every map on the full cube restricts, and an integer map for $k_{\min}(Q)\,Q$ on the full cube is checked exactly. These certificates do not use Theorems 10.13 and 11.3.

The first has no complementary pair of words, like Z_7:

$$\begin{aligned} G_9: \quad & 3A+5B+5C+5D+3E+4F+6G+6H+4I+4ABCDEFGHI \\ \geq\ & ABCDG+ABCHI+ABEFG+ACDEF+ACEGH \\ & +BCDEH+BCFGH+CDEGI+DFGHI\ . \end{aligned} \tag{11.36}$$

With the coins in the order of its RHS terms, the point $c=\frac{1}{32}(2,25,28,31,10,8,1,19,4)$ attains all ten radii, the ten constraint rows (10.6) have rank 9, and they have a linear dependence with positive coefficients, so, as for Z_7, $\Pi(G_9)=\{c\}$ and $k_{\min}(G_9)=32$, against $k_{\min}(H_9)=18$ on the same number of parties. By Lemma 10.9, $32G_9=\sum_K[(32-32c_K)\,\mathrm{SA}^1_K+32c_K\,\mathrm{SA}^2_K]$ with $32c=(2,25,28,31,10,8,1,19,4)$.

Status of $k_{\min}(G_9)=32$: certified (DRAT, 2 encoders, for $k\leq 31$; integer map for $k=32$).

The other three are coin-weighing stars $Q_{\mathcal{T}}$. For H_{11}, $m=8$ and the ten weighings are

$$\{1,3,4,5,6\},\ \{1,2,3,5,7\},\ \{1,4,6,7\},\ \{2,3,4,8\},\ \{2,3,6,7\},$$
$$\{1,2,6,8\},\ \{2,4,5,7\},\ \{3,7,8\},\ \{5,6,8\},\ \{1,4,5,8\}\ ,$$

a half-regular list in which every coin lies in exactly five sets, with $\omega=(1,2,2,3,3,4,5,6)$ and $q=26$. For H_{15}, $m=8$, the list consists of the seven sets

$$\{3,7,8\},\ \{4,6,8\},\ \{2,3,5,8\},\ \{5,6,7\},\ \{2,3,6,7\},\ \{2,4,5,7\},\ \{3,4,5,6\}$$

followed by their complements in the same order, with $\omega=(1,2,3,4,5,6,7,8)$ and $q=36$. For H_{17}, $m=9$, the list consists of the eight sets

$$\{2,3,4,8\},\ \{3,4,5,6\},\ \{2,4,6,7\},\ \{2,3,5,7\},\ \{3,4,6,8,9\},\ \{2,4,5,7,9\},\ \{2,3,5,6,9\},\ \{3,4,7,9\}$$

followed by their complements, with $\omega=(6,4,6,5,2,3,4,1,1)$ and $q=32$. These eight sets are the rows of an 8×9 matrix of the construction of Alon and Vũ, read as weighings as in Section 11.9. By Theorem 11.3, $k_{\min}=q$ in each case.

Status of $k_{\min}(H_{11})=26$, $k_{\min}(H_{15})=36$, $k_{\min}(H_{17})=32$: certified (DRAT, 2 encoders each, for $k<k_{\min}$; integer maps for $k=k_{\min}$), and also consequences of the argued Theorem 11.3.

Table 1 collects the star inequalities of this chapter with their certificates.

11.9. Superexponential least scaling of star inequalities

Theorem 11.3 turns every list of weighings that determines the weights into a star inequality with $k_{\min}=q$, the total weight in coprime integers. To make q large on few parties we need few weighings that force very unequal weights. This is a question about matrices, and Alon and Vũ answer it [**3**].

Weighing matrices. Let B be an $n\times(n+1)$ matrix with entries ±1 and rank n. In this section and the next, the letter n denotes a matrix dimension. Each row of B is a weighing of $n+1$ coins, with the coins of entry -1 on one pan and those of entry $+1$ on the other, and a weight vector x balances the row b exactly when $b\cdot x=0$. Since B has rank n, its kernel is a line $\mathbb{R}x$: the weighings determine the weights up to a common factor, although some weights may be 0 or negative. Alon and Vũ measure how unequal they are by

$$\nu(B)=\max\left\{|x_K/x_{K'}|\colon K,K'\in[n+1],\ x_{K'}\neq 0\right\}\ , \tag{11.37}$$

	N	m	point of $\Pi(Q)$	$k_{\min}$	certificates
D_0	7	4	$\frac{1}{4}(1,1,1,1)$	4	no map for $k \le 3$; map for $k = 4$
D_1	11	6	$\frac{1}{8}(1,1,1,1,2,2)$	8	no map for $k \le 2$; map for $k = 8$
D_2	15	8	$\frac{1}{16}(1,1,1,1,2,2,4,4)$	16	none beyond Theorem 11.18
Z_7	7	7	$\frac{1}{8}(2,1,1,6,3,1,2)$	8	no map for $k \le 7$; map for $k = 8$
G_9	9	9	$\frac{1}{32}(2,25,28,31,10,8,1,19,4)$	32	no map for $k \le 31$; map for $k = 32$
H_9	9	8	$\frac{1}{18}(1,1,2,2,2,3,3,4)$	18	no map for $k \le 17$; map for $k = 18$
H_{11}	11	8	$\frac{1}{26}(1,2,2,3,3,4,5,6)$	26	no map for $k \le 25$; map for $k = 26$
H_{15}	15	8	$\frac{1}{36}(1,2,3,4,5,6,7,8)$	36	no map for $k \le 35$; map for $k = 36$
H_{17}	17	9	$\frac{1}{32}(6,4,6,5,2,3,4,1,1)$	32	no map for $k \le 31$; map for $k = 32$

TABLE 1. Star inequalities with large least scaling. The point is the unique point of the star polytope, in the coin order of the construction for D_j and of Section 11.8 for H_{11}, H_{15}, H_{17}, in the order of the RHS terms of (11.29), (11.36) and (11.11) for Z_7, G_9 and H_9, and $k_{\min}$ is the least common denominator of its coordinates (Theorem 10.13). "No map" means no integer map on the full cube, certified by DRAT proofs for each listed k, from three encoders for D_0, D_1 and Z_7 and from two for the others, on the full cube for D_0, D_1 and Z_7, and on the $N + 2$ points 0, z and the occurrence words for G_9 and the H_N. "Map" means an integer map on the full cube, checked exactly by two checkers, and by three for G_9 and H_9.

