ALGEBRAICITY OF WEIL CLASSES ON NON-SPLIT ABELIAN SIXFOLDS OF WEIL TYPE OVER $\mathbb{Q}(\sqrt{-3})$ WITH HERMITIAN DETERMINANT -2

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Abstract. Let $K = \mathbb{Q}(\sqrt{-3})$. We give a proof that every Weil class is algebraic on every complex abelian sixfold of Weil type for K whose Hermitian form has signature $(3,3)$ and determinant -2 modulo norms. These forms are not split, so neither Schoen's theorem on sixfolds nor the theorem on sixfolds claimed in a preprint of Markman, both of which concern split forms, applies. We model the sixfolds by the part of the Jacobian of a cover $C \to \mathbb{P}^1$, branched at thirteen points, on which its group, the complex reflection group G_{32}, acts through its reflection representation. A degeneration argument with a ball-arithmetic certificate shows that this family dominates the nine-dimensional Shimura variety. On every member, an integral relation among characters induced from linear characters of subgroups, realized by correspondences with coefficients in the group algebra, reduces the Weil classes to the determinants of eigenspaces of eleven cyclic subcovers. A theorem of Schoen makes these determinants algebraic for the three étale subcovers; for the eight ramified ones we prove it for all cyclic covers of curves of degree at least 3 with at least 3 branch points and fully paired branching data, through a cone over a Fermat variety and Shioda's results. Specialization and isogenies then reach every sixfold. The determinant of the Hermitian form and the relation among characters are exact computations.

1. Introduction

For a smooth complex projective variety S, a class in $H^{2p}(S,\mathbb{Q})$ is a *Hodge class* if its image in $H^{2p}(S,\mathbb{C})$ has type (p,p), and it is *algebraic* if it is a rational linear combination of fundamental classes of subvarieties of codimension p. Algebraic classes are Hodge classes, and the Hodge conjecture asserts the converse. It holds for $p = 1$ by the Lefschetz $(1,1)$ theorem (see e.g. [1]) and is open in general, already for abelian varieties.

Let A be a complex abelian variety of dimension $2n$ and K an imaginary quadratic field acting on A up to isogeny so that both embeddings of K occur with multiplicity n on $H^{1,0}(A)$. Weil observed that the two-dimensional space $\mathrm{HW}(A) = \bigwedge_K^{2n} H^1(A,\mathbb{Q}) \subset H^{2n}(A,\mathbb{Q})$ consists of Hodge classes, and that for a general such A these classes do not lie in the subring generated by divisor classes [2]. These *Weil classes* are a standard test case of the Hodge conjecture for abelian varieties. A polarization compatible with K makes $H^1(A,\mathbb{Q})$ a Hermitian space over K of signature (n,n), and by Landherr's theorem [3] its isometry class is determined by the class of its determinant in $\mathbb{Q}^\times / \operatorname{Nm} K^\times$. The form is split (Section 2.1) exactly when this class is that of $(-1)^n$.

For $n = 2$ the Weil classes are known to be algebraic for $K = \mathbb{Q}(\sqrt{-3})$ and every determinant by Schoen [4], and a preprint of Markman claims a proof for every K [5, Corollary 1.6.1], obtained from its claim for sixfolds and Schoen's degeneration argument. For $n = 3$, Schoen proved algebraicity on a family of sixfolds for $K = \mathbb{Q}(\sqrt{-3})$ [4], which Markman identifies as the split sixfolds [6]; his earlier preprint describes them as having trivial discriminant [5], and in either description their determinant is not -2. That preprint claims a proof for the split sixfolds for every K [5, Theorem 1.5.1]. In March 2026, Mostaed's preprint described the question for the non-split sixfolds as "completely open" [7]. The theorem below was claimed first in a preprint of Severn [8], posted on 14 October 2026, unrefereed and declared to be primarily AI-generated, which uses the same family of covers by a different route (see What is new); in a later preprint the same author claims, through

Date: October 2026.

split abelian eightfolds of Weil type and products with elliptic curves with complex multiplication, the algebraicity of the Weil classes on every abelian sixfold of Weil type for $\mathbb{Q}(\sqrt{-3})$, of every determinant [9]. A preprint of Bouali claims a proof of the Hodge conjecture for all smooth projective varieties [10], and a preprint of OpenAI claims a proof of the Hodge conjecture for the self-powers of the Jacobian at the tensor-Hodge-generic points of families of abelian covers of curves [11, Theorem 1.1], using two companion preprints [12, 13]; we have not assessed these claims. In this note we give a proof of the following statement.

Theorem 1.1 (computer-assisted)**.** *Let $K = \mathbb{Q}(\sqrt{-3})$, let A be a complex abelian variety of dimension 6, and let $\iota\colon K \to \mathrm{End}(A) \otimes \mathbb{Q}$ be a homomorphism. Suppose that A has a polarization λ such that*

(a) the Rosati involution of λ restricts to complex conjugation on $\iota(K)$,
(b) both eigenspaces of $\iota(\sqrt{-3})$ on $H^{1,0}(A)$ have dimension 3, and
(c) the Hermitian form h_λ of λ (Section 2.2) satisfies $\det h_\lambda \equiv -2 \bmod \mathrm{Nm}\, K^\times$.

Then every class in $\mathrm{HW}(A,\iota) = \bigwedge^6_K H^1(A,\mathbb{Q})$ is algebraic.

Remark 1.2 (scope). Since 2 is not a norm from K, condition (c) excludes the split forms. No condition is imposed on the type of λ, on the order $\iota(K) \cap \mathrm{End}(A)$ or on $\mathrm{End}(A) \otimes \mathbb{Q}$; in particular abelian varieties with complex multiplication are included. Condition (c) is a property of (A,ι,λ) and not of (A,ι). If B_1 is an abelian threefold with an action ι_1 of K of signature $(2,1)$ and a compatible polarization λ_1 with Hermitian form h_1, then the Hermitian form of $(B_1, \bar\iota_1, c\lambda_1)$ is $-c\bar h_1$; so $A = B_1 \times B_1$ with the action $(\iota_1, \bar\iota_1)$ and the polarization $\lambda_1 \times c\lambda_1$, $c \in \mathbb{Z}_{>0}$, has determinant $-c^3(\det h_1)^2 \equiv -c$, which is split for $c = 1$ and satisfies (c) for $c = 2$. This is why the theorem asks for one polarization with (a)–(c).

Besides the arguments of this note and results cited where they are used, the main published inputs are Schoen's theorem on Hodge classes of generalized Prym varieties [14, Theorem 2.0, Corollary 3.1], Shioda's results on Fermat varieties [15], and the Lefschetz standard conjecture for abelian varieties [16, 17]. The proof is computer-assisted in two ways: exact computations in GAP [18] and Python, and a rank certificate in ball arithmetic [19], whose last step is the evaluation of a 9×9 determinant. Appendix A describes both.

Use of large language models and status of the proofs. This note was produced by large language models, instances of the Claude models of Anthropic running as autonomous agents, in October 2026. They found the construction, wrote the proofs and the programs of Appendix A, ran the computations and wrote the text. The arguments were checked only by other instances of the same models, acting as referees and recomputing the numbers with programs of their own; such checks may share the blind spots of the models that wrote the proofs. No human expert has checked the proofs. The steps we regard as least secure are the application of Theorem 4.1 to the large subcovers of Table 1, the rank certificate of Section 6, and the use of the character table and subgroup data of G_{32} computed by GAP.

While this work was in progress, the models read the statements of the preprint [11] and the constructions used in its proofs, but not the proofs themselves. That suitably branched covers of $\mathbb{P}^1$ with a primitive complex reflection group can have isotypic parts with the dimension and signature of Weil sixfolds was observed before that reading. The plan of obtaining the Weil classes from the determinant lines of eigenspaces of cyclic subcovers, with [11, Lemma 3.4] as the intended input for these lines, combined by a contraction under an integrality condition on multiplicity vectors, was formulated during it. The choice of G_{32} and of the branching type, the relation of Proposition 5.2, the correspondence in $\mathbb{C}[G]$, the dominance proof and the proof of Theorem 4.1 came afterwards; Theorem 4.1 was written from a summary of the statement of [11, Lemma 3.4] and of the construction in its proof (Remark 4.8). No result of [11, 12, 13] is used. The author and

the models learned of the preprint [8] on 18 October 2026, after the proof and the first versions of this note were complete.

The note has not been refereed. It is made public together with this account of how it was produced and checked, and if an error is found, a correction will be posted.

Code availability. The programs for the computations of Appendix A, with the tuple of Appendix B and the data of Tables 1 and 2, are available under the MIT licence at `https://github.com/ahippelainen/weil-nonsplit-abelian-sixfolds-public`.

Conventions. From here on, $K=\mathbb{Q}(\zeta)$ with $\zeta=e^{2\pi i/6}$, so that $K=\mathbb{Q}(\sqrt{-3})$ and $\zeta^2=e^{2\pi i/3}$, and $\zeta_m=e^{2\pi i/m}$. Cohomology has rational coefficients unless other coefficients are written. For a smooth projective variety S, $\mathrm{Alg}^k(S)_{\mathbb{Q}}\subset H^{2k}(S,\mathbb{Q})$ is the span of the classes of algebraic cycles of codimension k and $\mathrm{Alg}(S)_{\mathbb{Q}}=\bigoplus_k \mathrm{Alg}^k(S)_{\mathbb{Q}}$; a complex class lies in $\mathrm{Alg}(S)_{\mathbb{Q}}\otimes\mathbb{C}$ if it is a complex linear combination of algebraic classes. This space is stable under complex conjugation, and $(\mathrm{Alg}(S)_{\mathbb{Q}}\otimes\mathbb{C})\cap H^*(S,\mathbb{Q})=\mathrm{Alg}(S)_{\mathbb{Q}}$. A finite group acting on a variety acts on its cohomology and on functions by $g\cdot x=(g^{-1})^*x$, except in Sections 2.5 and 4, where groups act by pullback. For a finite group G acting on a curve C and an irreducible complex representation ρ' of G we put $M_{\rho'}=\mathrm{Hom}_G(\rho',H^1(C,\mathbb{C}))$, so that $H^1(C,\mathbb{C})=\bigoplus_{\rho'}\rho'\otimes M_{\rho'}$; for a subgroup H and a character ψ of H, a superscript (H,ψ) denotes the subspace on which H acts through ψ. We write ω_C for the canonical sheaf of a smooth curve and for the dualizing sheaf of a nodal curve. Section 4 has its own notation for a single cyclic cover, and Sections 5 and 6 introduce auxiliary letters that are used only there.

The method. Let $G=G_{32}$ be the complex reflection group of order 155520, number 32 in the list of Shephard and Todd [20], let $\rho\colon G\to\mathrm{GL}_4(K)$ be its reflection representation, and let V_ρ be ρ viewed as a $\mathbb{Q}[G]$-module, so that $\mathrm{End}_{\mathbb{Q}[G]}(V_\rho)=K$. For a G-cover $C\to\mathbb{P}^1$ the space

$$B_C=\mathrm{Hom}_{\mathbb{Q}[G]}\left(V_\rho,H^1(C,\mathbb{Q})\right) \tag{1.3}$$

is the first cohomology of an abelian variety A_C with an action of K, defined up to isogeny. For covers branched at thirteen points whose local monodromies are six reflections of determinant ζ^2, six of determinant ζ^4 and one involution with a two-dimensional (-1)-eigenspace, A_C is a sixfold of Weil type of signature $(3,3)$ (Lemma 3.1). We fix the Hurwitz component $\mathcal{H}$ of such covers given by the tuple of Appendix B and show by an exact computation that $\det h\equiv -2$ on all of $\mathcal{H}$ (Proposition 3.2).

The Weil classes of A_C are reached through cyclic subcovers. For a subgroup $H\le G$ that contains the centre $Z(G)$ and a character ψ of H that agrees on $Z(G)$ with the central character of ρ, the cover $C/\ker\psi\to C/H$, the *cyclic subcover* of (H,ψ), has cyclic group $H/\ker\psi$, and the eigenspace of ψ in the first cohomology of $C/\ker\psi$, pulled back to C, is $W_{H,\psi}=\bigoplus_{\rho'}\rho'^{(H,\psi)}\otimes M_{\rho'}$. Its multiplicity vector $v_{H,\psi}=(\dim\rho'^{(H,\psi)})_{\rho'}$ is indexed by the fourteen irreducible representations ρ' with the central character of ρ. We find eleven pairs (H_j,ψ_j) and signs c_j with

$$e_\rho=\sum_{j=1}^{11}c_j\,v_{H_j,\psi_j}\ , \tag{1.4}$$

where e_ρ is the indicator vector of ρ (Proposition 5.2 and Tables 1, 2). Eight of the subcovers are ramified. For them, Theorem 4.1, which we prove for cyclic covers of curves of degree $m\ge 3$ with any base genus, $r\ge 3$ branch points and fully paired branching data (Section 2.5), shows that the top exterior power Λ_j of $W_j=W_{H_j,\psi_j}$, its determinant line, is a complex combination of algebraic classes on the Jacobian $J(C)$; for the three étale subcovers this is Schoen's theorem [14]. By (1.4), for $0\ne w'\in\rho$ there is a matrix Ψ with entries in $\mathbb{C}[G]$ that maps $U_+=\bigoplus_{c_j=+1}W_j$, placed in seven copies of $J(C)$, isomorphically onto $(w'\otimes M_\rho)\oplus U_-$, placed in five further copies,

with $U_- = \bigoplus_{c_j=-1} W_j$. Complex combinations of homomorphisms of abelian varieties realize Ψ on cohomology, and on top exterior powers

$$\bigwedge \Psi^*\big(\bigwedge U_+\big) = \det\big(\Psi^*|_{U_+}\big)\cdot \Lambda_{w'}\otimes\Lambda_- \ , \qquad \Lambda_{w'} = \bigwedge{}^6(w'\otimes M_\rho)\ , \quad \Lambda_- = \bigwedge U_- \ . \tag{1.5}$$

Contracting with the complex conjugate of the algebraic class Λ_- and a power of a polarization isolates $\Lambda_{w'}$, and the rational Weil classes of A_C follow, for every member of $\mathcal{H}$ (Theorem 5.8).

A degeneration to an admissible cover and a ball-arithmetic evaluation of a 9×9 minor of a limit of the codifferential show that the period map of $\mathcal{H}$ has rank 9 (Theorem 6.1). Since the Shimura variety of these sixfolds has dimension 9, the family dominates it, and specialization, isogenies and Landherr's theorem reach every sixfold of Theorem 1.1 (Section 7).

Section 2 collects preliminaries, Section 3 defines the family and computes its determinant, Section 4 proves Theorem 4.1, Section 5 proves algebraicity on every member of the family, Section 6 proves dominance, and Section 7 proves Theorem 1.1. Appendix A describes the computations and Appendix B gives the tuple.

What is new. Theorem 1.1 was claimed first in the preprint [8]. The proof in this note was obtained independently of [8] and completed before it appeared. Both use the ρ-isotypic factors of G_{32}-covers of $\mathbb{P}^1$ with the same branching type, and the two routes differ in the supply of Weil classes and in the proof of dominance. In [8] the Weil classes come from Schoen's algebraic classes on fifty-one cyclic quotients, chosen by an integral identity among characters and combined by correspondences given by Lefschetz classes through a theorem of Milne, and dominance is shown by an exact computation: along an explicit isomonodromic deformation of an explicit cover, a 9×9 minor of the codifferential of the period map has a non-zero Taylor coefficient of order seven. Here eleven cyclic subcovers are combined through a signed integral relation among their multiplicity vectors, realized by correspondences with coefficients in $\mathbb{C}[G]$ and a contraction (Section 5); Theorem 4.1 gives a complete proof for the eight ramified subcovers, all of degree 6 or 12, which the printed proof of Schoen's Lemma 2.8 in [14] does not reach, and for which his later paper [21] sketches a different argument (Remark 4.7); and dominance is proved by a degeneration to a boundary point with a ball-arithmetic certificate (Section 6). The current version of [8] also gives a different proof of Schoen's Lemma 2.8 (Remark 4.7). We have not checked the proof of [8]. Kani and Rosen derive isogenies between products of Jacobians of quotient curves from relations among idempotents of $\mathbb{Q}[G]$ [22]; here the characters are twisted by linear characters ψ, and the relation serves to extract a determinant line rather than an isogeny. Van Geemen and Verra applied Schoen's cyclic theorem to a cyclic subcover of a quaternionic étale cover [23, Corollary 4.10], and the tools of [11] include factorizations of determinant lines across direct sums and contractions of paired determinants, for abelian covers. Theorem 4.1 follows Schoen's approach [14, 21].

