Weil classes on non-split Weil sixfolds over $\mathbb{Q}(\sqrt{-3})$

The proof explained, from the basics

A companion to the note *Algebraicity of Weil classes on non-split abelian sixfolds of Weil type over* $\mathbb{Q}(\sqrt{-3})$ *with Hermitian determinant* -2

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Preface

Status

This companion, like the note it accompanies, was produced by large language models, instances of the Claude models of Anthropic running as autonomous agents, in October 2026. The steps of the proof were checked only by other instances of the same models, and such checks may share the blind spots of the models that wrote them. The author of the note has not reviewed this companion, and no human expert has checked either the proof or the companion. It is published together with the note as an explanation of its proof, and it may contain errors in its explanations, examples and remarks, in addition to any errors in the proof itself.

The note *Algebraicity of Weil classes on non-split abelian sixfolds of Weil type over* $\mathbb{Q}(\sqrt{-3})$ *with Hermitian determinant* -2 (October 2026) states the proof concisely, for specialists; the code for the computations on which the proof relies is published with it. This companion explains the same proof from the basics. It is meant to be read on its own: it states everything it uses, proves what can be proved in a few pages, and cites the published literature for the rest.

What the proof claims

The note claims the following theorem, with a proof that has not been refereed: every Weil class is algebraic on every complex abelian sixfold A with an action of $K = \mathbb{Q}(\sqrt{-3})$ that admits a compatible polarization of signature $(3, 3)$ whose Hermitian form has determinant -2 modulo norms from K (discriminant 2). These are the sixfolds of the *non-split* class with the smallest discriminant (the classes are $\operatorname{disc} h = 1, 2, 5, 10, 11, \ldots$, and only 1 is split): the Hermitian space contains no three-dimensional isotropic subspace, while Schoen's theorem on sixfolds and the result on sixfolds claimed in a preprint of Markman (Section 1.5) concern the split class. The precise statement is Theorem 1.5. Besides its own arguments and standard facts, the proof uses Schoen's 1988 theorem on generalized Prym varieties [Sch88], Shioda's results on Fermat varieties [Shi79] and the Lefschetz standard conjecture for abelian varieties [Lie68; Kle68]. It uses no unrefereed result. The computer-assisted parts of the proof are exact computations in GAP (the tuple: product 1, membership in G_{32}, generation, classes; the invariant Hermitian form from braid monodromy, of determinant -128; the characters, subgroups and subcovers and the integral relation; the Dehn-twist relation on M_ρ; and the fixed spaces on the non-abelian component) and one ball-arithmetic certificate (the 9×9 minor, with the two 18×18 pivot minors before it).

Who this is for

The companion is written for a master's student in mathematics who knows linear algebra, groups, rings and fields, complex analysis and basic topology, and has met Riemann surfaces and some algebraic geometry. It does not assume Hodge theory, abelian varieties or the representation theory of finite groups beyond the definitions: the facts it needs from them are stated where they are used, with a proof when the proof is short and a precise reference to the published literature otherwise (Chapter B lists the sources). The less standard material that the proof uses is developed in full: representations of finite groups over $\mathbb{Q}$, K and $\mathbb{C}$ (Chapter 3), complex reflection groups (Chapter 4), G-covers of the line and Hurwitz

spaces (Chapter 5), period maps near the boundary and interval arithmetic (Chapter 6), symmetric products and compactly supported cohomology of local systems (Chapter 8), and the Shimura-variety facts at level $N \geq 3$ (Chapter 12). The main objects come with a motivation, a worked example with actual numbers, a precise statement, and either a proof or a precise citation; most chapters add intuition boxes. A reader who wants to know what is claimed and what the claim rests on can take the fast path below and then Chapter 13.

Two ways to read

Fast path, about thirty pages. Chapter 1, then the first section of every following chapter, then Chapter 13. The first sections are written so that they read consecutively as one proof sketch; each says what its step proves, the one idea, and one picture or table:

step	section	topic	pages
statement	Section 1.1	the theorem and its proof	pp. 2–3
(i)(a), (viii)	Section 2.1	the Hermitian form and the class 2	pp. 11–12
(ii), (vi)	Section 3.1	finite-group representation theory	pp. 23–24
(i)	Section 4.1	the group G_{32}	pp. 35–38
(i)	Section 5.1	the family $\mathcal{H}$	pp. 43–44
(i)(b)	Section 6.1	dominance	pp. 54–55
(ii), (v)	Section 7.1	the eleven pieces	pp. 66–67
(iii)	Section 8.1	the cyclic-cover theorem, first half	pp. 75–77
(iii)	Section 9.1	the cyclic-cover theorem, second half	pp. 85–86
(iv)	Section 10.1	the étale pieces	pp. 94–95
(vi)	Section 11.1	the assembly	pp. 102–103
(vii), (viii)	Section 12.1	globalization and isogenies	pp. 113–114
all	Chapter 13	the whole proof, and its delicate steps	pp. 122–125

Full path. Chapter 1 to Chapter 13 in order. Part I states the theorem; Part II builds the group, the covers and the family, and readers who know finite groups and covers of curves can read only the first sections of Chapters 3 to 5; Part III produces the algebraic cycles; Part IV reaches every sixfold and reviews the whole argument. A reader who wants to examine the most delicate steps first can take Chapters 6 to 9 and 11 and then Chapter 13.