which does not depend on the choice of x in the line (they write $\xi(B)$, a letter used here for points of the polytope). The paper writes w for the kernel vector, a letter that the companion uses for vertices of graph models. Expanding along its first row the determinant of the $(n+1) \times (n+1)$ matrix formed by a row of B placed on top of B, which vanishes because it has two equal rows, shows that the vector with entries $(-1)^K \det B_K$, where B_K is B with the column K deleted, lies in the kernel. It is nonzero because B has rank n, so it spans the kernel, and $\nu(B)$ is a ratio of two determinants of $n \times n$ matrices with entries ± 1. Hadamard's inequality bounds the numerator by $n^{n/2}$, and a nonzero determinant of such a matrix is divisible by 2^{n-1}, as the proof of Proposition 11.42 below shows, so $\nu(B) \le n^{n/2} 2^{1-n}$. This is the argument by which Alon and Vũ bound the entries of inverses of ± 1 matrices [**3**].

What Alon and Vũ prove. Their Main Theorem determines, up to a factor $2^{O(n)}$, the largest possible absolute value of an entry of the inverse of an invertible $n \times n$ matrix with entries in $\{0, 1\}$ or in $\{-1, 1\}$, and it constructs matrices that come within such a factor of the maximum [**3**]. They state that their construction is based on ideas of Håstad. From it they obtain, in a proposition, for every n an $n \times (n+1)$ matrix B with entries ± 1 and rank n with

$$\nu(B) \;\ge\; 2^{\frac{1}{2} n \log_2 n - n(2+o(1))} \,, \tag{11.38}$$

so the bound $n^{n/2} 2^{1-n} = 2^{\frac{1}{2} n \log_2 n - n + 1}$ above is attained up to a factor $2^{-(1+o(1))n}$. They use these matrices for coin-weighing problems (see also [**37**]). In the weighing process they build from B, after the signs of some columns are changed, each column K of B is repeated ω_K times, once for each of ω_K coins, where ω is the nonnegative integer vector of least sum on the kernel line. So the process has $\sum_K \omega_K$ coins, and they note that this number is at least $\nu(B)$ [**3**]. In Theorem 11.3 the same number is the least scaling q.

COROLLARY 11.39 (Superexponential least scaling of star inequalities).

(i) *For every $n \times (n+1)$ matrix B with entries ± 1 and rank n there is a valid star inequality Q on $2n+1$ parties, implied by subadditivity and not balanced, with $\nu(B) \le k_{\min}(Q) < \infty$.*

(ii) *For every $N \geq 3$ there is a valid star inequality Q_N on N parties, implied by subadditivity and not balanced, with $k_{\min}(Q_N) < \infty$ and*

$$\log_2 k_{\min}(Q_N) \;\geq\; \frac{N}{4}\log_2 N - \Big(\frac{5}{4} + o(1)\Big)N \qquad (N \to \infty)\ . \tag{11.40}$$

Status: argued; part (ii) rests on the proposition of Alon and Vũ quoted in (11.38), and through it on their main theorem [**3**].

Proof. (i) *Making the weights nonnegative.* Let x span the kernel of B. Changing the sign of column K of B and of x_K keeps $Bx = 0$, keeps the entries ± 1 and the rank, and does not change $\nu(B)$. In the weighing language it moves coin K to the other pan in every weighing. After such changes, $x \geq 0$.

Removing the coins of weight zero. Let $F = \{K\colon x_K > 0\}$. If F had a single element K, then $Bx = x_K \cdot (\text{column } K)$ would vanish, which is impossible for a column with entries ± 1. So $|F| \geq 2$. Let B_F be the $n \times |F|$ matrix of the columns of B in F. A solution y of $B_F y = 0$, extended by zeros outside F, solves $By = 0$, so it is a multiple of x. Hence the solutions of $B_F y = 0$ form the line spanned by x_F, the restriction of x to F, which has positive entries.

The list of weighings. Number the coins of F by $[m]$, $m = |F| \geq 2$, and let ω be the vector of positive integers without a common factor on the ray of x_F. For each row b of B_F let $T_b = \{K\colon b_K = -1\}$, so that $b = \mathbf{1} - 2\cdot\mathbf{1}_{T_b}$ and $b\cdot\xi = \xi(\overline{T_b}) - \xi(T_b)$. Take the paired list of the n sets T_b and their complements, which has $2n$ members. A vector balances T_b, or $\overline{T_b}$, exactly when $b\cdot\xi = 0$, so the balancing vectors are the solutions of $B_F\xi = 0$, the line $\mathbb{R}\omega$, and (W1) holds. Condition (W2) holds with all $\gamma_i = 1$ because the list is paired.

The bound. Theorem 11.3 gives a valid star inequality on $N = 2n+1$ parties with $k_{\min} = q = \sum_K \omega_K$. The ratios of the entries of ω are those of the nonzero entries of x, so $\nu(B) = \max_K \omega_K / \min_K \omega_K$, and since $\min_K \omega_K \geq 1$,

$$k_{\min} = \sum_K \omega_K \;\geq\; \max_K \omega_K \;\geq\; \nu(B)\ . \tag{11.41}$$

(ii) *Odd N.* Let $N = 2n+1$ with $n \geq 1$, take B as in (11.38), and apply (i). With $n = (N-1)/2$,

$$\frac{1}{2} n \log_2 n = \frac{N-1}{4}\log_2\frac{N-1}{2} = \frac{N}{4}\log_2 N - \frac{N}{4} + O(\log N)\ , \qquad n(2+o(1)) = (1+o(1))N\ ,$$

and (11.41) with (11.38) gives (11.40).

Even N. Let $N = 2n+2$. Add to the list of the case $2n+1$ a second copy of its first set T_1. The balance conditions are the same as before, so (W1) still holds, and (W2) holds with $\gamma = \frac{1}{2}$ on the two copies of T_1 and $\gamma = 1$ on all other sets, because the two copies together contribute $\mathbf{1} - 2\cdot\mathbf{1}_{T_1}$, which cancels against $\overline{T_1}$. Theorem 11.3 gives a star inequality on N parties with the same q, and the computation above with $n = (N-2)/2$ gives (11.40). □

Padding with a party that lies in no term (Section 11.11) would also give an even number of parties with the same $k_{\min}$, but the padded inequality is no longer a star inequality, which matters for the comparison with Proposition 11.42 below, a statement about star inequalities. The repeated weighing keeps it one. For $n = 8$, the matrix behind H_{17} of Section 11.8 has $\nu(B) = 6$ and $q = 32$. This case is far from the asymptotic regime of (11.38).