2. Preliminaries

2.1. **Hermitian forms.** A Hermitian form h on a finite-dimensional K-vector space V is K-linear in the first variable and satisfies $h(y,x)=\overline{h(x,y)}$. Its determinant $\det h$ is the class in $\mathbb{Q}^\times/\operatorname{Nm}K^\times$ of the determinant of a Gram matrix; a change of basis multiplies the Gram determinant by a norm, and rescaling h by $c\in\mathbb{Q}^\times$ multiplies it by $c^{\dim V}$. A form of dimension $2n$ is *split* if V has a totally isotropic subspace of dimension n. Landherr's theorem [3], in the form of [24, Proposition 4.1, Corollary 4.2], says that (V,h) is determined up to isometry by its dimension, its signature and $\det h$, and that a form of dimension $2n$ is split if and only if it has signature (n,n) and $\det h\equiv(-1)^n$. Deligne and Markman call the class $\det h$ the discriminant [24, 5], so the sixfolds of Theorem 1.1 are those of discriminant -2 in their terms. Norms from K are positive and have even valuation at the prime 2, which is inert in K; so neither 2 nor -2 is a norm, and a form with $\det h\equiv -2$ is not split.

2.2. **Weil type.** Let A be an abelian variety of dimension $2n$ with a homomorphism $\iota\colon K \to \mathrm{End}(A)\otimes\mathbb{Q}$ and a polarization λ with Riemann form E on $H^1(A,\mathbb{Q})$. We normalize the complex structure J of $H^1(A,\mathbb{R})$ given by the Hodge decomposition and the sign of E so that $E(x,Jx)>0$ for $x\neq 0$. The triple (A,ι,λ) is *of Weil type of signature* (n,n) if the Rosati involution restricts to complex conjugation on K, that is, $E(ax,y)=E(x,\bar a y)$ for $a\in K$, and both eigenspaces of $\iota(\sqrt{-3})$ on $H^{1,0}(A)$ have dimension n. There is then a unique Hermitian form h_λ on the K-vector space $H^1(A,\mathbb{Q})$ with $E=\mathrm{Tr}_{K/\mathbb{Q}}(\sqrt{-3}\,h_\lambda)$, and by the Riemann relations it has signature (n,n). Let $H^1(A,\mathbb{C})_\pm$ be the subspaces on which $\iota(\sqrt{-3})$ acts by $\pm\sqrt{-3}$. They are exchanged by complex conjugation, so the sum of the lines $\bigwedge^{2n} H^1(A,\mathbb{C})_+$ and $\bigwedge^{2n} H^1(A,\mathbb{C})_-$ is the complexification of a two-dimensional subspace $\mathrm{HW}(A,\iota)\subset H^{2n}(A,\mathbb{Q})$, which is $\bigwedge^{2n}_K H^1(A,\mathbb{Q})$. Each $H^1(A,\mathbb{C})_\pm$ has Hodge numbers (n,n), so both lines have type (n,n) and $\mathrm{HW}(A,\iota)$ consists of Hodge classes [2].

Lemma 2.1. *Let $f\colon A\to A'$ be an isogeny with $f\circ\iota(a)=\iota'(a)\circ f$ for $a\in K$. Then* $\mathrm{HW}(A,\iota)$ *consists of algebraic classes if and only if* $\mathrm{HW}(A',\iota')$ *does.*

Proof. The map $f^*\colon H^1(A',\mathbb{Q})\to H^1(A,\mathbb{Q})$ is a K-linear isomorphism of Hodge structures, so $\bigwedge^{2n} f^*$ maps $\mathrm{HW}(A',\iota')$ onto $\mathrm{HW}(A,\iota)$. Since f^* and f_* preserve algebraic classes and f_*f^* is multiplication by $\deg f$, the claim follows. □

For a Hermitian space (V,h) over K of dimension 6 and signature $(3,3)$, with $E=\mathrm{Tr}_{K/\mathbb{Q}}(\sqrt{-3}\,h)$, let $\mathfrak{D}$ be the set of complex structures J on $V\otimes\mathbb{R}$ that commute with K, preserve E and satisfy $E(x,Jx)>0$ for $x\neq 0$. It is a connected Hermitian symmetric domain of dimension 9, isomorphic to $U(3,3)/(U(3)\times U(3))$. A point of $\mathfrak{D}$ is determined by the 3-dimensional subspace F^1 of type $(1,0)$ in the eigenspace $V\otimes_K\mathbb{C}$ of one embedding of K, and the tangent space of $\mathfrak{D}$ there is $\mathrm{Hom}(F^1, V\otimes_K\mathbb{C}/F^1)$.

2.3. **Standard facts.** We use the following standard facts; (T1) and (T7) we state without reference or proof.

(T1) For a finite group G and $r\geq 3$, the G-covers of $\mathbb{P}^1$ branched at r points, up to isomorphism, form a finite topological covering of the configuration space of r distinct unordered points of $\mathbb{P}^1$. Its fibre over a configuration is the set of tuples up to simultaneous conjugation, and its monodromy is the action of the braid group. By Riemann's existence theorem it is a smooth quasi-projective variety, finite étale over the configuration space, so each Hurwitz component is a smooth irreducible quasi-projective variety of dimension r, on which PGL_2 acts.

(T2) For a smooth family of curves, the differential of the period map along a tangent vector ∂ is cup product with the Kodaira–Spencer class $\mathrm{KS}(\partial)\in H^1(C,T_C)$, as a map $F^1H^1(C)\to H^1(C)/F^1$, and $E_C(\mathrm{KS}(\partial)\cup\phi,\eta)=\pm\langle\mathrm{KS}(\partial),\phi\eta\rangle$ for $\phi,\eta\in H^0(\omega_C)$, with the cup-product form E_C on $H^1(C)$, the Serre duality pairing $H^1(C,T_C)\times H^0(\omega_C^{\otimes 2})\to\mathbb{C}$ and a universal sign; see e.g. [25].

(T3) For a flat proper family of nodal curves over a reduced base with $h^0(\omega)$ constant on the fibres, $\pi_*\omega_{\mathcal{C}/\Delta}$ is locally free and commutes with base change [26]; see e.g. [27].

(T4) On a nodal curve, the sections of the dualizing sheaf, the *Rosenlicht differentials*, are the meromorphic differentials on the normalization with at most simple poles at the preimages of the nodes and opposite residues on the two branches; see e.g. [28].

(T5) For a family of G-covers of $\mathbb{P}^1$ in which the branch point x_i moves, and a G-invariant quadratic differential $q=\pi^*(f(x)\,dx^2)$ on C, $\langle\mathrm{KS}(\partial_{x_i}),q\rangle = c_{\mathrm{KS}}\,\mathrm{Res}_{x=x_i} f(x)\,dx$, with a constant $c_{\mathrm{KS}}\neq 0$ that does not depend on i. Indeed, $\mathrm{KS}(\partial_{x_i})$ is represented on small discs around the preimages of x_i by the lift of ∂_x, and at each of the $|G|/e$ preimages, e the ramification index, the residue of the lift of ∂_x contracted with q is $e\,\mathrm{Res}_{x=x_i} f(x)\,dx$. For the opening of a node the analogous formula is [28, Lemma 4].

(T6) The monodromy of the plumbing $zz' = t$ around $t = 0$ is the Dehn twist about the vanishing circle, the case of dimension one of the Picard–Lefschetz formula; see e.g. [29, 30]. A Dehn twist about a circle δ acts on H_1 by $a \mapsto a + (a \cdot \delta)\,\delta$, with the intersection number $a \cdot \delta$, in the conventions of [29]; see e.g. [31]. By Poincaré duality it acts on H^1 by $u \mapsto u + N_\delta u$ with $N_\delta u = \pm\big(\int_\delta u\big)\,\delta^\vee$, where $\delta^\vee \in H^1$ is the class with $E_C(\delta^\vee, u') = \int_\delta u'$, so that $E_C(N_\delta u, u') = \pm \int_\delta u \int_\delta u'$, with a universal sign.

(T7) The rank of a holomorphic family of matrices does not drop under small deformations of the parameter; the rank of a composite of holomorphic maps is at most the rank of each factor.

2.4. **Covers of curves.** Paths are composed from left to right: $\ell\ell'$ is ℓ followed by ℓ'. For a local system $\mathcal{F}$ on a connected space with base point b, a loop ℓ at b acts on the stalk $\mathcal{F}_b$ by transport along ℓ^{-1}; this makes $\mathcal{F}_b$ a left module over the fundamental group, and $\mathcal{F} \mapsto \mathcal{F}_b$ is an equivalence between local systems and such modules. For real points $x_1 < \cdots < x_r$ of $\mathbb{P}^1$ and $U = \mathbb{P}^1 \setminus \{x_1, \dots, x_r\}$ we take the base point b in the upper half-plane and loops $\ell_1, \dots, \ell_r$, where ℓ_i crosses the real line first in (x_{i-1}, x_i) and then in (x_i, x_{i+1}), so that it encircles x_i once counterclockwise; then $\pi_1(U, b)$ is generated by the ℓ_i with the single relation $\ell_1 \cdots \ell_r = 1$ (see e.g. [32], whose conventions we follow).

For a finite group G, a *G-cover* of $\mathbb{P}^1$ is a smooth projective connected curve C with a faithful action of G such that $C/G = \mathbb{P}^1$; let $\pi\colon C \to \mathbb{P}^1$ be the quotient map, branched at $x_1, \dots, x_r$, and π° its restriction over U. The stalk of $\pi^\circ_*\mathbb{C}$ at b is the space of functions on $\pi^{-1}(b)$. After the choice of a point over b, which identifies $\pi^{-1}(b)$ with G, the group G acts by $(g \cdot F)(g') = F(g^{-1}g')$, and $\pi_1(U, b)$ acts by an action that commutes with it, hence by right translations: $(\ell \cdot F)(g') = F(g'\gamma_C(\ell))$ for a homomorphism $\gamma_C\colon \pi_1(U, b) \to G$, determined by the cover up to conjugation. We call $(\gamma_C(\ell_1), \dots, \gamma_C(\ell_r))$ the *tuple* of the cover; its entries are non-trivial, generate G and have product 1. As the branch points move, the braid group acts on tuples, and a *Hurwitz component* is a connected component of the space of covers (T1).

For an irreducible representation ρ' of G, the local system $\operatorname{Hom}_G(\rho', \pi^\circ_*\mathbb{C})$ on U corresponds to the module $\rho'^\vee$ with ℓ acting by $\rho'^\vee(\gamma_C(\ell))$: an equivariant map $\Upsilon\colon \rho' \to \operatorname{Map}(G, \mathbb{C})$ is determined by $a = \Upsilon(\cdot)(1) \in \rho'^\vee$, through $\Upsilon(v)(g') = a(g'^{-1}v)$, and $(\ell \cdot \Upsilon)(\cdot)(1) = a(\gamma_C(\ell)^{-1} \cdot) = \rho'^\vee(\gamma_C(\ell))a$. Since π is finite, $\pi_*\mathbb{C} = j_*\pi^\circ_*\mathbb{C}$ for the inclusion $j\colon U \to \mathbb{P}^1$, and so $M_{\rho'} = H^1(\mathbb{P}^1, j_* \operatorname{Hom}_G(\rho', \pi^\circ_*\mathbb{C}))$.

For $\rho' \neq 1$ the Chevalley–Weil formula gives

$$\dim M_{\rho'} = (r-2)\dim \rho' - \sum_{i=1}^{r} \dim \operatorname{Fix}(\gamma_i|_{\rho'})\ , \tag{2.2}$$

where $(\gamma_1, \dots, \gamma_r)$ is the tuple of the cover, and the Hodge numbers $h^{1,0}$, $h^{0,1}$ of $M_{\rho'}$ are the two numbers

$$-\dim \rho' + \sum_{i=1}^{r}\sum_{k} \tau_{ik}\ , \qquad -\dim \rho' + \sum_{i=1}^{r}\sum_{k} (1 - \tau_{ik})\ , \tag{2.3}$$

where $e^{2\pi i \tau_{ik}}$, $0 < \tau_{ik} < 1$, runs over the eigenvalues $\neq 1$ of γ_i on ρ' [33]; see e.g. [34, Theorem 2.10]. For a connected cyclic cover $X \to Y$ of a curve of genus g with group $\mu_m = \langle\sigma\rangle$, branched at r points $b_1, \dots, b_r$ with local monodromies $\sigma^{a_1}, \dots, \sigma^{a_r}$, and a character $\chi \neq 1$ of μ_m with $\chi(\sigma) = \zeta_m^k$, the subspace of $H^1(X, \mathbb{C})$ on which σ^* acts by $\chi(\sigma)$ has dimension $2g - 2 + \#\{i : ka_i \not\equiv 0 \bmod m\}$, and its Hodge numbers are the two numbers $g - 1 + \sum_i \langle ka_i/m\rangle$ and $g - 1 + \sum_i \langle -ka_i/m\rangle$, where $\langle\cdot\rangle$ is the fractional part [33]. We include a proof. The part $\mathcal{O}_\chi$ of the direct image of $\mathcal{O}_X$ on Y on which σ^* acts by $\chi(\sigma)$ is a line bundle on Y with $H^0(Y, \mathcal{O}_\chi) = 0$, since $\chi \neq 1$ and $H^0(X, \mathcal{O}_X)$ consists of the constants. The product $\mathcal{O}_\chi^{\otimes m} \to \mathcal{O}_Y$ is injective and an isomorphism away from the branch points, and it vanishes at b_i to order $m\langle \pm ka_i/m\rangle$, with a sign fixed by the conventions and the same for all i, as one sees on a disc over b_i with a coordinate w such that w^e, e the order of σ^{a_i},

is a coordinate of Y at b_i and σ^{a_i}, the local monodromy at b_i, multiplies w by $\zeta_e^{\pm 1}$, with this sign. So $\deg \mathcal{O}_\chi = -\sum_i \langle \pm k a_i/m \rangle$, and by Riemann–Roch the part of $H^{0,1}(X) = H^1(X, \mathcal{O}_X)$ on which σ^* acts by $\chi(\sigma)$, which is $H^1(Y, \mathcal{O}_\chi)$ since $X \to Y$ is finite, has dimension $g - 1 + \sum_i \langle \pm k a_i/m \rangle$. Complex conjugation maps the part of $H^{1,0}(X)$ on which σ^* acts by $\chi(\sigma)$ onto the part of $H^{0,1}(X)$ on which it acts by $\overline{\chi(\sigma)}$, and $\langle y \rangle + \langle -y \rangle$ is 1 for $y \notin \mathbb{Z}$ and 0 for $y \in \mathbb{Z}$. We use these Hodge-number formulas only in cases where the two numbers are equal, so it does not matter which of them is $h^{1,0}$.

For a tuple $\gamma = (\gamma_1, \dots, \gamma_r)$ in $\mathrm{GL}_4(K)$ with product 1, let L_γ be the local system of K-vector spaces of rank 4 on U corresponding to the module K^4 with ℓ_i acting by γ_i. Its *parabolic cohomology* is $H^1_{\mathrm{par}} = H^1(\mathbb{P}^1, j_* L_\gamma)$. The group $H^1(U, L_\gamma)$ is the cohomology of the group $\pi_1(U, b)$ with coefficients in K^4, and H^1_{par} is the subspace of classes of cocycles u with $u(\ell_i) \in \mathrm{Im}(\gamma_i - 1)$ for all i (see e.g. [32], where it is stated for right modules). If γ is the tuple of a G-cover with $G \subset \mathrm{GL}_4(K)$ and ρ is the inclusion, then the module of $L_\gamma \otimes_K \mathbb{C}$ is $\rho \circ \gamma_C = (\rho^\vee)^\vee \circ \gamma_C$, so $L_\gamma \otimes_K \mathbb{C} \cong \mathrm{Hom}_G(\rho^\vee, \pi_*^\circ \mathbb{C})$ and $H^1_{\mathrm{par}} \otimes_K \mathbb{C} \cong M_{\rho^\vee}$.