Each chapter opens with a blue box, *Where we are*, that says what the chapter contributes, which step of the proof it supports and what it needs; every chapter but the last closes with a remark on where its results are used. Yellow boxes, *Intuition*, say in plain words what an object or a step means; green boxes, *What just happened*, sum up a long argument; cyan boxes, *Physics aside*, give an analogy from physics and say where it stops. Red boxes, *Remark on the proof*, mark places where the argument is terse, where it relies on a standard fact that is recalled rather than proved, or where it rests on a single computation; Chapter 13 discusses those that matter for checking the proof.

How to use the exercises

The exercises stand right after the concept they need, six to ten per chapter, and most of them are computations with the numbers of the proof: the genus 505441 of a curve of the family, the dimension 312 of piece 9, the determinant class of a diagonal Hermitian form. They are meant to be done when met, with pen and paper. Each has a full solution in Chapter D, preceded by a one-paragraph hint, so that a reader can look at the hint first and the solution only if the hint is not enough. The two exercises of Chapter 13 are of a different kind: they ask which parts of the proof survive if a step fails.

Conventions

The table collects the conventions that cause most confusion; Section 2.1 and Remark 2.2 fix the conventions for Hermitian forms in detail.

object	convention
the field	$K = \mathbb{Q}(\sqrt{-3}) = \mathbb{Q}(\zeta) = \mathbb{Q}(\omega)$, $\zeta = e^{2\pi i/6}$, $\omega = \zeta^2 = e^{2\pi i/3}$, $\theta = \sqrt{-3}$
Hermitian form	h_λ on $H^1(A, \mathbb{Q})$, K-linear in the first variable, with $E_\lambda = \mathrm{Tr}_{K/\mathbb{Q}}(\theta h_\lambda)$ (Definition 2.1); a form on $H_1(A, \mathbb{Q})$, as in much of the literature, has the same determinant class
determinant, discriminant	$\mathrm{disc}\, h = (-1)^n \det h$ in dimension $2n$; the theorem's class is $\det h \equiv -2$, $\mathrm{disc}\, h = 2$; Deligne and Markman call $\det h$ the discriminant
Shimura variety	$\mathrm{Sh}_3^{(2)}$, indexed by $\mathrm{disc}\, h$
the family	$\mathcal{H}$, the Hurwitz component of the branching tuple $\boldsymbol{\gamma}$ of Section 4.5
proof steps	(i)–(viii) as in Figure 1.1
pieces	numbered 1–11 as in Table 7.1
group actions on H^1	a left action, for instance $g \cdot x = (g^{-1})^* x$; pull-back reverses products (Remark 11.2)
reserved letters	θ is always $\sqrt{-3}$ and ω always $e^{2\pi i/3}$; Θ is the class of a polarization; Weil lines are Λ (Λ_j, Λ_u, ...); $\mathcal{B}_C$ is the Hodge structure and A_C the sixfold with $H^1(A_C, \mathbb{Q}) = \mathcal{B}_C$
algebraic classes	$\mathrm{Alg}^p(X)$, the $\mathbb{Q}$-span of classes of codimension-p cycles; "in $\mathrm{Alg} \otimes \mathbb{C}$" means in its complex span

What this companion does and does not establish

The proof has not been refereed, and no expert has checked it. This companion explains the argument; it does not certify it. Writing a proof out in more detail makes it easier to check, but it does not make it checked. The worked examples test special cases, small groups and small covers; they can expose an error in a formula or a number, but they do not test the steps whose content lies in large computations, the exact computations with G_{32} and the ball-arithmetic certificate of Chapter 6, which the companion explains but which can only be checked by rerunning them. Chapter 13 puts the whole proof on a few pages and lists the steps that a reader checking it may want to examine first.

Part 1

The statement

CHAPTER 1

The theorem and the plan of its proof

Proof steps: statement; overview of (i)–(viii)

Where we are. This chapter states the theorem, says how it relates to earlier results and claims on Weil classes, describes how its proof is organized in eight steps, which chapter of this companion explains which step, and which published results and computations the proof relies on. Needs: the definition of a Hodge class, recalled in Section 1.2 with references; everything else is introduced here at the level of a summary and explained in the later chapters.