All these inequalities have $\rho_O = 1$. For a star inequality with $\rho_O > 0$, the point z of Theorem 10.13, which has only the bit of $S([N])$ set, lies in the realizable domain $\mathcal{A}(Q)$, together with 0 and the occurrence words: it has a single bit set, and the only LHS term that contains $[N]$ is $[N]$ itself. So Step 3 of the proof of Theorem 10.13 works on $\mathcal{A}(Q)$, $k^{\mathcal{A}}_{\min}(Q) = k_{\min}(Q)$, and the bound (11.40) holds for $k^{\mathcal{A}}_{\min}$ as well.

11.10. An upper bound

The least scaling of a star inequality, when finite, is also bounded from above. Let $\Delta(n)$ be the largest absolute value of the determinant of an $n \times n$ matrix with entries in $\{0,1\}$, with $\Delta(0) = 1$. For instance $\Delta(1) = \Delta(2) = 1$, and $\Delta(3) = 2$, attained by the matrix with rows $(1,1,0)$, $(0,1,1)$, $(1,0,1)$. We call a star inequality *complement-closed* if, for the word η of every party and of the purifier, the complementary word $\beta - \eta$ is also the word of a party or of the purifier. A coin-weighing star with a paired list is complement-closed, since its weighings come in complementary pairs and the word $\mathbf{1}$ of the party N is the complement of the purifier's word 0. So are Q' and the inequalities of Corollary 11.39, also for even N.

Proposition 11.42 (An upper bound for star inequalities). *Let Q be a star inequality on N parties with $\Pi(Q) \neq \emptyset$. Then*

$$k_{\min}(Q) \;\leq\; 2\Delta(N) \;\leq\; 2(N+1)^{(N+1)/2}2^{-N} \;. \tag{11.43}$$

If Q is complement-closed, then $k_{\min}(Q) \leq 2\Delta(\lfloor (N-1)/2 \rfloor)$.

Status: argued.

Proof. *Step 1: the constraints.* By (10.6), the constraint of $i \in [N] \cup \{O\}$ reads $s^{(i)} \cdot c \leq h_i$ on the box, with $s^{(i)}_K = 1$ if $\eta^{(i)}_K = 0$, $s^{(i)}_K = -1$ if $\eta^{(i)}_K = \beta_K$, and the integer $h_i = \rho_i - \sum_{K\colon \eta^{(i)}_K = \beta_K} \beta_K$.

Step 2: a vertex. As in Step 5 of the proof of Theorem 10.13, $\Pi(Q)$ is a nonempty bounded polyhedron and has a vertex v, and $k_{\min}(Q)$ is at most the least common denominator of the coordinates of v, because v lies in $\frac{1}{k}\mathbb{Z}^m$ for that k. Let $F = \{K\colon 0 < v_K < \beta_K\}$ and $t = |F|$. The coordinates of v outside F are 0 or β_K, integers. At a vertex the constraints that hold with equality have rank m. The box constraints among them involve only coordinates outside F, so the equalities $s^{(i)} \cdot v = h_i$, restricted to the coordinates in F after the integer contributions of the other coordinates are moved to the right, have rank t. Choosing t of them with independent rows gives

$$M v_F = h' \;, \tag{11.44}$$

with M an invertible $t \times t$ matrix with entries ± 1 and $h' \in \mathbb{Z}^t$. There are $N+1$ constraints, so $t \leq N+1$. If $t \leq 1$, then v is integral and $k_{\min}(Q) = 1$. Let $t \geq 2$.

Step 3: divisibility. As in Alon and Vũ [**3**], for a square matrix with $\ell \geq 1$ rows and entries ± 1, adding the first row to each other row does not change the determinant and leaves $\ell - 1$ rows with entries in $\{0, \pm 2\}$. Taking a factor 2 out of each of them shows that the determinant is divisible by $2^{\ell-1}$.

Step 4: Cramer's rule. The solution of (11.44) is $v_F = \operatorname{adj}(M)h' / \det M$, where the entries of the adjugate $\operatorname{adj}(M)$ are, up to sign, the determinants of the $(t-1) \times (t-1)$ submatrices of M. By Step 3 they are divisible by 2^{t-2}, so every coordinate of v_F is an integer divided by $|\det M|/2^{t-2}$, which is itself an integer by Step 3. Hence the least common denominator of v divides $|\det M|/2^{t-2}$.

Step 5: from ± 1 to $0, 1$. We use the known correspondence between ± 1 matrices of order t and $\{0,1\}$ matrices of order $t-1$, as Alon and Vũ describe it [**3**]. Multiplying a row or a column of M by -1 does not change $|\det M|$. Multiply the columns so that the first row consists of 1s, and then the rows after the first so that the first column consists of 1s. Subtracting the first row from the others leaves the first column $(1, 0, \ldots, 0)$ and the block $-2B'$ in the other rows and columns, where B' is a $(t-1) \times (t-1)$ matrix with entries in $\{0,1\}$. Expanding along the first column, $|\det M| = 2^{t-1}|\det B'|$. With Step 4,

$$k_{\min}(Q) \;\leq\; \frac{|\det M|}{2^{t-2}} \;=\; 2|\det B'| \;\leq\; 2\Delta(t-1) \;\leq\; 2\Delta(N) \;, \tag{11.45}$$

since Δ does not decrease: bordering an $(n-1) \times (n-1)$ matrix with a row and a column that are 0 except for a 1 on the diagonal keeps the determinant.

Step 6: Hadamard's inequality. Let B'' be an $n \times n$ matrix with entries in $\{0,1\}$, and let M'' be the matrix of order $n+1$ whose first row and first column consist of 1s and whose other entries are $1-2B''_{ab}$. Its entries are ± 1, and Step 5 run backwards gives $|\det M''| = 2^n |\det B''|$. Hadamard's inequality bounds the absolute value of a determinant by the product of the Euclidean lengths of the rows, here $(\sqrt{n+1})^{n+1}$ (see e.g. [**3**]). So $|\det B''| \le (n+1)^{(n+1)/2} 2^{-n}$, which is the second inequality of (11.43) for $n = N$.