2.5. **Fermat varieties.** For $m \geq 2$ and $s \geq 0$ let $F_m^s = \{u_1^m + \cdots + u_{s+2}^m = 0\} \subset \mathbb{P}^{s+1}$, and let $\mathfrak{T}_m^s = \mu_m^{s+2}/\mu_m$, the quotient by the diagonal, act by $u_i \mapsto \zeta_m^{c_i} u_i$. A character of $\mathfrak{T}_m^s$ is a vector $\alpha \in (\mathbb{Z}/m)^{s+2}$ with $\sum_i \alpha_i = 0$, acting by $(\zeta_m^{c_i})_i \mapsto \zeta_m^{\sum_i \alpha_i c_i}$; let $H^s(F_m^s, \mathbb{C})_\alpha$ be the subspace on which $\mathfrak{T}_m^s$ acts through α, and $\mathfrak{A}_m^s$ the set of α with all $\alpha_i \neq 0$. By Shioda's Theorem I [15], for $\alpha \in \mathfrak{A}_m^s$ the space $H^*(F_m^s, \mathbb{C})_\alpha$ is a line in degree s, since the cohomology outside the primitive middle part is spanned by powers of the hyperplane class, on which $\mathfrak{T}_m^s$ acts trivially. For s even let $\mathfrak{C}_m^s \subset \mathfrak{A}_m^s$ be the set of α with $H^s(F_m^s, \mathbb{C})_\alpha \subset \mathrm{Alg}^{s/2}(F_m^s)_\mathbb{Q} \otimes \mathbb{C}$. Shioda proves [15, Corollary to Theorem II]: if $s, s' \geq 1$ are odd, $\beta \in \mathfrak{C}_m^{s-1}$ and $\beta' \in \mathfrak{C}_m^{s'-1}$, then the concatenation (β, β') lies in $\mathfrak{C}_m^{s+s'}$. Since this statement is unchanged when all characters are replaced by their negatives, it holds for $\mathfrak{C}_m^s$ as defined here. A character is *fully paired* if it is a permutation of $(n_1, -n_1, \dots, n_k, -n_k)$ with all $n_i \neq 0$; a tuple of elements of $\mathbb{Z}/m$ is fully paired in the same sense.

Lemma 2.4. *Every fully paired character of length $2k$ lies in $\mathfrak{C}_m^{2k-2}$.*

Proof. Permutations of the coordinates are automorphisms of F_m^{2k-2} and permute the lines $H^{2k-2}(F_m^{2k-2}, \mathbb{C})_\alpha$ accordingly, so $\mathfrak{C}_m^{2k-2}$ is stable under permutations. For $k = 1$, F_m^0 is a set of m points and its cohomology consists of algebraic classes, so $(n_1, -n_1) \in \mathfrak{C}_m^0$. If $(n_1, -n_1, \dots, n_{k-1}, -n_{k-1}) \in \mathfrak{C}_m^{2k-4}$, Shioda's statement with $(s, s') = (2k-3, 1)$ gives $(n_1, -n_1, \dots, n_k, -n_k) \in \mathfrak{C}_m^{2k-2}$. □

3. The G_{32} family

Let $G = G_{32}$ and $\rho\colon G \to \mathrm{GL}_4(K)$ be as in the introduction. The representation ρ is absolutely irreducible, since the commutant in $\mathrm{GL}_4(\mathbb{C})$ of the generating tuple γ of Appendix B consists of scalars (Appendix A), and its character field is K, since its character takes the value $2 - \zeta^2 \notin \mathbb{Q}$ at γ_1 (Table 2). So V_ρ, which is K^4 viewed as a $\mathbb{Q}[G]$-module of dimension 8, is irreducible with $\mathrm{End}_{\mathbb{Q}[G]}(V_\rho) = K$, and $V_\rho \otimes \mathbb{C} = \rho \oplus \bar\rho$, where K acts on ρ through its inclusion in $\mathbb{C}$ and on $\bar\rho$ through the conjugate. A G-invariant Hermitian inner product identifies $\bar\rho$ with the dual $\rho^\vee$. We identify G with $\rho(G)$.

Let $\gamma = (\gamma_1, \dots, \gamma_{13})$ be the tuple of Appendix B. It consists of twelve reflections of order 3, those with $i \in \{2, 3, 4, 7, 8, 12\}$ of determinant ζ^2 and the others of determinant ζ^4, and an involution γ_{13} with $\mathrm{rk}(\gamma_{13} - 1) = 2$ in the conjugacy class of size 90; it satisfies $\gamma_1 \cdots \gamma_{13} = 1$, generates G, and its commutant in $\mathrm{GL}_4(\mathbb{C})$ consists of scalars (Appendix A). Let $\mathcal{H}$ be the Hurwitz component of the covers with tuple γ, of dimension 13 by (T1), and for C in $\mathcal{H}$ let B_C be as in (1.3). The field $K = \mathrm{End}_{\mathbb{Q}[G]}(V_\rho)$ acts on B_C by $(af)(u) = f(au)$, and $B_C \otimes \mathbb{C} = M_\rho \oplus M_{\bar\rho}$, the two eigenspaces of K.

Lemma 3.1. *Let C be in $\mathcal{H}$, let E_C be the cup-product form on $H^1(C,\mathbb{Q})$, let $\epsilon \in \{\pm 1\}$ be the sign for which ϵE_C is the Riemann form of the principal polarization of $J(C)$ in the normalization of Section 2.2, and let $w \in V_\rho$ be rational and non-zero.*

(i) B_C is a K-vector space of dimension 6 and a rational Hodge structure of weight one on which K acts by morphisms, and M_ρ, $M_{\bar\rho}$ have Hodge numbers $(3,3)$.
(ii) Evaluation $f \mapsto f(w)$ embeds B_C into $H^1(C,\mathbb{Q})$ as a sub-Hodge structure, and $E_w(f,f') = \epsilon E_C(f(w),f'(w))$ is a polarization of B_C with $E_w(af,f') = E_w(f,\bar a f')$ for $a \in K$.
(iii) The Hermitian form h_w of E_w has signature $(3,3)$, and it is invariant under the monodromy of B_C along loops in $\mathcal{H}$.

So the abelian variety A_C with $H^1(A_C,\mathbb{Q}) = B_C$, with this action of K and the polarization E_w, is of Weil type of signature $(3,3)$.

Proof. By (2.2), $\dim M_\rho = 11\cdot 4 - (12\cdot 3+2) = 6$. A reflection of determinant ζ^2 has the eigenvalue $e^{2\pi i/3}$ on ρ, one of determinant ζ^4 has $e^{4\pi i/3}$, and γ_{13} has -1 twice, so both numbers (2.3) equal $-4 + 6\cdot\frac13 + 6\cdot\frac23 + 2\cdot\frac12 = 3$; the same holds for $\bar\rho$. Since $B_C \otimes \mathbb{C} = M_\rho \oplus M_{\bar\rho}$ has dimension 12, $\dim_K B_C = 6$, which gives (i), the Hodge structure being induced from $H^1(C,\mathbb{Q})$.

If $f(w) = 0$, then $\ker f$ is a non-zero $\mathbb{Q}[G]$-submodule of the irreducible V_ρ, so $f = 0$; hence evaluation at w is injective, and its image is a sub-Hodge structure. The restriction of the polarization ϵE_C of $J(C)$ to it is a polarization. For fixed f, f' the bilinear form $(u,u') \mapsto E_C(f(u),f'(u'))$ on V_ρ is G-invariant, so it equals $\langle bu, u'\rangle_0$ for some $b \in K$, where $\langle\cdot,\cdot\rangle_0$ is a fixed G-invariant positive definite symmetric form on V_ρ. The adjoint for $\langle\cdot,\cdot\rangle_0$ is a positive involution of the imaginary quadratic field K, hence complex conjugation, and so $E_w(af,f') = \epsilon\langle baw, w\rangle_0 = \epsilon\langle bw, \bar a w\rangle_0 = E_w(f,\bar a f')$. This proves (ii), and the signature in (iii) follows from (i) as in Section 2.2. The monodromy of $H^1(C,\mathbb{Q})$ along loops in $\mathcal{H}$ preserves E_C and commutes with G, up to the automorphisms of the covers, which are the elements of $Z(G)$ and act on B_C by scalars in $\mu_6 \subset K$; such scalars preserve h_w. □

Proposition 3.2 (computer-assisted)**.** *For every C in $\mathcal{H}$ and every rational $w \neq 0$, $\det h_w \equiv -2 \bmod \operatorname{Nm} K^\times$. In particular h_w is not split.*

Proof. Since $\mathcal{H}$ is connected and h_w is invariant under the monodromy (Lemma 3.1(iii)), the class of $\det h_w$ is constant on $\mathcal{H}$, so we may take C with real branch points $x_1 < \cdots < x_{13}$ and $\gamma_C(\ell_i) = \gamma_i$. Let $\pi\colon C \to \mathbb{P}^1$ be the quotient map. By Section 2.4, $B_C = H^1(\mathbb{P}^1, j_*\mathcal{M})$ for the local system $\mathcal{M} = \operatorname{Hom}_{\mathbb{Q}[G]}(V_\rho, \pi_*^\circ\mathbb{Q})$ of K-vector spaces of rank 4 on U, with K acting through V_ρ. The summand of $\mathcal{M}\otimes_{\mathbb{Q}}\mathbb{C}$ on which K acts through its inclusion in $\mathbb{C}$ is $\operatorname{Hom}_G(\rho, \pi_*^\circ\mathbb{C})$, whose module is $\rho^\vee \circ \gamma_C \cong \bar\rho\circ\gamma_C$, the complex conjugate of the module $\rho\circ\gamma_C$ of $L_\gamma \otimes_K \mathbb{C}$. So $\mathcal{M}$ and the local system $\bar L_\gamma$, which is L_γ with K acting through complex conjugation, become isomorphic after $\otimes_K\mathbb{C}$. Since the commutant of γ consists of scalars, $\operatorname{Hom}_K(\mathcal{M}, \bar L_\gamma)$ is a line over K, and its non-zero elements are isomorphisms. This gives a K-antilinear isomorphism $\tau\colon B_C \to H^1_{\mathrm{par}}$, the parabolic cohomology of L_γ.

A cocycle u is determined by $u_i = u(\ell_i) \in K^4$, subject to $\sum_i \gamma_1\cdots\gamma_{i-1}u_i = 0$, and it is parabolic if $u_i \in \operatorname{Im}(\gamma_i - 1)$ for all i; the coboundaries are $((\gamma_i - 1)v)_i$, $v \in K^4$. The braid generator σ_j acts on $\pi_1(U,b)$ by Artin's automorphism $\ell_j \mapsto \ell_j\ell_{j+1}\ell_j^{-1}$, $\ell_{j+1} \mapsto \ell_j$, on tuples by $(\gamma_j,\gamma_{j+1}) \mapsto (\gamma_j\gamma_{j+1}\gamma_j^{-1}, \gamma_j)$, and on cocycles, by pullback, by

$$u_j \mapsto u_j + \gamma_j u_{j+1} - \gamma_j\gamma_{j+1}\gamma_j^{-1}u_j\ , \qquad u_{j+1} \mapsto u_j\ , \tag{3.3}$$

the other u_i being unchanged. Whether σ_j corresponds to a half-twist or to its inverse is immaterial below. For a braid β that fixes γ, let A_β be the map induced on H^1_{par}. By [32], applied to the local systems $\mathcal{M}$ of the covers along the corresponding loop in $\mathcal{H}$, τ carries the monodromy of B_C along this loop to $c_\beta A_\beta$, where c_β comes from an element of GL_4 that commutes with all γ_i, hence is a

scalar. The monodromy preserves the lattice $\mathrm{Hom}_{\mathbb{Z}[G]}(L_\rho, H^1(C,\mathbb{Z}))$ for a G-stable lattice $L_\rho \subset V_\rho$, so the norm of its determinant over K is ± 1 and the determinant has absolute value 1.

For a braid β and an index j, let $k \geq 1$ be minimal such that σ_j^k fixes the pair in positions $j, j+1$ of the tuple obtained from γ by β; then β, followed by σ_j^k and by β^{-1}, fixes γ. For 67 such braids with $A_\beta \neq 1$, the space of sesquilinear forms Ω on H^1_{par} with $A_\beta^* \Omega A_\beta = \Omega$ for all 67 is one-dimensional over K, spanned by a Hermitian form Ω_0 of signature $(3,3)$ and, in the basis of Appendix A, determinant $-128 = -2 \cdot 8^2$. Since A_β preserves the non-degenerate form Ω_0, $|\det A_\beta| = 1$, and so $|c_\beta| = 1$. The form $h'(x,y) = \overline{h_w(\tau^{-1}x, \tau^{-1}y)}$ on H^1_{par} is Hermitian and invariant under the monodromy (Lemma 3.1), hence under the A_β, so $h' = c_0 \Omega_0$ with $c_0 \in K^\times$, and $c_0 \in \mathbb{Q}^\times$ because both forms are Hermitian. Its determinant is the complex conjugate of the rational number $\det h_w$, so $\det h_w \equiv c_0^6 \cdot (-128) \equiv -2$. □

4. Determinant lines of cyclic covers

This section proves that, for a cyclic cover of curves with fully paired branching data, the top exterior power of a faithful eigenspace of H^1, its determinant line, is a complex combination of algebraic classes on the Jacobian. The mechanism goes back to Schoen: the eigenspace is carried by a quotient of a self-product of the curve, whose relevant cohomology lives on one fibre of an Abel–Jacobi map [14]. That fibre is a cone over a quotient of a Fermat variety, which brings in the cyclic covers of projective space branched along hyperplanes studied in [21]. We give a proof uniform in the degree $m \geq 3$, the genus of the base and the number $r \geq 3$ of branch points, and compare it with Schoen's in Remark 4.7.

Throughout this section, Y is a smooth projective curve of genus $g \geq 0$ and $\pi\colon X \to Y$ is a connected Galois cover with group $\mu_m = \langle \sigma \rangle$, $m \geq 3$, branched exactly over distinct points $b_1, \dots, b_r$, $r \geq 3$, with local monodromy σ^{a_i} at b_i, $a_i \in \mathbb{Z}/m \setminus \{0\}$, so that $\sum_i a_i = 0$. Let ψ be the character of μ_m with $\psi(\sigma) = \zeta_m$. A subscript ψ on the cohomology of a variety with an action of μ_m, or on a sheaf with such an action, denotes the part on which σ acts through ψ. Let $W_\psi = H^1(X,\mathbb{C})_\psi$, of dimension $d = 2g - 2 + r$ (Section 2.4). Put $e = \gcd(m, a_1, \dots, a_r)$, $m' = m/e$, $a_i' = a_i/e \in \mathbb{Z}/m'$, and $\alpha' = (a_1', \dots, a_r') \in \mathfrak{A}_{m'}^{r-2}$. We identify $H^1(X)$ with $H^1(J(X))$, so that $\Lambda_\psi = \bigwedge^d W_\psi$ is a line in $H^d(J(X),\mathbb{C})$.

Theorem 4.1. *In this setting the following hold.*

(a) Suppose that r is even. If α' lies in $\mathfrak{C}_{m'}^{r-2}$, then $\Lambda_\psi \subset \mathrm{Alg}^{d/2}(J(X))_\mathbb{Q} \otimes \mathbb{C}$.

(b) If $(a_1, \dots, a_r)$ is fully paired, then r is even, the hypothesis of (a) holds for every faithful character ψ^t, $t \in (\mathbb{Z}/m)^\times$, every $H^1(X,\mathbb{C})_{\psi^t}$ has Hodge numbers $(d/2, d/2)$, and the subspace of $H^d(J(X),\mathbb{Q})$ whose complexification is $\bigoplus_t \bigwedge^d H^1(X,\mathbb{C})_{\psi^t}$ consists of algebraic classes.

Proof. Let $\mathcal{T} = \mu_m^d \rtimes \mathfrak{S}_d$ act on X^d, with μ_m acting through σ in each factor and the symmetric group $\mathfrak{S}_d$ permuting the factors, and let $\mathcal{N}$ be the kernel of $\mathcal{T} \to \mu_m$, $(c,s) \mapsto \prod_j c_j$. Put $Z = X^d/\mathcal{N}$ with quotient map $q\colon X^d \to Z$, and let $\pi_Z\colon Z \to \mathrm{Sym}^d Y$ be the quotient by $\mathcal{T}/\mathcal{N} \cong \mu_m$. For a finite group acting on a compact variety, pullback along the quotient map identifies the cohomology of the quotient with the invariants (transfer).

Step 1 (the ψ-part of Z). By transfer and the Künneth formula, $H^*(Z,\mathbb{C})_\psi = \bigwedge^d W_\psi$, a line in degree d, and q^* maps it onto the line $\mathfrak{l}_\psi \subset H^d(X^d,\mathbb{C})$ of alternating tensors in $W_\psi^{\otimes d}$. Indeed, a Künneth component $H^*(X)_{\chi_1} \otimes \cdots \otimes H^*(X)_{\chi_d}$, with characters χ_j of μ_m, is invariant under the subgroup $\{c \in \mu_m^d : \prod_j c_j = 1\}$ only if all χ_j are equal; the $\mathfrak{S}_d$-invariants of $H^*(X)_\chi^{\otimes d}$ form its graded symmetric power; $H^0(X)$ and $H^2(X)$ carry the trivial character; and σ in one factor acts on the χ-summand by $\chi(\sigma)$, so only $\chi = \psi$ contributes to the ψ-part, where the graded symmetric power of the odd space W_ψ is $\bigwedge^d W_\psi$.

Step 2 (the ψ-part of $\pi_{Z*}\mathbb{C}$). Let $\Pi_i = \{D' \in \mathrm{Sym}^d Y : D' \geq b_i\}$ and $U = \mathrm{Sym}^d Y \setminus \bigcup_i \Pi_i$, with inclusion $j\colon U \to \mathrm{Sym}^d Y$. Then $(\pi_{Z*}\mathbb{C}_Z)_\psi = j_!\mathcal{L}_\psi$ for a rank-one local system $\mathcal{L}_\psi$ on U whose monodromy around each Π_i is non-trivial. Indeed, $\mathcal{T}/\mathcal{N}$ acts freely over U: if (c,s) fixes a point P of X^d with no coordinate over a branch point, following a cycle of s shows that the c_j along each cycle multiply to 1, so $(c,s) \in \mathcal{N}$. Over Π_i a point of Z is represented by some P with a coordinate over b_i, and σ^{a_i} in that factor fixes P and has non-trivial image in $\mathcal{T}/\mathcal{N}$; since ψ is faithful, the ψ-part of the stalk vanishes there. Moving one point of a divisor once around b_i moves its lifts by σ^{a_i}, and $\psi(\sigma^{a_i}) \neq 1$.

Step 3 (support on one Abel–Jacobi fibre). Let $D = b_1 + \cdots + b_r$ and $\mathcal{L}_0 = \omega_Y(D)$, of degree d. If $g = 0$, then $\mathrm{Sym}^d Y = \mathbb{P}^d = |\mathcal{L}_0|$; let $g \geq 1$. Since $d \geq 2g-1$, the Abel–Jacobi map $\mathrm{aj}\colon \mathrm{Sym}^d Y \to \mathrm{Pic}^d(Y)$ is a projective bundle with fibres $|\mathcal{L}|$. For $\mathcal{L} \ncong \mathcal{L}_0$ of degree d, $\mathcal{L}(-D)$ has degree $2g-2$ and is not isomorphic to ω_Y, so $h^0(\mathcal{L}(-D)) = g-1 = h^0(\mathcal{L})-r$ by Riemann–Roch: the r evaluations at the b_i are linearly independent on $H^0(\mathcal{L})$. Each $\Pi_i \cap |\mathcal{L}|$ is the reduced hyperplane of sections vanishing at b_i, so $|\mathcal{L}| \cap U \cong (\mathbb{C}^*)^{r-1} \times \mathbb{C}^{g-1}$, on which $\mathcal{L}_\psi$ is an exterior product of rank-one systems that is non-trivial on a factor $\mathbb{C}^*$; hence $H^*_c(|\mathcal{L}| \cap U, \mathcal{L}_\psi) = 0$ by the Künneth formula. By proper base change $R\mathrm{aj}_*(j_!\mathcal{L}_\psi)$ is supported at the point $\mathcal{L}_0$, and

$$H^*(Z,\mathbb{C})_\psi = H^*(\mathrm{Sym}^d Y, j_!\mathcal{L}_\psi) \cong H^*_c(U_0, \mathcal{L}_\psi)\ , \qquad U_0 = U \cap |\mathcal{L}_0|\ . \tag{4.2}$$

Step 4 (the special fibre is a cone over a Fermat quotient). Sections of $\mathcal{L}_0$ are differentials with at most simple poles at the b_i. The residue map $H^0(\omega_Y(D)) \to R = \{y \in \mathbb{C}^r : \sum_i y_i = 0\}$ is onto, with kernel $H^0(\omega_Y)$, by the residue theorem and $h^0(\mathcal{L}_0) = g + r - 1$. So after the choice of a splitting $|\mathcal{L}_0| = \mathbb{P}(H^0(\omega_Y) \oplus R)$, $\Pi_i \cap |\mathcal{L}_0| = \{y_i = 0\}$ (reduced), and all these hyperplanes contain the vertex $\mathbb{P}(H^0(\omega_Y)) \cong \mathbb{P}^{g-1}$. Projection from the vertex makes $\mathcal{V} = |\mathcal{L}_0| \setminus \mathbb{P}(H^0(\omega_Y))$ a vector bundle of rank g over $\mathbb{P}(R) \cong \mathbb{P}^{r-2}$, and U_0 is its restriction to the complement $U_R \subset \mathbb{P}(R)$ of the r hyperplanes $\{y_i = 0\}$, any $r-1$ of which are independent on R. Hence $\pi_1(U_0) = \pi_1(U_R)$, $H_1(U_R,\mathbb{Z}) = \mathbb{Z}^r/\mathbb{Z}(1,\dots,1)$ is generated by the meridians, $\mathcal{L}_\psi$ is pulled back from a rank-one system $\mathcal{L}'$ on U_R with the same meridian monodromies, and by the Thom isomorphism $H^k_c(U_0,\mathcal{L}_\psi) \cong H^{k-2g}_c(U_R,\mathcal{L}')$.

Let $F = F^{r-2}_{m'} \subset \mathbb{P}^{r-1}$ and $\mathfrak{T}_F = \mathfrak{T}^{r-2}_{m'}$, with generators δ_i (the factor $\mu_{m'}$ in position i, at $\zeta_{m'}$). The map $u \mapsto (u_1^{m'}, \dots, u_r^{m'})$ maps F onto $\mathbb{P}(R)$, and the preimage F° of U_R is a $\mathfrak{T}_F$-torsor whose meridian around $\{y_i = 0\}$ is δ_i. Define $\varsigma\colon \mathfrak{T}_F \to \mu_m$ by $\delta_i \mapsto \sigma^{a_i}$; it is well defined because $\sum_i a_i = 0$, and its image is $\langle \sigma^e \rangle \cong \mu_{m'}$. The character $\alpha = \psi \circ \varsigma$ of $\mathfrak{T}_F$ is the vector α', since $\psi(\sigma^{a_i}) = \zeta_{m'}^{a'_i}$. The local system $\mathcal{L}'$ and the α-part of the pushforward of $\mathbb{C}_{F^\circ}$ arise by the same construction from torsors whose meridians map to σ^{a_i} and to δ_i, with $\varsigma(\delta_i) = \sigma^{a_i}$, so $\mathcal{L}'$ is that α-part. Points of F over $\{y_i = 0\}$ are fixed by the i-th factor $\mu_{m'}$, on which α is non-trivial, so

$$H^*_c(U_R,\mathcal{L}') = H^*(F,\mathbb{C})_\alpha = H^{r-2}(F,\mathbb{C})_\alpha\ , \tag{4.3}$$

a line by Section 2.5.

Step 5 (a smooth model and a lift). Let $Q = \mathbb{P}_F(\mathcal{O}_F^{\oplus g} \oplus \mathcal{O}_F(-m'))$, the bundle of lines in $\mathcal{O}_F^{\oplus g} \oplus \mathcal{O}_F(-m')$, with projection $\varpi\colon Q \to F$ and $\xi = c_1(\mathcal{O}_Q(1))$; it is smooth and projective of dimension $g + r - 2 = d - g$, and $Q = F$ if $g = 0$. Define $\kappa\colon Q \to |\mathcal{L}_0|$ by $(u, [\phi : \nu]) \mapsto [\phi + \nu \cdot (u_1^{m'}, \dots, u_r^{m'})]$, where $\phi \in H^0(\omega_Y)$ and $\nu \in \mathcal{O}(-m')_u = \mathbb{C}\,u^{\otimes m'}$ is read as a multiple of $(u_i^{m'})_i \in R$. It is well defined and $\mathfrak{T}_F$-invariant. Then $\kappa^{-1}(\mathbb{P}(H^0(\omega_Y))) = \{\nu = 0\} = F \times \mathbb{P}^{g-1}$, and $Q^\circ = \{\nu \neq 0\}$ is the total space of $\mathcal{O}_F(m')^{\oplus g}$, which κ maps to $\mathcal{V}$ linearly on the fibres over $F \to \mathbb{P}(R)$. The restriction $Q^{\circ\circ}$ of Q° to F° is a $\mathfrak{T}_F$-torsor over U_0, connected because $r \geq 3$. By Steps 2 and 4 the μ_m-torsor $\pi_Z^{-1}(U_0) \to U_0$ is the pushout of $Q^{\circ\circ}$ along ς, both having the meridian images σ^{a_i}, so there is a ς-equivariant lift $\hat{q}^\circ\colon Q^{\circ\circ} \to \pi_Z^{-1}(U_0)$ over U_0. The closure of its graph in $Q \times_{\mathrm{Sym}^d Y} Z$ is finite and birational over the normal variety Q, hence isomorphic to Q by Zariski's main theorem, and we obtain $\hat{q}\colon Q \to Z$ with $\pi_Z \circ \hat{q} = \kappa$.

Step 6 (injectivity). We claim that $\hat{q}^*$ is injective on $H^d(Z,\mathbb{C})_\psi$, with image $\xi^g \cup \varpi^* H^{r-2}(F,\mathbb{C})_\alpha$, and likewise on $H^d(Z,\mathbb{C})_{\bar\psi}$, with image $\xi^g \cup \varpi^* H^{r-2}(F,\mathbb{C})_{-\alpha}$. On the summand $H^*(\mathrm{Sym}^d Y, j_!\mathcal{L}_\psi)$ of $H^*(Z) = H^*(\mathrm{Sym}^d Y, \pi_{Z*}\mathbb{C})$, the map $\hat{q}^*$ is induced by the adjunction $j_!\mathcal{L}_\psi \to R\kappa_*\mathbb{C}_Q$, so it is the composite

$$H^*(\mathrm{Sym}^d Y, j_!\mathcal{L}_\psi) \xrightarrow{\sim} H_c^*(U_0,\mathcal{L}_\psi) \longrightarrow H_c^*(Q^{\circ\circ},\mathbb{C}) \longrightarrow H^*(Q,\mathbb{C}) \ . \tag{4.4}$$

The first map is the isomorphism (4.2). The second is induced by $\mathcal{L}_\psi \to \kappa^\circ_*\mathbb{C}$, where $\kappa^\circ\colon Q^{\circ\circ} \to U_0$ is finite étale; $\kappa^\circ_*\mathbb{C}$ is semisimple, and $\mathcal{L}_\psi$ maps isomorphically onto its α-isotypic summand for $\mathfrak{T}_F$, because $\hat{q}^\circ$ is ς-equivariant, so the second map is injective with image in the α-part. The third is extension by zero. On α-parts it is injective in degree d. First, every point of $Q^\circ \setminus Q^{\circ\circ}$ has a stabilizer in $\mathfrak{T}_F$ on which α is non-trivial, so $H_c^*(Q^{\circ\circ})_\alpha \cong H_c^*(Q^\circ)_\alpha$. Second, $Q \setminus Q^\circ = F \times \mathbb{P}^{g-1}$, with $\mathfrak{T}_F$ acting trivially on the second factor, and $H^{d-1}(F\times\mathbb{P}^{g-1})_\alpha = \bigoplus_k H^{d-1-2k}(F)_\alpha = 0$, because $H^l(F)_\alpha = 0$ for $l \neq r-2$ and $d-1-2k = r-2$ would force $2(g-k)=1$. By the long exact sequence of the pair, extension by zero is injective on $H_c^d(Q^\circ)_\alpha$, which is the Thom class of $Q^\circ \to F$ cupped with $\varpi^* H^{r-2}(F)_\alpha$. The extension by zero of the Thom class is the class of the section $Q_\infty = \mathbb{P}_F(\mathcal{O}_F(-m')) \subset Q$, which is the transversal zero locus of the section of $\mathcal{O}_Q(1)^{\oplus g}$ induced by the projection to $\mathcal{O}_F^{\oplus g}$, hence of class ξ^g. This proves the claim.

Step 7 (a correspondence). Let $\tilde{Q}$ be a resolution of an irreducible component of $Q \times_Z X^d$ that dominates Q, with $p_1\colon \tilde{Q} \to Q$ generically finite and surjective, $p_2\colon \tilde{Q} \to X^d$, and $\hat{q}\circ p_1 = q \circ p_2$. Put $\Gamma_* = p_{2*}p_1^*\colon H^k(Q) \to H^{k+2g}(X^d)$ and

$$\varepsilon_\psi = \frac{1}{|\mathcal{T}|} \sum_{(c,s)\in\mathcal{T}} \psi\Big(\prod_j c_j\Big)^{-1} (c,s)^* \in \mathbb{C}[\mathcal{T}] \ ;$$

in degree d it projects onto $\mathfrak{l}_\psi$, and its adjoint for the cup product is $\varepsilon_{\bar\psi}$. For $0 \neq \beta_+ \in H^{r-2}(F,\mathbb{C})_\alpha$ and $0 \neq \beta_Z \in H^d(Z,\mathbb{C})_{\bar\psi}$, Step 6 gives $\hat{q}^*\beta_Z = \xi^g \cup \varpi^*\beta_-$ with $0 \neq \beta_- \in H^{r-2}(F,\mathbb{C})_{-\alpha}$, and

$$\int_{X^d} \varepsilon_\psi \Gamma_* \varpi^*\beta_+ \cup q^*\beta_Z = \int_{\tilde{Q}} p_1^*\varpi^*\beta_+ \cup p_1^*\hat{q}^*\beta_Z = \deg(p_1)\int_F \beta_+ \cup \beta_- \neq 0 \ , \tag{4.5}$$

since ξ^g is the class of a section of ϖ and the lines $H^{r-2}(F)_{\pm\alpha}$ are dual under the cup product. Hence $\varepsilon_\psi\Gamma_*\varpi^*\beta_+$ spans $\mathfrak{l}_\psi$. If $\beta_+ \in \mathrm{Alg}(F)_\mathbb{Q} \otimes \mathbb{C}$, this class lies in $\mathrm{Alg}(X^d)_\mathbb{Q} \otimes \mathbb{C}$, because ε_ψ is a combination of graphs of automorphisms and cycle class maps commute with pullbacks between smooth varieties and with proper pushforwards [35]; see e.g. [36].

Step 8 (transfer to the Jacobian). Fix $P_0 \in X$ and let $f\colon X^d \to J(X)$, $(P_1,\dots,P_d) \mapsto \sum_j (P_j - P_0)$. Then f^* maps Λ_ψ isomorphically onto $\mathfrak{l}_\psi$: for $\eta_1,\dots,\eta_d \in W_\psi$, $f^*(\eta_1\wedge\cdots\wedge\eta_d) = \prod_k \sum_j \mathrm{pr}_j^*\eta_k$, and the terms with two η_k in the same factor vanish because $\eta_k \cup \eta_l \in H^2(X)_{\psi^2} = 0$, as $\psi^2 \neq 1$ for $m \geq 3$. The spaces $H^1(X,\mathbb{C})_{\psi^t}$, $t \in (\mathbb{Z}/m)^\times$, are distinct of dimension d, and there are at least two of them, so $2d \leq 2g(X)$. Hence f factors through $\mathrm{Sym}^d X$, which maps birationally onto its image in $J(X)$, and by the projection formula and Poincaré's formula (see e.g. [37])

$$f_* f^* x = x \cup f_* 1 = d!\, x \cup \frac{\theta^{g(X)-d}}{(g(X)-d)!} \ , \qquad x \in H^d(J(X)) \ , \tag{4.6}$$

where θ is the class of a theta divisor. Cup product with $\theta^{g(X)-d}$ is an isomorphism $H^d(J(X)) \to H^{2g(X)-d}(J(X))$ whose inverse is induced by an algebraic correspondence, because the Lefschetz standard conjecture holds for abelian varieties [16, 17]. So if $\alpha' \in \mathfrak{C}_{m'}^{r-2}$, Steps 6–8 applied to an algebraic β_+ spanning $H^{r-2}(F,\mathbb{C})_\alpha$ give $\Lambda_\psi \subset \mathrm{Alg}(J(X))_\mathbb{Q} \otimes \mathbb{C}$. This proves (a).