Step 7: complement-closed stars. If $\eta^{(i')} = \beta - \eta^{(i)}$, then $s^{(i')} = -s^{(i)}$ by Step 1. No word is its own complement, because $\beta_K \ge 1$. If Q is complement-closed, the distinct words among the $N+1$ words of the parties and the purifier therefore fall into at most $\lfloor (N+1)/2 \rfloor$ complementary pairs. Two constraints from the same word or from the same pair give rows of M that are equal or opposite, which an invertible M cannot contain. So $t \le \lfloor (N+1)/2 \rfloor$, and (11.45) gives $k_{\min}(Q) \le 2\Delta(t-1) \le 2\Delta(\lfloor (N-1)/2 \rfloor)$. □

For Q', which is complement-closed with $N = 3$, the second bound is $2\Delta(1) = 2 = k_{\min}(Q')$. For D_0, complement-closed with $N = 7$, it is $2\Delta(3) = 4 = k_{\min}(D_0)$. So the bound is attained in both cases. In general, by (11.43) and Corollary 11.39, the largest finite value of $k_{\min}$ over N-party star inequalities (which are implied by subadditivity and are not balanced, Section 11.12) lies between $N^{(1/4-o(1))N}$ and $N^{(1/2+o(1))N}$. For complement-closed star inequalities the second bound of the proposition is $2\Delta(\lfloor (N-1)/2 \rfloor) \le N^{(1/4+o(1))N}$, by Step 6 with $n = \lfloor (N-1)/2 \rfloor$, so within this class, which contains the inequalities of Corollary 11.39, the exponent $\frac{1}{4}$ cannot be improved. For star inequalities in general the exponents $\frac{1}{4}$ and $\frac{1}{2}$ leave a gap, which the paper leaves open (Section 12.2).

11.11. Every number of parties

Adding parties that lie in no term changes neither validity nor the maps. By the isolated-party lemma (Lemma 2.16), an N-party inequality read as an $(N+1)$-party inequality with the same terms is valid if and only if the original is [**1, 4**]: a graph model for the larger set of parties is one for the smaller set once the boundary vertices of the new party are labeled O. The padded inequality also has the same integer maps for every k (Chapter 4), because its cube, its weights and the pins of the old parties are unchanged, and the occurrence word of the new party is 0, so its pin is the pin of the purifier, which is already imposed. Hence the padded inequality has the same $k_{\min}$. (A padded star inequality is no longer a star inequality in the sense of Definition 10.1, since its term $S([N])$ no longer contains all parties, but nothing here needs it to be one.)

Proposition 11.46. *The supremum of the finite values of $k_{\min}$ over valid N-party inequalities does not decrease with N. It is at least 8 for $N = 7, 8$, at least 32 for $9 \le N \le 14$ and at least 36 for every $N \ge 15$, and, through star inequalities, which are implied by subadditivity and are not balanced, it is at least $N^{N/4} 2^{-(5/4+o(1))N}$ as $N \to \infty$.*

Status: the value 8 is certified at $N = 7$ and derived at $N = 8$, the value 32 is certified at $N = 9$ and derived for $10 \le N \le 14$, the value 36 is certified at $N = 15$ and derived for $N \ge 16$, and the asymptotic bound is argued and rests on the theorem of Alon and Vũ (Corollary 11.39).

Proof. Padding an N-party inequality with one party gives an $(N+1)$-party inequality with the same $k_{\min}$, by the paragraph above, so the supremum does not decrease. The certified values $k_{\min}(Z_7) = 8$ at $N = 7$, $k_{\min}(G_9) = 32$ at $N = 9$ and $k_{\min}(H_{15}) = 36$ at $N = 15$ (Table 1) and padding give the first three bounds, and Corollary 11.39(ii) gives the last. □

All the inequalities behind these bounds are star inequalities or padded star inequalities, so they are implied by subadditivity and are not balanced (Section 11.12). The explicit family D_j gives $2^{\lfloor (N+1)/4 \rfloor}$ by padding, which Corollary 11.39 exceeds for large N.

11.12. Scope of the star results

Facets. By Lemma 10.9, every star inequality with a nonempty polytope, in particular every one in Chapters 10 and 11 and every one behind the superexponential bound of Corollary 11.39, is implied by subadditivity. So the values of $k_{\min}$ of star inequalities found in Chapters 10 and 11 concern inequalities that subadditivity already implies, and they say nothing about how much scaling the facets of HEC_N need.

Balance. By the Bell-value formula (3.6), the Bell value of a star inequality on a pair i, O is $d_\alpha(x^{(i)}, 0) - d_\beta(y^{(i)}, 0)$. The occurrence word $x^{(i)}$ has the bits $\{i\}$ and $[N]$ set, with weights ρ_i and ρ_O, and $d_\beta(y^{(i)}, 0) = \sum_K \beta_K[i \in J_K] = |\eta^{(i)}|$, where $|\eta| = \sum_K \eta_K$. So

$$Q \cdot r_{iO} = \rho_i + \rho_O - |\eta^{(i)}| \ . \tag{11.47}$$

For a coin-weighing star $Q_{\mathcal{T}}$, a party $i \le N-1$ with word $\mathbf{1}_{T_i}$ has $|T_i| + 1 - |T_i| = 1$, and the party N with word $\mathbf{1}$ has $(m-1) + 1 - m = 0$. So the $N-1$ parties of the weighings have Bell value 1, and $Q_{\mathcal{T}}$ is not balanced. This covers D_j, whose $4j+6$ parties with a word $\mathbf{1}_T$ or $\mathbf{1}_{\overline{T}}$ have Bell value 1, the inequalities H_N, and those of Corollary 11.39. For Q' the Bell values with the purifier are $0, 1, 1$ for A, B, C, and for Z_7 they are $\rho_i + 2 - |X|$, that is, $3, 1, 1, 3, 1, 1, 2$ for $A, \ldots, G$. Neither is balanced, and the next lemma shows that no star inequality with $1 < k_{\min} < \infty$ can be.