Step 9 (fully paired data). If (a_i) is fully paired, then r is even and (a_i') is fully paired, because dividing by e keeps the pairs. By Lemma 2.4, α' lies in $\mathfrak{C}_{m'}^{r-2}$, so (a) applies. Multiplication by $t \in (\mathbb{Z}/m)^\times$ preserves full pairing, so (a) applies to every ψ^t, with the generator of μ_m on which

ψ^t takes the value ζ_m. Each pair contributes 1 to $\sum_i \langle t a_i/m\rangle$ and to $\sum_i \langle -t a_i/m\rangle$, so both Hodge numbers of $H^1(X,\mathbb{C})_{\psi^t}$ are $g-1+r/2=d/2$. Finally the subspace in (b) is rational, being stable under the Galois group, and its complexification $\bigoplus_t \bigwedge^d H^1(X,\mathbb{C})_{\psi^t}$ lies in $\mathrm{Alg}(J(X))_{\mathbb{Q}}\otimes\mathbb{C}$; so it consists of algebraic classes. □

Remark 4.7 (Schoen's theorem). For fully paired data, which Schoen calls simple, his Theorem 2.0 and Corollary 3.1 [14] state that the Weil Hodge structure of the generalized Prym variety of $X\to Y$, whose complexification is the space in Theorem 4.1(b), is spanned by algebraic classes, for every $m>2$ and every genus of Y; his transfer to the Prym variety uses Poincaré's formula and Lieberman's algebraic inverse of the Lefschetz operator, as in Step 8. His proof of the key Lemma 2.8 [14] chooses, for each pair (a_{2i-1},a_{2i}), a prime $p\mid m$ with $p\nmid a_{2i-1}$. Such a prime does not exist when a_{2i-1} is divisible by every prime factor of m, as for $m=12$ and the pair $(6,6)$, which occurs in the subcovers $j=4,10$ of Table 1. Moreover, the last display of that proof identifies the invariants of the subgroup of index p with those of μ_m, which excludes the characters of order p but not the faithful characters unless $m=p$. As printed, the argument is complete for m prime, whereas the ramified subcovers of Table 1 have $m\in\{6,12\}$. In the proof of Lemma 4.4 of [21], the analogue of Lemma 2.8 in that paper, Schoen sketches in two sentences a different argument for this step, with a prime $p\mid m$ such that the quotient by the subgroup of order p is ramified over b_{2i-1}, which always exists, and remarks that Lemma 2.8 should have been proved that way; since a faithful character is non-trivial on every non-trivial subgroup, this argument removes both difficulties. The sketch invokes the argument of [14, Sublemma 2.9], which uses that E_i and a quotient of F_i, minus a divisor, are $\mathbb{P}^1$-bundles with a section over the blown-up locus, where E_i is the exceptional divisor of the i-th blow-up and F_i its preimage in the cover; when $\gcd(m,a_{2i-1})>1$ the fibre of F_i over a general point of the blown-up locus consists of $\gcd(m,a_{2i-1})$ disjoint copies of $\mathbb{P}^1$, so the argument does not apply verbatim. This is the case for every pair in Table 1, whose exponents are all non-units modulo m, and Theorem 4.1 gives a complete proof by another route. The current version of [8] replaces this step by the statement that the χ-part of $H^*(F_i,\mathbb{C})$ vanishes for every faithful character χ of μ_m, which it derives by writing this part as the cohomology of E_i with coefficients in the extension by zero of a rank-one local system with non-trivial monodromy around the strict transforms of the branch divisors, and by applying proper base change to the projection of the $\mathbb{P}^1$-bundle E_i to the blown-up locus, on each fibre of which this cohomology vanishes. Schoen's 1989 paper uses the correspondence with Fermat varieties for branching data in which the exponents outside some inverse pairs generate $\mathbb{Z}/m$ [21]; Step 6 is a self-contained proof of injectivity.

Remark 4.8 (a related preprint). Lemma 3.4 of the preprint [11] claims, for every connected cyclic cover of curves and every faithful character, that the top exterior power of the eigenspace lies in the span of the images of the cohomology of CM abelian varieties under algebraic correspondences [11]. Its proof uses the construction of Steps 1–5, for which [11] names Schoen's work [14] as a direct precedent, and its injectivity step uses Deligne's weight argument [11], whereas Step 6 uses extension by zero.

5. Weil classes on every member of $\mathcal{H}$

In this section C is a member of $\mathcal{H}$. The data used below (local monodromies, numbers of branch points, genera of quotients) are topological and constant on $\mathcal{H}$, and no genericity is used.

5.1. **Cyclic subcovers and the relation.** The centre $Z(G)$ is cyclic of order 6; let χ_ρ be the central character of ρ. Exactly fourteen irreducible representations of G have central character χ_ρ (Appendix A). We label them $\rho_1=\rho,\rho_2,\dots,\rho_{14}$ as in Table 2, where they are distinguished by their degree, by $\dim M_{\rho_k}$, computed by (2.2) from the character values on the classes of the γ_i, and by the character value $\chi_{\rho_k}(\gamma_1)$ at the reflection γ_1 of order 3 and determinant ζ^4.

For a subgroup $H \leq G$ containing $Z(G)$ and a character $\psi\colon H \to \mathbb{C}^\times$ with $\psi|_{Z(G)} = \chi_\rho$, put

$$W_{H,\psi} = H^1(C,\mathbb{C})^{(H,\psi)} = \bigoplus_{k=1}^{14} \rho_k^{(H,\psi)} \otimes M_{\rho_k}\ , \qquad v_{H,\psi}(\rho_k) = \dim \rho_k^{(H,\psi)} = \langle \rho_k|_H, \psi\rangle_H\ ; \tag{5.1}$$

only the ρ_k contribute, since $Z(G)$ acts on $\rho'^{(H,\psi)}$ through $\psi|_{Z(G)} = \chi_\rho$, and the last equality is Frobenius reciprocity. Let $X = C/\ker\psi$ and $Y = C/H$. Then $X \to Y$ is a connected Galois cover with cyclic group $H/\ker\psi \cong \mu_m$, m the order of ψ, branched exactly at the r_ψ points of Y at which ψ of the local monodromy is non-trivial. By transfer, pullback identifies the eigenspace of $H^1(X,\mathbb{C})$ on which $H/\ker\psi$ acts through ψ with $W_{H,\psi}$, so $\dim W_{H,\psi} = 2g(Y) - 2 + r_\psi$ (Section 2.4). This eigenspace is $W_{\psi'}$ in the notation of Section 4, for the character ψ' of $H/\ker\psi$ induced by ψ^{-1} and the generator on which ψ' takes the value ζ_m.

By Brauer's induction theorem (see e.g. [38]) one may look for integral relations between e_ρ and the vectors $v_{H,\psi}$. We use the following one, found by lattice reduction and checked exactly.

Proposition 5.2 (computer-assisted)**.** *There are eleven pairs (H_j,ψ_j) as above, with the data of Table 1, such that for the signs c_j of Table 1 and $v_j = v_{H_j,\psi_j}$, given in Table 2,*

$$e_\rho = \sum_{j=1}^{11} c_j\, v_j\ . \tag{5.3}$$

For every j, the cover $X_j = C/\ker\psi_j \to Y_j = C/H_j$ is cyclic of degree m_j and branched at r_j points with the local exponents of Table 1, which are fully paired, and

$$d_j := \dim W_j = 2g(Y_j) - 2 + r_j = \sum_{k=1}^{14} v_j(\rho_k)\dim M_{\rho_k}\ , \qquad W_j = W_{H_j,\psi_j}\ . \tag{5.4}$$

The proof is the computation of Appendix A. Taking the scalar product of (5.3) with the vector $(\dim M_{\rho_k})_k$ gives the check $\sum_j c_j d_j = 28374 - 28368 = 6 = \dim M_\rho$ on the column d of Table 1.

TABLE 1. The eleven cyclic subcovers $X_j \to Y_j$ of (5.3). Here $|H_j|$ is the order of the subgroup, m the order of ψ_j, r the number of branch points, $d = \dim W_j$, m' the degree of the Fermat variety in Theorem 4.1, and the exponents are listed as value : multiplicity in $\mathbb{Z}/m$. The last column names the result that makes Λ_j algebraic (Section 5.2).

j	c_j	$\lvert H_j\rvert$	m	r	$g(Y_j)$	d	$g(X_j)$	m'	exponents	input
1	-1	432	6	544	989	2520	7015	6	2:270, 3:4, 4:270	Thm. 4.1
2	$+1$	486	6	384	913	2208	6241	3	2:192, 4:192	Thm. 4.1
3	-1	324	6	448	1361	3168	9049	6	2:216, 3:16, 4:216	Thm. 4.1
4	$+1$	288	12	654	1537	3726	21043	6	4:324, 6:6, 8:324	Thm. 4.1
5	$+1$	216	24	0	2341	4680	56161	–	étale	[14]
6	$+1$	144	6	336	3397	7128	21043	6	2:162, 3:12, 4:162	Thm. 4.1
7	-1	120	24	0	4213	8424	101089	–	étale	[14]
8	$+1$	108	12	0	4681	9360	56161	–	étale	[14]
9	$+1$	3888	6	160	77	312	775	6	2:78, 3:4, 4:78	Thm. 4.1
10	-1	72	12	672	6793	14256	84169	6	4:324, 6:24, 8:324	Thm. 4.1
11	$+1$	972	6	64	449	960	2809	6	2:24, 3:16, 4:24	Thm. 4.1

TABLE 2. The fourteen irreducible representations ρ_k of G with central character χ_ρ, their degree, $\dim M_{\rho_k}$, the character value at the reflection γ_1 of order 3 and determinant ζ^4, and the multiplicity vectors v_j of Table 1, as columns: the entry in row k and column v_j is $v_j(\rho_k)$. With the signs c_j, the signed sum of the entries of row k is 1 for $k = 1$ and 0 otherwise, which is (5.3), and $d_j = \sum_k v_j(\rho_k) \dim M_{\rho_k}$.

k	$\dim \rho_k$	$\dim M_{\rho_k}$	$\chi_{\rho_k}(\gamma_1)$	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}
			c_j	-1	$+1$	-1	$+1$	$+1$	$+1$	-1	$+1$	$+1$	-1	$+1$
1	4	6	$2-\zeta^2$	0	0	0	0	0	0	0	0	0	0	1
2	4	42	$-1+2\zeta^2$	0	0	0	0	0	0	0	0	1	1	0
3	20	102	$7+7\zeta^2$	0	0	0	0	1	1	1	1	0	2	0
4	20	102	$1-5\zeta^2$	0	0	0	1	0	0	1	1	0	1	0
5	20	138	$-5-8\zeta^2$	2	0	0	2	1	0	1	1	0	1	0
6	20	174	$-5+\zeta^2$	0	1	1	1	0	1	1	1	0	2	0
7	20	174	$-8-5\zeta^2$	2	0	1	2	1	1	1	1	0	1	0
8	36	162	9	0	0	1	0	1	1	2	2	0	2	1
9	36	270	$9\zeta^2$	0	1	0	0	1	2	2	2	1	5	0
10	60	342	$3-6\zeta^2$	1	0	1	1	2	2	3	3	0	4	1
11	60	378	$3+3\zeta^2$	0	1	1	1	1	3	3	3	0	5	0
12	60	450	$-6+3\zeta^2$	1	1	1	1	2	3	3	3	0	6	1
13	64	384	$8+8\zeta^2$	0	1	1	0	2	3	3	4	0	6	0
14	80	552	$-8-8\zeta^2$	2	1	2	3	2	3	4	5	0	6	0

5.2. **The determinant lines of the subcovers.** For each j let $\Lambda_j = \bigwedge^{d_j} W_j \subset H^{d_j}(J(C), \mathbb{C})$.

Proposition 5.5. *For every j, $\Lambda_j \subset \mathrm{Alg}(J(C))_{\mathbb{Q}} \otimes \mathbb{C}$.*

Proof. For the eight ramified subcovers, $j \in \{1, 2, 3, 4, 6, 9, 10, 11\}$, the cover $X_j \to Y_j$ is connected and cyclic of degree $m_j \in \{6, 12\}$, with $r_j \geq 64$ branch points and fully paired exponents (Proposition 5.2), so Theorem 4.1(b) shows that the top exterior power of the eigenspace of $H^1(X_j, \mathbb{C})$ corresponding to W_j lies in $\mathrm{Alg}(J(X_j))_{\mathbb{Q}} \otimes \mathbb{C}$.

For the three étale subcovers, $j \in \{5, 7, 8\}$, we have $r_j = 0$, $m_j \in \{24, 24, 12\}$ and $g(Y_j) \in \{2341, 4213, 4681\}$. Schoen's theorem [14, Theorem 2.0, Corollary 3.1] applies with $r = 0$, for which his simplicity condition is empty [14], and shows that the rational subspace whose complexification is the sum of the top exterior powers of the faithful eigenspaces of $H^1(X_j)$ consists of algebraic classes; each of these top exterior powers, in particular the one corresponding to W_j, lies in the complexification of that subspace, hence in $\mathrm{Alg}(J(X_j))_{\mathbb{Q}} \otimes \mathbb{C}$. Patel and Zhang prove the same for étale covers with finite abelian group over a curve of genus at least 2 [37], which gives a second reference for these three subcovers.

The cover $C \to X_j$ induces the norm map $J(C) \to J(X_j)$, whose pullback on H^1 is the pullback $H^1(X_j) \to H^1(C)^{\ker \psi_j}$, an isomorphism that maps the eigenspace in question onto W_j. Pullback preserves $\mathrm{Alg}_{\mathbb{Q}} \otimes \mathbb{C}$, so $\Lambda_j \subset \mathrm{Alg}(J(C))_{\mathbb{Q}} \otimes \mathbb{C}$. □

5.3. **A correspondence in $\mathbb{C}[G]$ and a contraction.** We realize (5.3) by a matrix over $\mathbb{C}[G]$ acting on products of copies of $J(C)$, and isolate the line of ρ by a contraction. Take one copy of $J(C)$ for each j: let $J_+ = J(C)^7$, with one factor for each j with $c_j = +1$, let $J_- = J(C)^4$, with one factor for each j with $c_j = -1$, and let J_0 be one further copy of $J(C)$. Let $U_+ = \bigoplus_{c_j=+1} W_j \subset H^1(J_+, \mathbb{C})$, each W_j in its own factor, and write $U_+ = \bigoplus_k \mathfrak{u}_+(\rho_k) \otimes M_{\rho_k}$ with $\mathfrak{u}_+(\rho_k) = \bigoplus_{c_j=+1} \rho_k^{(H_j, \psi_j)} \subset \rho_k^{\oplus 7}$.

Fix $w' \in \rho$, $w' \neq 0$, and let

$$U_{w'} = (w' \otimes M_\rho) \oplus U_- \subset H^1(J_0 \times J_-, \mathbb{C}) \ , \qquad U_- = \bigoplus_{c_j=-1} W_j \ ,$$

with $w' \otimes M_\rho$ in the factor J_0. Then $U_{w'} = \bigoplus_k \mathfrak{u}_{w'}(\rho_k) \otimes M_{\rho_k}$ with $\dim \mathfrak{u}_{w'}(\rho_k) = \delta_{k1} + \sum_{c_j=-1} v_j(\rho_k) = \sum_{c_j=+1} v_j(\rho_k) = \dim \mathfrak{u}_+(\rho_k)$ by (5.3). In particular $n_+ := \dim U_+ = \dim U_{w'} = 28374$, and $n_- := \dim U_- = 28368$.