Lemma 11.48. *If a star inequality Q is balanced, then $\Pi(Q) \subseteq \{0\}$.*

Proof. Balance means $Q \cdot r_{iO} = 0$ for every party i, so by (11.47) $|\eta^{(i)}| = \rho_i + \rho_O$. Let $c \in \Pi(Q)$ and fix a party i. By the triangle inequality and the constraints of i and of the purifier,

$$|\eta^{(i)}| = \|\eta^{(i)} - 0\|_1 \le \|c - 0\|_1 + \|\eta^{(i)} - c\|_1 \le \rho_O + \rho_i = |\eta^{(i)}| \ , \tag{11.49}$$

so equality holds throughout. The ℓ_1 triangle inequality is the sum over the coordinates of $|\eta^{(i)}_K| \le |c_K| + |\eta^{(i)}_K - c_K|$, so equality of the sums forces equality in every coordinate. For a coordinate K with $i \notin J_K$ we have $\eta^{(i)}_K = 0$, and equality reads $0 = 2c_K$, so $c_K = 0$. Every J_K misses some party, because $|J_K| \le N-1$, so every coordinate of c vanishes. □

With Theorem 10.13 this gives $k_{\min}(Q) = 1$ if $0 \in \Pi(Q)$, and no integer map for any kQ otherwise. So no balanced star inequality has $1 < k_{\min} < \infty$. The balanced and superbalanced failures of Chapters 7 and 8 are not star inequalities, and the star results do not cover them. Their least scalings, where they are known, are determined in Chapters 7 and 8. More generally, none of the inequalities without an unscaled map treated in the companion is known to define a facet, and the one facet treated, F_7, has $k^{\mathcal{A}}_{\min}(F_7) = 1$. So these results do not bear on the empirical remark of Grimaldi, Headrick, Hubeny and Shteyner on primitive, that is facet-defining, inequalities [**5**] (Chapter 4).

CHAPTER 12

What is decided and what is open

Where we are. The paper asks whether valid balanced and superbalanced holographic entropy inequalities always have contraction proofs without rescaling, and how large the scaling must be when they do not. Chapters 7–11 answer the first question in the negative for both classes, follow one failure to a facet, and describe the least scaling exactly for star inequalities. This chapter collects the status of every result in one place and lists the questions that remain open.

12.1. The results and their status

Tables 1–4 list the results of Chapters 7–11, one row per statement, with the status words certified, derived and argued as defined in the Preface. Where the paper gives a statement no status word, as for superbalance, the descent identities and the comparison with the literature, the status column says how the statement is obtained. The column "certificate" names the kind of certificate (Chapter 6), with the domain and its number of points where it matters, and the last column counts the encoders or checkers that are separate programs. What a DRAT proof does and does not certify is explained in Section 6.4, and the steps the paper regards as least secure are named in Section 6.7.

statement	status	certificate	independent checks
B_7^* is valid	certified	identity with 43 instances of SA and MMI	2 programs
$2B_7^*$ has an integer map on the full cube	certified	integer map, full cube (256)	2 checkers
B_7^* has no integer map on the full cube	certified	DRAT proof, full cube (256)	3 encoders, and a hand proof
B_7^* has no integer map on $\mathcal{A}(B_7^*)$	argued	written proof	
$k_{\min}(B_7^*) = 2$	derived	from the second and third rows	
$k_{\min}^{\mathcal{A}}(B_7^*) = 2$	argued	rests on the written proof	
B_7 is valid; $2B_7$ has an integer map on the full cube	certified	identity with 46 instances; integer map, full cube (256)	exact check
B_7 has no integer map on the full cube	certified	DRAT proof, full cube (256)	2 encoders
$k_{\min}(B_7) = 2$	derived	from the two rows above	
Q_8^A has no integer map; $2Q_8^A$ has one	certified	DRAT proof; integer map, full cube (128)	3 encoders; all-pairs check

TABLE 1. Status of the results of Chapter 7 on balanced inequalities. Empty entries: no computer certificate is involved.

12.2. Open questions

The following questions are left open by the paper. For each we record what is known.

statement	status	certificate	independent checks
U_6 is superbalanced	exact computation	all 21 Bell values	
U_6 is valid	certified	identity (8.3) with instances of MMI	2 programs
U_6 has no integer map (full cube, $\mathcal{A}$)	certified	DRAT proofs on $\mathcal{A}$ (580); hand proof	2 encoders, and the hand proof
$2U_6$ has an integer map on $\mathcal{A}(U_6)$	certified	integer map on $\mathcal{A}$ (580)	2 checkers
$k^{\mathcal{A}}_{\min}(U_6) = 2$	derived	from the two rows above	
$Q_8, Q_8', Q_8'', E_0, \dots, E_5$ are superbalanced	exact computation	all Bell values	
Q_8 is valid	certified	rational map, full cube (16,384)	2 checkers
Q_8' is valid	certified	rational map, full cube (16,384)	3 checkers
Q_8'' is valid	certified	rational map, full cube (16,384)	2 checkers
E_0, E_1 are valid	certified	integer maps for $2E_0$, $2E_1$, full cube (16,384)	2 checkers each
E_3, E_4 are valid	certified	integer maps for $2E_3$, $2E_4$, full cube (32,768)	2 checkers each
E_2 is valid	derived	rational map on $\mathcal{A}$ (2,916)	2 checkers
E_5 is valid	derived	rational map on $\mathcal{A}$ (6,912)	2 checkers
Q_8, Q_8'' have no integer map (full cube, $\mathcal{A}$)	certified	DRAT proofs on $\mathcal{U}$ (135)	3 encoders each
Q_8' has no integer map (full cube, $\mathcal{A}$)	certified	DRAT proof on $\mathcal{U}$ (135)	2 encoders
E_0, E_1 have no integer map (full cube, $\mathcal{A}$)	certified	DRAT proofs on $\mathcal{U}$ (134)	3 encoders each
E_2, E_3, E_4 have no integer map (full cube, $\mathcal{A}$)	certified	DRAT proofs on $\mathcal{U}$ (134)	2 encoders each
E_5 has no integer map (full cube, $\mathcal{A}$)	certified	DRAT proof on $\mathcal{U}$ (134)	1 encoder
the descent identities (8.29)	definitions	exact integer identities	
$\sup \lambda = \frac{2}{3}$ for $Q_8 - \lambda R_8$	derived	validity of Q_8'; 37 graph models	2 max-flow codes
$\sup \lambda = 2$ for $E_s - \lambda \mu_s$, $s = 0, 1, 4$	derived	validity of E_{s+1}; graphs with 138, 168, 10 vertices	at least 3 max-flow codes
$Q_8, Q_8', E_0, \dots, E_5$ are not facets	derived	positive-combination lemma	
$3Q_8$ has no integer map (full cube, $\mathcal{A}$)	certified	DRAT proof on $\mathcal{U}$ (135)	1 encoder
$2E_0$, $2E_1$, $2E_3$, $2E_4$ have integer maps on the full cube	certified	integer maps, full cube (16,384; 32,768)	2 checkers each
$k_{\min}(E_s) = k^{\mathcal{A}}_{\min}(E_s) = 2$ for $s = 0, 1, 3, 4$	derived	from the maps for $2E_s$ and the absence of a map for E_s	
$2E_4$ has an integer map on $\mathcal{A}(E_4)$, and it is equivariant	certified	integer map on $\mathcal{A}$ (2,916)	3 checkers

TABLE 2. Status of the results of Chapter 8 on the superbalanced chain. "Full cube, $\mathcal{A}$" means that the non-existence holds on both, by the restriction lemma and the antichain property, and for U_6 because a map on the full cube restricts to $\mathcal{A}(U_6)$, where the DRAT proofs are.