Proposition 5.6. *For every non-zero $w' \in \rho$, the line $\Lambda_{w'} = \bigwedge^6(w' \otimes M_\rho) \subset H^6(J(C), \mathbb{C})$ lies in $\mathrm{Alg}(J(C))_\mathbb{Q} \otimes \mathbb{C}$.*

Proof. For each k choose a linear map $A_k \colon \rho_k^{\oplus 7} \to \rho_k^{\oplus 5}$ that maps $\mathfrak{u}_+(\rho_k)$ isomorphically onto $\mathfrak{u}_{w'}(\rho_k)$, and $A_{\rho'} = 0$ for the other irreducible ρ'. Since $\mathbb{C}[G] = \bigoplus_{\rho'} \mathrm{End}(\rho')$, the collection $\Psi = (A_{\rho'})_{\rho'}$ is an element of $\mathrm{Mat}_{5\times 7}(\mathbb{C}[G])$. For $\Psi_0 = (x_{ab}) \in \mathrm{Mat}_{5\times 7}(\mathbb{Z}[G])$, with $x_{ab} = \sum_g n_{ab,g}\, g$, let $f_{\Psi_0} \colon J_0 \times J_- \to J_+$ be the homomorphism whose b-th component is $\sum_a \sum_g n_{ab,g}\, g^{-1} \circ \mathrm{pr}_a$, with G acting on $J(C)$ through its action on C. Its pullback $H^1(J(C))^{\oplus 7} \to H^1(J(C))^{\oplus 5}$ is $(y_b)_b \mapsto (\sum_b x_{ab} \cdot y_b)_a$, with G acting on H^1 as in the Conventions. Extending $\Psi_0 \mapsto f_{\Psi_0}^*$ linearly, Ψ acts by a map $\Psi^* \colon H^1(J_+, \mathbb{C}) \to H^1(J_0 \times J_-, \mathbb{C})$, through $A_{\rho'}$ on the factors ρ', and Ψ^* maps U_+ isomorphically onto $U_{w'}$.

The map $\Phi \colon \Psi \mapsto \bigwedge^{n_+} \Psi^* \in \mathrm{Hom}(H^{n_+}(J_+, \mathbb{C}), H^{n_+}(J_0 \times J_-, \mathbb{C}))$ is a homogeneous polynomial map with rational coefficients in the coordinates of $\mathrm{Mat}_{5\times 7}(\mathbb{C}[G])$. For $\Psi_0 \in \mathrm{Mat}_{5\times 7}(\mathbb{Q}[G])$ and an integer $N \neq 0$ with $N\Psi_0$ integral, $\Phi(\Psi_0) = N^{-n_+} f_{N\Psi_0}^*$ maps algebraic classes to algebraic classes. Since $\mathrm{Mat}_{5\times 7}(\mathbb{Q}[G])$ is Zariski dense in $\mathrm{Mat}_{5\times 7}(\mathbb{C}[G])$, every linear form that vanishes on $\Phi(\mathrm{Mat}_{5\times 7}(\mathbb{Q}[G]))$ vanishes at $\Phi(\Psi)$, so $\Phi(\Psi)$ lies in the complex span of maps that preserve $\mathrm{Alg}_\mathbb{Q}$, and it maps $\mathrm{Alg}(J_+)_\mathbb{Q} \otimes \mathbb{C}$ into $\mathrm{Alg}(J_0 \times J_-)_\mathbb{Q} \otimes \mathbb{C}$. The class $\Lambda_+ = \bigwedge^{n_+} U_+$ is, up to sign, the product of the pullbacks of the Λ_j, $c_j = +1$, from the factors of J_+, so it lies in $\mathrm{Alg}(J_+)_\mathbb{Q} \otimes \mathbb{C}$ by Proposition 5.5. Hence

$$\Phi(\Psi)\Lambda_+ = \det\left(\Psi^*|_{U_+}\right) \cdot \Lambda_{w'} \otimes \Lambda_- \in \mathrm{Alg}(J_0 \times J_-)_\mathbb{Q} \otimes \mathbb{C} \ , \qquad \Lambda_- = \bigwedge^{n_-} U_- \ , \tag{5.7}$$

with $\det(\Psi^*|_{U_+}) \neq 0$ computed in bases of U_+ and $U_{w'}$, and $\Lambda_{w'} \otimes \Lambda_- = \mathrm{pr}_{J_0}^* \Lambda_{w'} \cup \mathrm{pr}_{J_-}^* \Lambda_-$.

Let θ be the class of a G-invariant product polarization on J_-, E its Riemann form on $H^1(J_-)$, and $g_- = \dim J_-$. Since E is G-invariant, it pairs the part of $H^1(J_-, \mathbb{C})$ with central character χ_ρ only with the part with central character $\bar\chi_\rho$; as χ_ρ is not real, U_- is isotropic and $U_- \cap \bar U_- = 0$. The space U_- is a sum of complex sub-Hodge structures, so $U_- = U_-^{1,0} \oplus U_-^{0,1}$, and the Hermitian form $iE(x, \bar y)$ is definite on $H^{1,0}$ and on $H^{0,1}$, with opposite signs, which are orthogonal for it. If $x = x' + x'' \in U_-$ satisfies $E(x, \bar y) = 0$ for all $y \in U_-$, then taking $y = x'$ and $y = x''$ gives $x' = x'' = 0$. So E pairs U_- perfectly with $\bar U_-$, $\widetilde U_- = U_- \oplus \bar U_-$ is a non-degenerate real sub-Hodge structure, and $H^1(J_-, \mathbb{C}) = \widetilde U_- \oplus \widetilde U_-^\perp$. Accordingly $\theta = \theta_1 + \theta_2$, with θ_1 and θ_2 the classes of the restrictions of E, and

$$\Lambda_- \cup \bar\Lambda_- \cup \theta^{g_- - n_-} = \Lambda_- \cup \bar\Lambda_- \cup \theta_2^{g_- - n_-} \neq 0 \ ,$$

a product of top-degree forms on $\widetilde U_-$ and on $\widetilde U_-^\perp$. The class $\Theta_- = \bar\Lambda_- \cup \theta^{g_- - n_-}$ lies in $\mathrm{Alg}(J_-)_\mathbb{Q} \otimes \mathbb{C}$, because Λ_- does by Proposition 5.5, $\mathrm{Alg}_\mathbb{Q} \otimes \mathbb{C}$ is stable under complex conjugation, and θ is algebraic. Hence

$$\mathrm{pr}_{J_0 *}\left((\Lambda_{w'} \otimes \Lambda_-) \cup \mathrm{pr}_{J_-}^* \Theta_-\right) = \pm\Big(\int_{J_-} \Lambda_- \cup \Theta_-\Big)\, \Lambda_{w'}$$

lies in $\mathrm{Alg}(J(C))_\mathbb{Q} \otimes \mathbb{C}$, with a non-zero coefficient. □

5.4. **Rational Weil classes.** It remains to pass from the lines $\Lambda_{w'}$ to the rational Weil classes.

Theorem 5.8. *For every C in $\mathcal{H}$, every class in $\mathrm{HW}(A_C)$ is algebraic.*

Proof. Let $w \in V_\rho \setminus \{0\}$ be rational and write $w = w' + w''$ in $V_\rho \otimes \mathbb{C} = \rho \oplus \bar\rho$, so that $w'' = \bar w'$ and $w' \neq 0$. By Lemma 3.1(ii), evaluation at w embeds B_C into $H^1(C, \mathbb{Q})$ as a sub-Hodge structure, which is $p^*H^1(B_w, \mathbb{Q})$ for a quotient $p\colon J(C) \to B_w$; we give B_w the transported action of K, so that B_w is K-isogenous to A_C. The two eigenspaces of K on $H^1(B_w, \mathbb{C})$ pull back to $w' \otimes M_\rho$ and $w'' \otimes M_{\bar\rho}$, so

$$p^*\mathrm{HW}(B_w) \otimes \mathbb{C} = \mathbb{C}\,\Lambda_{w'} \oplus \mathbb{C}\,\bar\Lambda_{w'} \subset \mathrm{Alg}(J(C))_{\mathbb{Q}} \otimes \mathbb{C}$$

by Proposition 5.6, and $p^*\mathrm{HW}(B_w) \subset (\mathrm{Alg}(J(C))_{\mathbb{Q}} \otimes \mathbb{C}) \cap H^6(J(C), \mathbb{Q}) = \mathrm{Alg}(J(C))_{\mathbb{Q}}$. With $p'\colon B_w \to J(C)$ such that $p \circ p'$ is multiplication by an integer $N \neq 0$ (Poincaré reducibility), $\mathrm{HW}(B_w) = N^{-6}p'^*p^*\mathrm{HW}(B_w)$ consists of algebraic classes, and by Lemma 2.1 so does $\mathrm{HW}(A_C)$. □

6. Dominance

Locally on $\mathcal{H}$, a flat K-isometry of (B_C, h_w) with a fixed Hermitian space (V, h) defines the period map $C \mapsto$ (Hodge structure of B_C) $\in \mathfrak{D}$; its rank at a point does not depend on the choices.

Theorem 6.1. *The period map of $\mathcal{H}$ has rank 9 at a general point.*

The rank is at most $\dim \mathfrak{D} = 9$. We bound it from below by the rank of a limit of the codifferential at a boundary point, using the standard facts (T2)–(T7).

6.1. **The boundary point and a plumbing family.** Let the branch points be real and ordered, and let them collide in the five clusters $\{1,2\}$, $\{3,4\}$, $\{6,7\}$, $\{8,9\}$, $\{11,12\}$. The stable model of the base is a curve with a main component $P_0 \cong \mathbb{P}^1$, carrying eight special points (the five nodes and the branch points x_5, x_{10}, x_{13}), and five components $P_\nu \cong \mathbb{P}^1$, one for each cluster ν, each carrying two branch points and a node. The admissible G-cover C_0 over it [39] has the local monodromies $\gamma_1\gamma_2, \gamma_3\gamma_4, \gamma_5, \gamma_6\gamma_7, \gamma_8\gamma_9, \gamma_{10}, \gamma_{11}\gamma_{12}, \gamma_{13}$ on P_0, which commute with each other (Appendix A). For the four clusters $\{1,2\}, \{6,7\}, \{8,9\}, \{11,12\}$ the two monodromies are inverse to each other, so the node has trivial monodromy; we call these nodes *trivial*. For the cluster $\{3,4\}$ the monodromies γ_3, γ_4 do not commute and $\gamma_3\gamma_4$ has order 6.

At each G-orbit of nodes of C_0 we plumb G-equivariantly, replacing a neighbourhood $zz' = 0$ of a node by $zz' = t_\nu$ in coordinates on which its stabilizer acts by inverse characters, so that zz' is invariant. For a trivial node ν with centre $c_\nu \in P_0$, the plumbing of the base is $x - c_\nu = t_\nu y$, with y the coordinate on P_ν in which the node is at ∞, so the plumbed curve is the G-cover of $\mathbb{P}^1$ branched at $x = c_\nu + t_\nu y$ for the two branch points y of P_ν. At the node of $\{3,4\}$, whose stabilizer has order 6, the base is plumbed by $x - c_{(3,4)} = t_{(3,4)}^6 y$. Together with the positions of the eight special points of P_0, this gives a flat proper family $\mathcal{C} \to \Delta^5 \times \mathcal{U}$, with $\mathcal{U} \subset \mathbb{C}^8$ a neighbourhood of the initial positions, whose fibres are nodal and are smooth exactly where all $t_\nu \neq 0$. Its smooth fibres near real positions with small $t_\nu > 0$ are covers branched at real ordered points with the tuple γ, so they lie in $\mathcal{H}$, and so do all its smooth fibres. By (T7), the rank of the period map of $\mathcal{C}$ at a point is at most the rank of the period map of $\mathcal{H}$ at a general point, so one point of rank 9 suffices.

6.2. **Two lemmas.** Fix a G-invariant Hermitian inner product on ρ, an orthonormal basis $\varepsilon_1, \dots, \varepsilon_4$, and identify $\bar\rho$ with $\rho^\vee$, with dual basis $\varepsilon^1, \dots, \varepsilon^4$. For a smooth or nodal fibre C write $F^1_\rho = \mathrm{Hom}_G(\rho, H^0(\omega_C))$ and $F^1_{\bar\rho} = \mathrm{Hom}_G(\rho^\vee, H^0(\omega_C))$, and for $\phi \in F^1_\rho$, $\eta \in F^1_{\bar\rho}$ put

$$\phi\eta = \sum_{i=1}^{4} \phi(\varepsilon_i)\,\eta(\varepsilon^i) \in H^0(\omega_C^{\otimes 2})\ , \tag{6.2}$$

which does not depend on the basis and is G-invariant, because $\sum_i \varepsilon_i \otimes \varepsilon^i$ is the invariant identity tensor of $\rho \otimes \rho^\vee$. For $f \in M_\rho$ and $f' \in M_{\bar\rho} = \mathrm{Hom}_G(\rho^\vee, H^1(C, \mathbb{C}))$ put $E_\rho(f, f') = \sum_i E_C(f(\varepsilon_i), f'(\varepsilon^i))$.

Since E_C is G-invariant and perfect, it pairs the ρ-isotypic part of $H^1(C,\mathbb{C})$ perfectly with the $\rho^\vee$-isotypic part, so E_ρ is a perfect pairing between M_ρ and $M_{\bar\rho}$. The point of $\mathfrak{D}$ attached to a smooth fibre C is $F^1_\rho \subset M_\rho$, and the tangent space there is $\mathrm{Hom}(F^1_\rho, M_\rho/F^1_\rho) \cong (F^1_\rho \otimes F^1_{\bar\rho})^\vee$, by E_ρ.

Lemma 6.3 (isotypic Griffiths formula)**.** *At a point with smooth fibre C, the codifferential of the period map sends $\phi\otimes\eta \in F^1_\rho \otimes F^1_{\bar\rho}$ to the linear form $\partial \mapsto \pm\langle \mathrm{KS}(\partial), \phi\eta\rangle$, with a sign that does not depend on ∂, ϕ, η.*

Proof. By (T2) the derivative of the Hodge filtration of $H^1(C)$ along a tangent vector ∂ is cup product with $\mathrm{KS}(\partial)$. The family is G-equivariant, so $\mathrm{KS}(\partial)$ is G-invariant and cup product with it commutes with G; hence the derivative of the period map along ∂ is composition with $\mathrm{KS}(\partial)\cup$, from F^1_ρ to $\mathrm{Hom}_G(\rho, H^1(C)/F^1) = M_\rho/F^1_\rho$. Since F^1 is isotropic for E_C and of half dimension, E_C pairs $H^1(C)/F^1$ perfectly with F^1, and E_ρ identifies M_ρ/F^1_ρ with the dual of $F^1_{\bar\rho}$. So the codifferential sends $\phi\otimes\eta$ to $\partial \mapsto E_\rho(\mathrm{KS}(\partial)\cup\phi, \eta) = \pm\sum_i \langle \mathrm{KS}(\partial), \phi(\varepsilon_i)\eta(\varepsilon^i)\rangle = \pm\langle \mathrm{KS}(\partial), \phi\eta\rangle$. □

On the collar of one lift of a node ν to C, write the $\rho^\vee$-valued form $\sum_i \phi(\varepsilon_i)\otimes\varepsilon^i$ as $\sum_n A_n\, z^n\, dz/z$ with $A_n \in \mathbb{C}^4$, and the ρ-valued form $\sum_i \eta(\varepsilon^i)\otimes\varepsilon_i$ as $\sum_n B_n\, z^n\, dz/z$; we write $A_n \cdot B_n$ for the contraction of (6.2). For holomorphic frames, the coefficients are integrals over a fixed circle of sections of $\pi_*\omega_{\mathcal{C}/\Delta}$, hence holomorphic on all of $\Delta^5\times\mathcal{U}$.