The least scaling of the superbalanced failures. For E_0, E_1, E_3 and E_4 the least scaling is 2, on the full cube and on the realizable domain (Proposition 8.42), so kQ has an integer map for every even k, because maps add. For E_2, E_5, Q_8, Q_8' and Q_8'' it is not known, and for the six-party U_6 it is 2 on the realizable domain and not known on the full cube, and $3Q_8$ has no integer map, on the full cube or on its realizable domain. The decided cases are consistent with the possibility that, for

statement	status	certificate	independent checks
F_7 is valid	derived	two rational maps on $\mathcal{A}$ (11,046), for F_7 and $2F_7$	4 checkers each
F_7 defines a facet of HEC_7	derived	two lists of 126 tight graph models; rank modulo primes	2 max-flow codes, 3 primes
F_7 is not among the 43 inequalities compared, from the literature searched	exact computation	exact comparison of canonical forms	
the decomposition (9.20)	certified	integer identity	2 programs
the identity (9.19)	certified	integer identity, checked from the coefficient vectors	
the face of E_0 (of $E_1, \dots, E_5$, Y_7) lies in the facet F_7	derived	positive-combination lemma	
F_7 has an integer map on $\mathcal{A}(F_7)$	certified	integer map on $\mathcal{A}$ (11,046)	2 checkers
$k^{\mathcal{A}}_{\min}(F_7) = 1$	derived	from the row above	
no integer map for F_7 on $\mathcal{A}(F_7)$ is equivariant	certified	DRAT proofs on $\mathcal{A}$ (11,046)	2 encoders
least equivariant k for F_7 lies in $[2, 6]$	derived	averaging over $\mathrm{Stab}(F_7)$	
V_7, V_7' are valid	derived	rational maps on $\mathcal{A}$ (10,032 each)	2 checkers each
V_7, V_7' have integer maps on their realizable domains	certified	integer maps on $\mathcal{A}$ (10,032 each)	2 checkers each
no integer map for V_7 on $\mathcal{A}(V_7)$ is equivariant	certified	DRAT proof on $\mathcal{A}$ (10,032)	1 encoder
X_7 is valid	derived	rational map on $\mathcal{A}$ (6,912)	2 checkers
Y_7 is valid	derived	identity (9.19)	
P_7 has an integer map on $\mathcal{A}(P_7)$	certified	integer map on $\mathcal{A}$ (6,912)	3 checkers
P_7 is valid	derived	from the row above	
the identities (9.33)	certified	integer identities	3 programs
the identities (9.34)	derived	from (9.33) and (8.29)	
$\frac{1}{2}(V_7' + P_7)$ has no integer map (full cube, $\mathcal{A}$)	certified	DRAT proofs on $\mathcal{U}$ (134)	2 encoders
X_7 is not a facet	derived	identity (9.33), positive-combination lemma	

TABLE 3. Status of the results of Chapter 9 on the facet F_7.

these inequalities, kQ has an integer map exactly when k is even. For the four inequalities with $k_{\min} = 2$ this would follow from the absence of integer maps for odd $k \geq 3$, which the paper does not establish, and an integer map for an odd multiple of any inequality of Chapter 8 would rule it out.

Facets without an unscaled map. Whether some facet of HEC_N has no unscaled map is open. The one facet reached here, F_7, has an unscaled map on its realizable domain. Of the other summands of the decomposition (9.20), V_7 and V_7' have unscaled maps on their realizable domains and have 125 known tight graph models with independent entropy vectors, one fewer than a facet needs, so whether they are facets is open, and for X_7 it is unknown whether it has an unscaled map. The inequality X_7 is not a facet, and up to instances of MMI every E_s is half the sum of V_7' and of the further inequality P_7, both with unscaled maps on their realizable domains, while their half-sum has none (Section 9.7). If V_7, V_7' or P_7 is not a facet, the facets into which it decomposes are not known, and any of them could lack an unscaled proof. The paper does not decide whether Q_8'' is a facet of HEC_8.

The full cube for F_7. Whether F_7 has an unscaled map on the full cube $\{0,1\}^{20}$ is open. It was not attempted, and the restriction lemma cannot decide it, because the map on $\mathcal{A}(F_7)$ restricts to $\mathcal{U}(F_7)$.

The smallest number of parties. The six-party inequality U_6 of Section 8.1 is superbalanced, hence balanced, and has no unscaled map. Whether a valid balanced or superbalanced inequality without an unscaled map exists with $N \leq 5$ is open. For $N \leq 4$ every facet is an instance of subadditivity or MMI [**1**], which have unscaled maps. The five-party facets were proved by contraction by Bao et al. [**1, 10**], and for a representative of each of the eight orbits of facets at $N = 5$ [**10**] the paper's computer search found an unscaled map on the full cube, which is not among its certificates. These statements concern only facets.

The least scaling of balanced inequalities. In every balanced valid inequality without an unscaled map whose least scaling the paper determines, namely B_7^*, B_7, Q_8^A and the superbalanced E_0, E_1, E_3 and E_4, the least scaling is 2. The superbalanced six-party inequality U_6 has least scaling 2 on its realizable domain. The paper states that the value 2 is not a general bound and leaves the constructions behind this statement to later work. By Lemma 11.48, a balanced inequality with $1 < k_{\min} < \infty$ cannot be a star inequality.

The growth of the least scaling. The supremum of the finite values of $k_{\min}$ over valid N-party inequalities is at least 32 for $9 \leq N \leq 14$ and at least 36 for $N \geq 15$ and grows at least as $N^{(1/4-o(1))N}$ (Proposition 11.46), through star inequalities, which are implied by subadditivity and are not balanced. For star inequalities the largest finite least scaling is at most $N^{(1/2+o(1))N}$, and at most $N^{(1/4+o(1))N}$ for complement-closed ones (Proposition 11.42). For star inequalities in general the gap between the exponents $\frac{1}{4}$ and $\frac{1}{2}$ is left open. For valid inequalities that are not star inequalities the paper gives no upper bound, and whether the least scaling of balanced or superbalanced inequalities can grow exponentially with N is open, as is how large the least scaling of a facet can be.

statement	status	certificate	independent checks
scaling spectrum of star inequalities (Theorem 10.13)	argued	written proof	
Q': no integer map for $k = 1, 3$	certified	DRAT proofs, full cube (16)	3 encoders, and a hand proof
Q': integer maps for $k = 2, 4$	certified	integer maps, full cube (16)	2 checkers
$k_{\min}(Q') = 2$	derived	from the two rows above	
coin-weighing stars (Theorem 11.3)	argued	written proof	
the family D_j (Theorem 11.18)	argued	written proof; (a), (b) also verified in exact rational arithmetic for $j = 0, 1, 2$	
D_0: no integer map for $k \le 3$; integer map for $k = 4$; so $k_{\min}(D_0) = 4$	certified	DRAT proofs; integer map, full cube (256)	3 encoders and the written proof; 2 checkers
D_1: no integer map for $k \le 2$; integer map for $k = 8$	certified	DRAT proofs; integer map, full cube (4,096)	3 encoders and the written proof; 2 checkers
$k_{\min}(D_1) = 8$	argued	the cases $k \le 2$ and $k = 8$ certified	
$k_{\min}(D_j) = 2^{j+2}$ for $j \ge 2$	argued	written proof	
Z_7: no integer map for $k \le 7$; integer map for $k = 8$; so $k_{\min}(Z_7) = 8$	certified	DRAT proofs; integer map, full cube (256)	3 encoders and an exact lattice-point computation; 2 checkers
G_9: no integer map for $k \le 31$; integer map for $k = 32$; so $k_{\min}(G_9) = 32$	certified	DRAT proofs on 0, z, $x^{(i)}$ (11); integer map, full cube (1,024)	2 encoders; 3 checkers
H_9: no integer map for $k \le 17$; integer map for $k = 18$; so $k_{\min}(H_9) = 18$	certified	DRAT proofs on 0, z, $x^{(i)}$ (11); integer map, full cube (1,024)	2 encoders; 3 checkers
H_{11}, H_{15}, H_{17}: no integer map for $k < k_{\min}$; integer map for $k = k_{\min}$; so $k_{\min} = 26$, 36, 32	certified	DRAT proofs on 0, z, $x^{(i)}$ (13, 17, 19); integer maps, full cube (4,096, 65,536, 262,144)	2 encoders; 2 checkers each
superexponential lower bound for star inequalities, which are implied by subadditivity and not balanced (Corollary 11.39)	argued	written proof; rests on a proposition and the main theorem of Alon and Vũ	
upper bound for star inequalities (Proposition 11.42)	argued	written proof	
valid N-party inequalities with $k_{\min} = 8$, $N = 7, 8$	certified at $N = 7$, derived at $N = 8$	padding Z_7	
valid N-party inequalities with $k_{\min} = 32$, $N \ge 9$	certified at $N = 9$, derived for $N \ge 10$	padding G_9	
valid N-party inequalities with $k_{\min} = 36$, $N \ge 15$	certified at $N = 15$, derived for $N \ge 16$	padding H_{15}	

TABLE 4. Status of the results of Chapters 10 and 11 on star inequalities. All maps are on the full cube. Non-existence is shown on the full cube or on the $N + 2$ points 0, z and $x^{(i)}$, to which every map on the full cube restricts.

APPENDIX A

Glossary of symbols

Each symbol is listed with its meaning and the section where it is introduced. Where the companion uses a different letter from the paper, the entry says so.

Parties, entropies, graph models and the cone		
N, $[N]$	the number of parties; the set $\{1, \ldots, N\}$ of parties	Ch. 1
$A, B, \ldots, H$	parties in examples	Ch. 1
O	the purifier, standing for the rest of the boundary	Ch. 1
i, j	parties or the purifier, elements of $[N] \cup \{O\}$; in Chs. 10–11 parties are i only, and j is the level of D_j	Ch. 1
I, J; I_l, J_r	subsets of $[N]$; the LHS and RHS terms of an inequality	Ch. 1
$S(I)$, S	the entropy of I; the entropy vector in $\mathbb{R}^{2^N-1}$	Ch. 1
$[P]$	Iverson bracket, 1 if P holds and 0 otherwise	Preface
Γ, S_Γ	a graph model and its entropy vector (the paper writes S_G)	Ch. 2
w; e, w_e	a vertex of a graph model (the paper writes v); an edge and its weight	Ch. 2
W, $C(W)$	a cut (a set of vertices) and its weight	Ch. 2
HEC_N	the holographic entropy cone	Ch. 2
$\mathfrak{S}_{N+1}$, $\mathrm{Stab}(Q)$	the relabelings of $[N] \cup \{O\}$; the stabilizer of Q	Ch. 2
σ	a generic relabeling (the paper writes g)	Ch. 2
g	the relabeling $(DF)(EG)$ only	Ch. 9
γ_a	coefficients in the positive-combination lemma	Ch. 2
r_{ij}	the Bell-pair vector of i and j	Ch. 3
W_I	the minimal min cut of I	Ch. 5
$U_r(t)$, t	the threshold cut and the threshold in the proof of the realizable-domain criterion	Ch. 5
$\mathbf{S}$	the matrix whose rows are tight entropy vectors	Ch. 6
p	a prime, for ranks modulo p	Ch. 6