Lemma 6.4 (Picard–Lefschetz constant)**.** *Let ν be a trivial node and N_ν the logarithm of the monodromy of the local system H^1 of the smooth fibres of $\mathcal{C}$ around $t_\nu = 0$. Since N_ν commutes with G, it acts on M_ρ by composition; write $N_\nu|_{M_\rho}$ for this action. At every point with smooth fibre,*

$$A^\nu_0(\phi)\cdot B^\nu_0(\eta) = c_{\mathrm{PL}}\, E_\rho(N_\nu\phi, \eta)\ , \qquad c_{\mathrm{PL}} = \frac{\pm 1}{(2\pi i)^2\,|G|}\ , \tag{6.5}$$

with a sign that does not depend on ν. Consequently, if $\sum_\nu k_\nu N_\nu|_{M_\rho} = 0$ at one smooth fibre, then $\sum_\nu k_\nu A^\nu_0 \cdot B^\nu_0 \equiv 0$ on $\Delta^5\times\mathcal{U}$.

Proof. Let C be a smooth fibre. The node ν has trivial monodromy, so its collar in the base lifts to $|G|$ disjoint collars in C, permuted simply transitively by G; let δ be the core circle of one of them and $\delta_g = g(\delta)$. The loop $t_\nu \mapsto e^{i\vartheta}t_\nu$, $\vartheta\in[0,2\pi]$, moves the two clustered branch points once around each other, and by (T6) its monodromy on C is the product of the Dehn twists about the vanishing circles δ_g. These are disjoint, so the twists commute and the classes $[\delta_g]$ have zero intersection numbers; the monodromy is $1+N_\nu$ with $N_\nu^2 = 0$ and

$$E_C(N_\nu u, u') = \pm\sum_{g\in G}\int_{\delta_g} u \int_{\delta_g} u'\ , \qquad u, u' \in H^1(C,\mathbb{C})\ ,$$

with the same sign for all trivial nodes, since each is plumbed by $zz' = t_\nu$ and encircled positively; the formula is quadratic in δ, so the orientation of δ does not matter. Apply it to $u = \phi(\varepsilon_i)$, $u' = \eta(\varepsilon^i)$ and sum over i. Since $\int_{g\delta}\phi(\varepsilon_i) = \int_\delta \phi(g^{-1}\varepsilon_i)$, and likewise for η, the contribution of g is $\sum_i \int_\delta \phi(g^{-1}\varepsilon_i)\int_\delta \eta(g^{-1}\varepsilon^i) = \sum_i \int_\delta\phi(\varepsilon_i)\int_\delta\eta(\varepsilon^i)$, by the invariance of $\sum_i \varepsilon_i\otimes\varepsilon^i$. So all $|G|$ terms are equal, and since $\int_\delta\phi(\varepsilon_i) = 2\pi i\,(A_0)_i$, we obtain $E_\rho(N_\nu\phi,\eta) = \pm|G|(2\pi i)^2 A_0\cdot B_0$, which is (6.5).

For the consequence, the formula involves N_ν only through $N_\nu|_{M_\rho}$. The monodromies around the coordinate hyperplanes commute with each other and with G, so each $N_\nu|_{M_\rho}$ is a flat endomorphism of the local system with fibres M_ρ over the connected set of points with smooth fibre; a relation at one smooth fibre therefore holds at all of them. There $\sum_\nu k_\nu A^\nu_0\cdot B^\nu_0 = c_{\mathrm{PL}}E_\rho(\sum_\nu k_\nu N_\nu\phi,\eta) = 0$, and since the A^ν_0, B^ν_0 are holomorphic on $\Delta^5\times\mathcal{U}$, the sum vanishes identically. □

By Appendix A, the maps that the full twists σ_6^2 and σ_8^2 of the pairs $\{6,7\}$ and $\{8,9\}$ induce on H^1_{par} by (3.3) are unipotent and have the same logarithm. At real positions with small $t_\nu > 0$

these full twists are the loops around $t_{(6,7)} = 0$ and $t_{(8,9)} = 0$. By the proof of Proposition 3.2, the monodromies of B_C along these loops correspond under τ to these maps times scalars, which are 1 because both sides are unipotent. So the logarithms of the two monodromies agree on B_C, hence on $B_C \otimes \mathbb{C} \supset M_\rho$:

$$N_{(6,7)}|_{M_\rho} = N_{(8,9)}|_{M_\rho} \ . \tag{6.6}$$

6.3. **The limit of the codifferential.** The sheaf $\pi_*\omega_{\mathcal{C}/\Delta}$ is locally free and commutes with base change by (T3), since $h^0(\omega)$ is constant on the connected nodal fibres. Its summands $\mathrm{Hom}_G(\rho, \pi_*\omega_{\mathcal{C}/\Delta})$ and $\mathrm{Hom}_G(\rho^\vee, \pi_*\omega_{\mathcal{C}/\Delta})$ are locally free, being images of idempotents, of rank 3 by Lemma 3.1 at a smooth fibre; so at C_0 the spaces F^1_ρ, $F^1_{\bar\rho}$ of Rosenlicht differentials (T4) have dimension 3. Fix holomorphic frames ϕ_1, ϕ_2, ϕ_3 and η_1, η_2, η_3 of these summands near C_0. Let $\mathfrak{M}$ be the 9×12 matrix with rows indexed by (a, b), $1 \le a, b \le 3$, and columns by the vector fields $t_\nu\partial_{t_\nu}$ for the four trivial clusters and ∂_x for the eight special points of P_0, with entries $\langle \mathrm{KS}(\cdot), \phi_a\eta_b\rangle$ at smooth fibres. By Lemma 6.3, $\mathfrak{M}$ is a column submatrix of the codifferential, so the rank of the period map at a smooth fibre is at least the rank of $\mathfrak{M}$ there.

The entries. Write $\phi_a\eta_b$ on P_0, and at smooth fibres on $\mathbb{P}^1$, as $\pi^*(f(x)\,dx^2)$. By (T5) the column of a branch point x_i is $c_{\mathrm{KS}} \operatorname{Res}_{x_i} f\,dx$. For a node c_ν, ∂_{c_ν} translates the clustered branch points, so its column is $c_{\mathrm{KS}}(2\pi i)^{-1} \oint f\,dx$ over a fixed circle around the collar, which is holomorphic on $\Delta^5 \times \mathcal{U}$ and equal at $t = 0$ to $c_{\mathrm{KS}} \operatorname{Res}_{x=c_\nu} f\,dx$ on P_0. For a trivial node, $t_\nu\partial_{t_\nu}$ is the scaling $\sum_i (x_i - c_\nu)\partial_{x_i}$ of its two branch points, and by the residue theorem on the collar its entry is $c_{\mathrm{KS}} \sum_i \operatorname{Res}_{x_i}(x - c_\nu) f\,dx = c_{\mathrm{KS}} \sum_n A_n \cdot B_{-n}$, the coefficient of $z^{-2}\,dz^2$ in $\phi_a\eta_b$ on the collar, $z = x - c_\nu$. All columns thus carry the same constant c_{KS}, which we drop. These entries are holomorphic on all of $\Delta^5 \times \mathcal{U}$.

The first-order term. On the other branch $z' = t_\nu/z$, so $-A_{-n}t_\nu^{-n}$ is the coefficient of $z'^n\,dz'/z'$, an integral over a fixed circle on the side of P_ν of a section of $\pi_*\omega_{\mathcal{C}/\Delta}$; hence A_{-n} and B_{-n} are t_ν^n times holomorphic functions for $n \ge 1$, compare [28], and

$$\sum_n A_n \cdot B_{-n} = A_0 \cdot B_0 - t_\nu\big(\phi^{\mathrm{m}} \cdot \eta^{\mathrm{c}} + \phi^{\mathrm{c}} \cdot \eta^{\mathrm{m}}\big)\big|_{t=0} + O(|t|^2) \ , \tag{6.7}$$

where $\phi^{\mathrm{m}}, \eta^{\mathrm{m}}$ are the values at the node on the side of P_0 (coefficients of dz) and $\phi^{\mathrm{c}}, \eta^{\mathrm{c}}$ those on the side of P_ν (coefficients of dz'). Put $\Xi_\nu = \phi^{\mathrm{m}} \cdot \eta^{\mathrm{c}} + \phi^{\mathrm{c}} \cdot \eta^{\mathrm{m}}$, a column indexed by (a, b).

The relation. By (6.6) and Lemma 6.4, with $\mathbf{k} = (k_\nu)$ given by $k_{(6,7)} = -1$, $k_{(8,9)} = 1$ and $k_\nu = 0$ for the other trivial nodes, $\sum_\nu k_\nu A_0^\nu \cdot B_0^\nu \equiv 0$, in any holomorphic frame. Fix $\beta_{(6,7)}, \beta_{(8,9)} \in \mathbb{C}^\times$, choose $\beta_\nu \in \mathbb{C}^\times$ arbitrarily for the other trivial nodes and for $\nu = (3, 4)$, and consider the curve $t_\nu = \beta_\nu s$, for all five nodes, at fixed positions, whose fibres are smooth for $s \ne 0$; and the vector field $\partial_{\mathbf{k}} = \sum_\nu k_\nu t_\nu \partial_{t_\nu}$. By (6.7), $\mathfrak{M}(s)\partial_{\mathbf{k}} = -s\,\Xi_{\mathbf{k}} + O(s^2)$ with $\Xi_{\mathbf{k}} = \sum_\nu k_\nu \beta_\nu \Xi_\nu$. Replacing the column $t_{(8,9)}\partial_{t_{(8,9)}}$ by $\partial_{\mathbf{k}}$, and this column by its quotient by $-s$, is right multiplication by an invertible matrix for $s \ne 0$. As $s \to 0$ the resulting matrix tends to

$$\mathfrak{M}^* = \big[\, R_\nu \ (\nu \text{ trivial},\ \nu \ne (8,9)) \ \big|\ T_i \ (\text{eight points of } P_0) \ \big|\ \Xi_{\mathbf{k}} \,\big] \ ,$$

where R_ν is the node column $A_0^\nu \cdot B_0^\nu$ at $t = 0$ and T_i the column $\operatorname{Res}_{x_i} f\,dx$ at $t = 0$. By (T7), $\operatorname{rk}\mathfrak{M}^* \le \operatorname{rk}\mathfrak{M}(s)$ for small $s \ne 0$. A change of holomorphic frames multiplies $\mathfrak{M}(s)$ on the left by an invertible holomorphic matrix $L(s)$, and $\mathfrak{M}^*$ by $L(0)$, so $\mathfrak{M}^*$ may be computed in any basis of the Rosenlicht differentials at C_0.

The component of the cluster $\{3, 4\}$. The two branch points of $P_{(3,4)}$ carry the reflections γ_3, γ_4, whose fixed hyperplanes are distinct, so $\dim \mathrm{Fix}\langle\gamma_3, \gamma_4\rangle = 2$. A ρ-form on the part of C_0 over $P_{(3,4)}$, with at most simple poles over the node, gives a class in the cohomology of the complement of the fibre over the node, and this map is injective on forms with at most logarithmic poles. By Section 2.4, the ρ-part of that cohomology is $H^1(\mathbb{P}^1 \setminus \{\infty\}, j_*L)$ for the local system L on $P_{(3,4)}$ minus its three special points with monodromies the inverse transposes of γ_3, γ_4; for $\rho^\vee$-forms the

monodromies are γ_3, γ_4. That group consists of the cocycles (u_3, u_4) with $u_i \in \mathrm{Im}(g_i - 1)$, where g_3, g_4 are these monodromies, modulo the coboundaries. Since $\mathrm{rk}(g_i - 1) = 1$ and the coboundaries form a space of dimension $4 - \dim \mathrm{Fix}\langle g_3, g_4\rangle = 2$, it is 0 in both cases. So every ρ- and $\rho^\vee$-form at C_0 vanishes on this component and, by (T4), has residue 0 at its node on the side of P_0. This is why the forms need to be written down only on P_0 and on the four components of the trivial nodes.

6.4. **The certificate.** The following computation completes the proof of Theorem 6.1.

Proposition 6.8 (computer-assisted)**.** *For the positions and the values $\beta_{(6,7)}$, $\beta_{(8,9)}$ of Appendix A, the 9×9 minor of $\mathfrak{M}^*$ formed by the columns T_i at the six special points of P_0 other than the node of $\{1, 2\}$ and x_{13}, $R_{(6,7)}$, $R_{(11,12)}$ and $\Xi_{\mathbf{k}}$ equals*

$$(2.6207710\ldots - 2.9287922\ldots i) \cdot 10^{-29} \pm 6 \cdot 10^{-106}\ , \tag{6.9}$$

and in particular is non-zero.

Proof of Theorem 6.1. By Proposition 6.8, $\mathrm{rk}\,\mathfrak{M}^* \geq 9$, so $\mathfrak{M}(s)$ has rank 9 for small $s \neq 0$, and the period map of $\mathcal{C}$ has rank 9 at the corresponding smooth fibres. By Section 6.1 the period map of $\mathcal{H}$ has rank at least 9 at a general point. □

7. Proof of Theorem 1.1

7.1. **Globalization.** Fix a base point $C_* \in \mathcal{H}$, a G-stable lattice $L_\rho \subset V_\rho$, a non-zero $w \in L_\rho$, so that E_w is integral on $V_\mathbb{Z}$ below, and let $(V, h) = (B_{C_*}, h_w)$ and $\mathfrak{D}$ the domain of Section 2.2 for (V, h). Let $V_\mathbb{Z} \subset V$ be the image of $\mathrm{Hom}_{\mathbb{Z}[G]}(L_\rho, H^1(C_*, \mathbb{Z}))$, and let Γ' be the intersection of $U(V_\mathbb{Z}, h)$ with the congruence subgroup of level $N \geq 3$ of the symplectic group of $(V_\mathbb{Z}, E_w)$, a neat arithmetic group. The covers in $\mathcal{H}$ have the automorphisms $Z(G)$, which act on B_C by scalars $\chi_\rho(z) \in \mu_6 \subset K$; these act trivially on $\mathfrak{D}$ and, since $\chi_\rho(z)^6 = 1$, on HW. The monodromy of B_C over $\mathcal{H}$, which is defined up to these scalars, preserves $V_\mathbb{Z}$ and h_w, and on a connected finite étale cover $\mathcal{H}' \to \mathcal{H}$ it lies in $\mu_6\Gamma'$. So the period map $\mathcal{P}\colon \mathcal{H}' \to \mathrm{Sh}' = \Gamma'\backslash\mathfrak{D}$ is well defined and holomorphic, and by Borel's extension theorem [40] it is algebraic. By Theorem 6.1 it has rank $9 = \dim \mathrm{Sh}'$ at a general point, so its image is constructible and contains a dense Zariski open subset U' of Sh'.

The map from Sh' to the moduli space of polarized abelian sixfolds with level-N structure is algebraic, again by [40], and pulling back the universal family gives an abelian scheme $\mathcal{A} \to \mathrm{Sh}'$ with K acting by endomorphisms up to isogeny, whose fibre over the class of $J \in \mathfrak{D}$ has $H^1 = (V_\mathbb{Z}, J)$. The fibre over $\mathcal{P}(t)$ is K-isogenous to A_{C_t}. The Weil classes of the fibres form a local system of rank 2, which becomes trivial on a finite étale cover $\mathrm{Sh}'' \to \mathrm{Sh}'$; let Sh''_0 be a connected component, which maps onto Sh'. For a flat section $\mathfrak{w}$ of the Weil classes over Sh''_0, the set of points at which $\mathfrak{w}$ is algebraic is a countable union of closed algebraic subsets of Sh''_0. Indeed, for Hilbert polynomials $P_1, \ldots, P_l$ of subschemes of the fibres and $q \in \mathbb{Q}^l$, let T run over resolutions of the irreducible components of the reduced fibre product over Sh''_0 of the relative Hilbert schemes $\mathrm{Hilb}^{P_1}, \ldots, \mathrm{Hilb}^{P_l}$ of $\mathcal{A} \times_{\mathrm{Sh}'} \mathrm{Sh}''_0$, which are proper over Sh''_0. Over the smooth T the classes of the fibres of the universal families are flat sections, so the locus where $\mathfrak{w} = \sum_i q_i[\mathcal{Z}_i]$ is a union of connected components of T, and its image in Sh''_0 is closed; see [41] for this argument. By Theorem 5.8 and Lemma 2.1 this set contains the preimage of U', which has non-empty interior in the analytic topology, so by Baire's theorem one of the closed algebraic subsets has non-empty interior and is therefore all of the irreducible Sh''_0. With $\mathfrak{w}$ running over a basis, the Weil classes are algebraic at every point of Sh'.