Inequalities and maps		
Q, kQ	an inequality and its coefficient vector, $Q \cdot S \geq 0$; its k-fold multiple	Ch. 1
L, R; l, r	the numbers of LHS and RHS terms; their indices	Ch. 1
α_l, β_r; $\lvert\alpha\rvert$, $\lvert\beta\rvert$	the LHS and RHS coefficients; their sums	Ch. 1
SA, MMI, MMI(X:Y:Z)	subadditivity, monogamy of mutual information, an instance of it	Ch. 1
$x^{(i)}$, $y^{(i)}$, $\overline{y}^{(i)}$	occurrence words of i: LHS, RHS, split RHS	Chs. 3, 4
d_α, d_β, d_1	weighted Hamming distances on the LHS cube, the RHS cube and the split cube	Chs. 3, 4
$\{0,1\}^L$	the full cube	Ch. 4
u, v, z	points of a cube; z is a distinguished point defined in each chapter where it occurs	Chs. 4, 7, 10
$x(w)$	the word of a vertex w (the paper writes $x(v)$)	Ch. 4
Ω	a domain, a subset of the cube containing 0 and the occurrence words	Ch. 4
$n = (n_r)$	a count map; real, or integer (unscaled)	Ch. 4
δ_α	the map that repeats bit l exactly α_l times (the paper writes δ)	Ch. 4
s	the copy index of the split RHS and the threshold index of the order encoding; the step index of E_s, μ_s	Chs. 4, 6, 8
M	a common denominator of a rational map	Ch. 4
k; $k_{\min}$, $k^{\mathcal{A}}_{\min}$	a scaling; the least scaling on the full cube and on the realizable domain	Ch. 4
$\mathcal{A}(Q)$; (R1), (R2)	the realizable domain; its two conditions	Ch. 5
$\mathcal{U}(Q)$, $[0, x^{(A)}]$, L_A	the restriction subdomain; the interval of A; the number of LHS terms containing A	Ch. 5
λ	the step of a splitting, as in $Q - \lambda X$ (the paper writes t or ε)	Ch. 6
$\sigma \cdot u$, $\sigma(l)$, $\sigma(r)$, n^σ	the action of a relabeling fixing O on points and indices; the conjugate map	Ch. 9

Named inequalities		
Q'	the three-party star $A + B + C + ABC \geq AB + AC$	Ch. 10
B_7^*, B_7	the balanced seven-party inequalities without unscaled maps	Ch. 7
Q_8^A	the balanced core, $N = 8$	Ch. 8
U_6	the superbalanced six-party inequality without an unscaled map	Ch. 8
$\psi_1, \ldots, \psi_4$	sums of instances of MMI in the identity for $6U_6$	Ch. 8
Q_8, Q_8', Q_8''	the superbalanced eight-party inequalities	Ch. 8
R_8	the sum of the Fano complements against $2(B + \cdots + H) + 2BCDEFGH$	Ch. 8
$E_0, \ldots, E_5$	the seven-party chain	Ch. 8
$\mu_0, \ldots, \mu_4$; μ_5	instances and orbit sums of MMI	Ch. 8; Ch. 9
F_7	the superbalanced facet of HEC_7	Ch. 9
V_7, V_7', X_7, Y_7	the other summands of the decomposition of $24E_0$	Ch. 9
P_7; M_s	the inequality with $2X_7 = V_7' + P_7 + 2g(P_7)$; the MMI combinations in $2E_s = V_7' + P_7 + 2M_s$	Ch. 9
D_j	the star family with $k_{\min} = 2^{j+2}$	Ch. 11
Z_7	the seven-party star with $k_{\min} = 8$	Ch. 11
G_9; H_9, H_{11}, H_{15}, H_{17}	the nine-party star without complementary words with $k_{\min} = 32$; certified coin-weighing stars	Ch. 11
$Q_{\mathcal{T}}$	the coin-weighing star of a list $\mathcal{T}$ of weighings	Ch. 11
SA_K^1, SA_K^2; P_I	the subadditivity chains of a star; positivity $S(I) \geq 0$	Ch. 10

The balanced example B_7^*		
$\mathcal{E}$; ϵ; $\mathcal{E}_i$	the six edges BC, BD, CD, EF, EG, FG of the two triangles; an edge; the two edges at the vertex i	Ch. 7
$\mathcal{C}$	a set of edges, the candidate for an integer value	Ch. 7
θ_ϵ, τ	the candidate values on $S(A\epsilon)$; the value on $S(ABCDEFG)$ is $14-\tau$ (the paper writes t)	Ch. 7
θ', τ'	the same for the doubled problem $2B_7^*$	Ch. 7

Star inequalities		
m; K	the number of RHS terms of a star; the index of an RHS term, a coin	Ch. 10
J_K, β_K	the RHS terms and coefficients of a star	Ch. 10
ρ_i, ρ_O	the radii	Ch. 10
$\eta^{(i)}$; $\lvert\eta\rvert$	the word of a party in a star; the sum of its entries	Ch. 10
$\Pi(Q)$	the star polytope	Ch. 10
$\mathcal{B}$, $\star$, e_i, $e_\star$, d	the star cube with one bit per singleton and the bit $\star$ of $S([N])$; its unit vectors; its weighted distance	Ch. 10
π	the projection that deletes the bits of weight 0	Ch. 10
c; ξ	a point of $\Pi(Q)$, the value $n(z)/k$; a generic point of $\Pi(Q)$	Chs. 10, 11
ϕ_K^1, ϕ_K^2; a_K, b_K	the maps of the subadditivity chains; their multiplicities	Ch. 10
$\omega^{(j)}$, q_j, m_j, N_j	the coin weights of D_j; their total; the number of coins; the number of parties	Ch. 11
T, $\mathcal{T}_j$, $T_1^{(j)}$, $\overline{T}$	a half; the list of halves; its first member; the complement of T	Ch. 11
u_j, v_j	the two coins added at level $j+1$	Ch. 11
$\mathbf{1}$, $\mathbf{1}_T$	the all-ones vector; the indicator vector of T	Ch. 11
$\omega(T)$, $\xi(X)$	the total weight of the coins in T, of the coordinates in X	Ch. 11
Λ_j	the solution space of the balance equations	Ch. 11
κ	the scalar in $\xi = \kappa\omega$ (the paper writes t)	Ch. 11
$\mathcal{T} = (T_1, \ldots, T_{N-1})$; γ_i	a list of weighings; the positive weights of condition (W2)	Ch. 11
$\nu(B)$; x	the largest ratio of two entries of a kernel vector of B (Alon and Vũ write $\xi(B)$); a kernel vector of B (the paper writes w)	Ch. 11
$\Delta(n)$	the largest absolute value of the determinant of an $n \times n$ matrix with entries in $\{0,1\}$	Ch. 11
$s^{(i)}$, h_i; M, B'	the coefficient vector and right-hand side of the constraint of i (the paper writes b_i); the matrices in the proof of the upper bound	Ch. 11

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