7.2. **Every sixfold.** Let (A, ι, λ) satisfy (a)–(c) of Theorem 1.1. The forms h_λ on $H^1(A, \mathbb{Q})$ and h on V are K-Hermitian of dimension 6, signature $(3, 3)$ and determinant $\equiv -2$, by (c) and Proposition 3.2, so by Landherr's theorem (Section 2.1) there is a K-linear isometry $f\colon H^1(A, \mathbb{Q}) \to V$. It maps the Riemann form $E_\lambda = \mathrm{Tr}_{K/\mathbb{Q}}(\sqrt{-3}\, h_\lambda)$ to $E = \mathrm{Tr}_{K/\mathbb{Q}}(\sqrt{-3}\, h)$, and the transported complex

structure J_A commutes with K, preserves E and is positive for it, because E_λ is the Riemann form of a polarization; so $J_A \in \mathfrak{D}$. The fibre of $\mathcal{A}$ over the class of J_A has rational first cohomology $(V, J_A) \cong H^1(A, \mathbb{Q})$, as Hodge structures with K-action. Abelian varieties up to isogeny are equivalent to polarizable rational Hodge structures of weight one (see e.g. [42, Corollary 6.9]), so this fibre is K-isogenous to A. By Section 7.1 and Lemma 2.1, $\mathrm{HW}(A, \iota)$ consists of algebraic classes. This proves Theorem 1.1.

Appendix A. The computations

This appendix describes the computations that the proof uses. The group-theoretic computations are exact: they are done in GAP 4.16.1 [18] with its required packages, in which elements of K and of cyclotomic fields are exact numbers, and in Python with exact integers and rationals. The rank certificate of Proposition 6.8 computes the eigenbases of the local monodromies, the local exponents and the binomial coefficients of the series expansions exactly; all products, the constraint matrices and their solutions, the expansions and the minors are computed in ball arithmetic [19], through python-flint 0.9.0, at a precision of 320 bits. A ball is a midpoint with a radius, and every operation returns a ball that contains the exact result. The programs are those of the repository named in the introduction.

The group and the tuple. The group G_{32} is built as the subgroup of $\mathrm{GL}_4(K)$ generated by four explicit reflections of order 3; it has order 155520. The thirteen matrices of Appendix B lie in it and generate a subgroup of order 155520, so they generate G_{32}. Their product is 1, $\gamma_1, \dots, \gamma_{12}$ are reflections of order 3 in the two conjugacy classes of size 40 of reflections of determinant ζ^2 and ζ^4, with the determinants stated in Section 3, and γ_{13} has order 2, determinant 1 and $\mathrm{rk}(\gamma_{13} - 1) = 2$, in the conjugacy class of size 90. The commutant of the tuple consists of scalars. Also $\gamma_1\gamma_2 = \gamma_6\gamma_7 = \gamma_8\gamma_9 = \gamma_{11}\gamma_{12} = 1$, $\gamma_3\gamma_4$ has order 6, the eight monodromies on P_0 of Section 6.1 commute, and γ_3, γ_4 fix a common subspace of dimension 2.

The determinant (Proposition 3.2). The parabolic cocycles of Section 3 are computed as the solutions of a linear system over K; they form a space of dimension 10, the coboundaries one of dimension 4, and a complement of the coboundaries gives a basis of H^1_{par}, of dimension 6. For 67 braids of the form used in the proof of Proposition 3.2, each checked to fix γ, (3.3) gives their matrices A_β on H^1_{par}, all different from 1. The equations $A_\beta^* \Omega A_\beta = \Omega$ form a linear system in the 36 entries of Ω, whose solution space has dimension 1 over K. A spanning form Ω_0 is made Hermitian (as $\Omega_0 + \Omega_0^*$, or as $(\zeta^2 - \zeta^4)\Omega_0$ if $\Omega_0^* = -\Omega_0$); its determinant, in the basis of the program, is -128, and exact Gaussian elimination gives the signature $(3, 3)$.

The Dehn twists (equation (6.6)*).* For each of the five clusters $\{j, j+1\}$, the smallest power σ_j^k that fixes γ is computed together with its matrix A on H^1_{par}, and $N = A^o - 1$ for the smallest $o \geq 1$ with $(A^o - 1)^2 = 0$. For the four trivial clusters $k = 2$ and $o = 1$, so A is unipotent and $N = A - 1$ is its logarithm, of rank 1; these four matrices N span a space of dimension 3 over K, with the single relation $N_{(6,7)} = N_{(8,9)}$.

The characters (Table 2). The group G is converted to a permutation group, and GAP computes its character table, the fourteen irreducible characters with central character χ_ρ, their degrees, and their values on the classes of the γ_i. These values give $\dim M_{\rho_k}$ by (2.2). The programs identify the numbering of the characters in each run of GAP with that of Table 2 through $\dim M_{\rho_k}$, requiring that exactly one such identification contains all eleven pairs of Proposition 5.2, and check that it respects the degrees and the values $\chi_{\rho_k}(\gamma_1)$.

The subcovers (Proposition 5.2). The subgroups $H \supseteq Z(G)$ are taken up to conjugacy as the preimages of representatives of the conjugacy classes of subgroups of $G/Z(G)$. For each H, the action of the γ_i on the cosets of H gives $g(C/H)$ by the Riemann–Hurwitz formula and the local monodromies of $C \to C/H$ as elements of H. For each linear character ψ of H with $\psi|_{Z(G)} = \chi_\rho$, taken from the character table of H, this gives the order m, the exponents, r_ψ, and the vector

$v_{H,\psi}$ by scalar products of characters. A Python program then checks in exact arithmetic that the eleven pairs of Table 1 occur, with the vectors of Table 2 and every entry of Table 1, including $g(X_j)$ from the Riemann–Hurwitz formula and the exponents, and checks the relation (5.3), the identity (5.4) and the full pairing of the exponents of ψ_j^t for every $t \in (\mathbb{Z}/m_j)^\times$.

The rank certificate (Proposition 6.8). The eight special points of P_0, in the order of the clusters $\{1,2\},\{3,4\},x_5,\{6,7\},\{8,9\},x_{10},\{11,12\},x_{13}$, are placed at $x = \frac{1}{7}(-13,-9,-5,-2,1,4,8,12)$. On the components of the trivial clusters $\{1,2\}$, $\{6,7\}$, $\{8,9\}$, $\{11,12\}$ the two branch points are at $y = -\frac{3}{5}, \frac{2}{7}$, at $y=-1, \frac{4}{7}$, at $y = -\frac{6}{5}, \frac{5}{7}$ and at $y = -\frac{7}{5}, \frac{6}{7}$ respectively, with the node at $y=\infty$. The values are $\beta_{(6,7)} = \frac{2+i}{3}$ and $\beta_{(8,9)} = \frac{-1+2i}{5}$, and the program also uses $\beta_{(1,2)} = 1$ and $\beta_{(11,12)} = \frac{3-i}{4}$, whose coefficients in $\Xi_{\mathbf{k}}$ are balls around 0.

On P_0, whose monodromies commute, a basis of ρ that diagonalizes all of them, and the dual basis, are computed exactly over K. In such a basis the l-th coordinate of a form on P_0 of one of the two types is a multivalued differential $\prod_j (x-p_j)^{a_{jl}}\, Q_l(x)\, dx$, where p_j are the special points, $a_{jl} = \tau_{jl} - 1$ if the l-th eigenvalue of the monodromy at p_j is $e^{2\pi i \tau_{jl}}$ with $0 < \tau_{jl} < 1$, and $a_{jl} = -1$ at a node and 0 at a branch point if it is 1, and Q_l is a polynomial whose degree bound makes the form holomorphic at ∞; the forms of the other type are obtained in the same way from the inverse transposes, and similarly on the four components of the trivial nodes. Which of the two types is ρ does not matter: exchanging them exchanges the two indices of the rows of $\mathfrak{M}^*$. The coefficients of the Q_l on P_0 in the basis $(x+\frac{20}{7})^n$, $n \geq 0$, and those of the forms on the four components, one on each, are the unknowns, 21 of them for each type. The constraints are the opposite-residue conditions (T4) at the four trivial nodes, four for each, and the vanishing of the residue on the side of P_0 at the node of $\{3,4\}$, which holds by Section 6.3; the program imposes it as the four conditions given by its exact computation that $\mathrm{Fix}(\gamma_3\gamma_4) \cap (\mathrm{Im}(\gamma_3 - 1) + \mathrm{Im}(\gamma_4 - 1)) = 0$, 20 in all. For each type, an 18×18 minor of the 20×21 constraint matrix is a ball that excludes 0, so the solution space has dimension at most 3; it contains the restrictions of the Rosenlicht differentials, of dimension 3 (Section 6.3), so it equals them, and a basis is obtained by solving the 18×18 system.

Each basis form is expanded at each special point of P_0 to order 3. The columns R_ν, T_i and Ξ_ν are formed as the coefficient of $(x-p_j)^{-2}$ in f, the coefficient of $(x-p_j)^{-1}$, and the expression of Section 6.3 in the values of the forms at the node on both sides. The relation vector is computed in ball arithmetic: the four columns R_ν of the trivial nodes have a 3×3 minor that is a ball excluding 0, so they have rank at least 3, and by (6.6) and Lemma 6.4 at most 3; so they satisfy a unique relation up to scale, which is $\mathbf{k}$, and a ball solve for it, normalized at $(8,9)$, gives balls of radius below $2 \cdot 10^{-82}$ around $0, -1, 1, 0$. The 9×9 minor of Proposition 6.8 is then evaluated as a ball; its value is (6.9), the real and imaginary parts being enclosed with radii $3.7 \cdot 10^{-106}$ and $4.6 \cdot 10^{-106}$.

Appendix B. The tuple

The tuple $\gamma = (\gamma_1, \ldots, \gamma_{13})$ of Section 3 consists of the following matrices in $G \subset \mathrm{GL}_4(K)$, acting on column vectors; γ_i is the local monodromy at the i-th of the real branch points $x_1 < \cdots < x_{13}$, in the convention of Section 2.4.

$$\gamma_1 = \tfrac{1}{36}\begin{pmatrix} 12-12\zeta^2 & 6+12\zeta^2 & 18 & -24-12\zeta^2 \\ 6-6\zeta^2 & 30-3\zeta^2 & 9\zeta^2 & 6-6\zeta^2 \\ 18+18\zeta^2 & -9\zeta^2 & 18-9\zeta^2 & 18+18\zeta^2 \\ -24-12\zeta^2 & 6+12\zeta^2 & 18 & 12-12\zeta^2 \end{pmatrix} \qquad \gamma_2 = \tfrac{1}{36}\begin{pmatrix} 24+12\zeta^2 & 12+6\zeta^2 & -18\zeta^2 & -12+12\zeta^2 \\ -6-12\zeta^2 & 33+3\zeta^2 & 9+9\zeta^2 & -6-12\zeta^2 \\ 18 & -9-9\zeta^2 & 27+9\zeta^2 & 18 \\ -12+12\zeta^2 & 12+6\zeta^2 & -18\zeta^2 & 24+12\zeta^2 \end{pmatrix}$$

$$\gamma_3 = \tfrac{1}{36}\begin{pmatrix} 24+12\zeta^2 & 6-6\zeta^2 & 6+30\zeta^2 & 0 \\ 6-6\zeta^2 & 33+3\zeta^2 & -3-15\zeta^2 & 0 \\ -30-6\zeta^2 & 15+3\zeta^2 & 15+21\zeta^2 & 0 \\ 0 & 0 & 0 & 36 \end{pmatrix} \qquad \gamma_4 = \tfrac{1}{36}\begin{pmatrix} 24+12\zeta^2 & 24+12\zeta^2 & 24+12\zeta^2 & 0 \\ -12-24\zeta^2 & 24+12\zeta^2 & -12+12\zeta^2 & 0 \\ -12-24\zeta^2 & -12+12\zeta^2 & 24+12\zeta^2 & 0 \\ 0 & 0 & 0 & 36 \end{pmatrix}$$

$$\gamma_5 = \tfrac{1}{36}\begin{pmatrix} 12-12\zeta^2 & 30+24\zeta^2 & -6-12\zeta^2 & 0 \\ 30+6\zeta^2 & -6-21\zeta^2 & 12+15\zeta^2 & 0 \\ -6+6\zeta^2 & 12-3\zeta^2 & 30-3\zeta^2 & 0 \\ 0 & 0 & 0 & 36 \end{pmatrix} \qquad \gamma_6 = \tfrac{1}{36}\begin{pmatrix} 12-12\zeta^2 & 18\zeta^2 & -12-6\zeta^2 & -12-24\zeta^2 \\ -18\zeta^2 & 18-9\zeta^2 & -9\zeta^2 & 18 \\ -12-6\zeta^2 & 9\zeta^2 & 30-3\zeta^2 & -6-12\zeta^2 \\ -12+12\zeta^2 & 18+18\zeta^2 & -6+6\zeta^2 & 12-12\zeta^2 \end{pmatrix}$$

$$\gamma_7 = \tfrac{1}{36}\begin{pmatrix} 24+12\zeta^2 & 18+18\zeta^2 & -6+6\zeta^2 & -24-12\zeta^2 \\ -18-18\zeta^2 & 27+9\zeta^2 & -9-9\zeta^2 & -18\zeta^2 \\ -6+6\zeta^2 & 9+9\zeta^2 & 33+3\zeta^2 & -12-6\zeta^2 \\ 12+24\zeta^2 & 18 & 6+12\zeta^2 & 24+12\zeta^2 \end{pmatrix} \qquad \gamma_8 = \tfrac{1}{36}\begin{pmatrix} 24+12\zeta^2 & -12+12\zeta^2 & -12+12\zeta^2 & 0 \\ -12+12\zeta^2 & 24+12\zeta^2 & -12+12\zeta^2 & 0 \\ -12+12\zeta^2 & -12+12\zeta^2 & 24+12\zeta^2 & 0 \\ 0 & 0 & 0 & 36 \end{pmatrix}$$

$$\gamma_9 = \tfrac{1}{36}\begin{pmatrix} 12-12\zeta^2 & -24-12\zeta^2 & -24-12\zeta^2 & 0 \\ -24-12\zeta^2 & 12-12\zeta^2 & -24-12\zeta^2 & 0 \\ -24-12\zeta^2 & -24-12\zeta^2 & 12-12\zeta^2 & 0 \\ 0 & 0 & 0 & 36 \end{pmatrix} \qquad \gamma_{10} = \tfrac{1}{36}\begin{pmatrix} 12-12\zeta^2 & 30+24\zeta^2 & -6-12\zeta^2 & 0 \\ 30+6\zeta^2 & -6-21\zeta^2 & 12+15\zeta^2 & 0 \\ -6+6\zeta^2 & 12-3\zeta^2 & 30-3\zeta^2 & 0 \\ 0 & 0 & 0 & 36 \end{pmatrix}$$

$$\gamma_{11} = \tfrac{1}{36}\begin{pmatrix} 36 & 0 & 0 & 0 \\ 0 & 12-12\zeta^2 & 24+12\zeta^2 & -24-12\zeta^2 \\ 0 & 24+12\zeta^2 & 12-12\zeta^2 & 24+12\zeta^2 \\ 0 & -24-12\zeta^2 & 24+12\zeta^2 & 12-12\zeta^2 \end{pmatrix} \qquad \gamma_{12} = \tfrac{1}{36}\begin{pmatrix} 36 & 0 & 0 & 0 \\ 0 & 24+12\zeta^2 & 12-12\zeta^2 & -12+12\zeta^2 \\ 0 & 12-12\zeta^2 & 24+12\zeta^2 & 12-12\zeta^2 \\ 0 & -12+12\zeta^2 & 12-12\zeta^2 & 24+12\zeta^2 \end{pmatrix}$$

$$\gamma_{13} = \tfrac{1}{36}\begin{pmatrix} 0 & 36+18\zeta^2 & 18\zeta^2 & 0 \\ 18-18\zeta^2 & -9 & 9+18\zeta^2 & 0 \\ -18-18\zeta^2 & -9-18\zeta^2 & -27 & 0 \\ 0 & 0 & 0 & 36 \end{pmatrix}$$

The product $\gamma_1\gamma_2\cdots\gamma_{13}$, taken from left to right, is 1; the other properties of the tuple used in the text are listed in Appendix A.

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