1.1. The theorem and its proof in two pages

The Hodge conjecture predicts which cohomology classes of a smooth complex projective variety come from subvarieties: exactly the rational classes of type (p,p), the *Hodge classes* [Del06, §1]. It is known for divisors and open in general, already for abelian varieties. On an abelian variety of even dimension $2n$ on which an imaginary quadratic field K acts with balanced multiplicities (n,n), Weil found a two-dimensional space of Hodge classes in H^{2n}, the *Weil classes* $\mathrm{HW}(A) = \bigwedge^{2n}_K H^1(A,\mathbb{Q})$, which for a general such A are not products of divisor classes [Wei79], [Gor99, Thm. 4.11]. They are a standard test case: for fourfolds Schoen proved them algebraic over $\mathbb{Q}(\sqrt{-3})$, and a preprint of Markman claims this for every K, while Schoen's theorem on sixfolds and the result on sixfolds claimed in Markman's preprint concern the case where a Hermitian form attached to the polarization is *split* (Section 1.5).

The theorem explained in this companion is about one non-split case. Take $K = \mathbb{Q}(\sqrt{-3}) = \mathbb{Q}(\omega)$. A Weil sixfold A with a compatible polarization λ carries a Hermitian form h_λ on the six-dimensional K-space $H^1(A,\mathbb{Q})$, of signature $(3,3)$, and its isometry class is decided by one invariant, the class of $\det h_\lambda$ modulo norms from K (Landherr). The split class is $\det h \equiv -1$; the theorem is about $\det h \equiv -2$, which is not split because 2 is inert in K (Section 1.3). The precise statement is Theorem 1.5: *if A admits one polarization with $\det h_\lambda \equiv -2$, every Weil class on A is algebraic.* Its proof is computer-assisted (Section 1.4).

The proof runs through eight steps (Figure 1.1, Section 1.6). (i) Curves C with an action of the complex reflection group $G = G_{32}$ of order 155520, branched over 13 points of $\mathbb{P}^1$, form a family over a space $\mathcal{H}$ of dimension 13, a finite cover of the space of sets of 13 points in $\mathbb{C}$; up to the action of $\mathrm{PGL}_2(\mathbb{C})$ on the branch points the curves have $13-3=10$ moduli. For each of them the part $\mathcal{B}_C$ of $H^1(C,\mathbb{Q})$ belonging to the reflection representation ρ is H^1 of a Weil sixfold A_C with $\det h \equiv -2$, and the period map of the family has rank 9, so it dominates the nine-dimensional moduli space of these sixfolds. (ii) Eleven subspaces of $H^1(C)$, the *pieces*, each an eigenspace of a cyclic cover of curves $X_j \to Y_j$ sitting under C, combine with signs ± 1 to the multiplicity vector of ρ. (iii)–(iv) The top exterior power of each piece lies in the complex span of algebraic classes: for the eight ramified pieces by a theorem on cyclic covers proved with a cone over a Fermat variety, which we call the *cyclic-cover theorem* (Theorem 8.1), and for the three étale pieces by Schoen's 1988 theorem. (v) These classes pull back to the Jacobian $J(C)$. (vi) A correspondence with complex coefficients in the group algebra $\mathbb{C}[G]$, followed by a contraction, turns them into the Weil classes of $\mathcal{B}_C$, on every member of the family. (vii) Because the family dominates, algebraicity spreads to

every point of the moduli space, and (viii) Landherr's theorem and an isogeny reach every sixfold of the theorem.

The ingredients specific to this proof are the integral relation of step (ii), the assembly step (vi), a proof of the cyclic-cover theorem that follows Schoen's construction with an injectivity step of its own, and the proof of dominance by a degeneration; the same family of G_{32}-covers is used in Severn's preprint, which claims the theorem first (Sections 1.5 and 1.7). The steps that a reader checking the proof may want to examine first are the cyclic-cover theorem at the scale of the ramified pieces and the dominance certificate (Section 1.8).

1.2. The Hodge conjecture and Weil classes

The conjecture. For a smooth complex projective variety M of dimension N, a closed subvariety Z of codimension p has a fundamental class $[Z] \in H^{2p}(M, \mathbb{Q})$: the Poincaré dual of integrating over Z. Integrating a closed form of type (a, b), $a + b = 2N - 2p$, over Z gives zero unless $(a, b) = (N - p, N - p)$, because Z is a complex variety of dimension $N - p$; by Poincaré duality this says that $[Z]$ is a rational class whose image in $H^{2p}(M, \mathbb{C})$ has type (p, p). The Hodge conjecture asserts the converse, up to rational coefficients: every class in $\mathrm{Hdg}^p(M) = H^{2p}(M, \mathbb{Q}) \cap H^{p,p}(M)$ is a rational linear combination of fundamental classes [Del06, §1]. For $p = 1$ this is the Lefschetz $(1, 1)$ theorem [Del06, §2, Remark (iii)], and by the hard Lefschetz theorem [Dem12, Ch. VI, (8.17)] it follows in degree $2N - 2$. Already for abelian varieties the general case is open.

Where new Hodge classes come from. On a very general principally polarized abelian variety the Hodge classes are the multiples of the powers Θ^k of the polarization class Θ, which are algebraic [Voi26, §1]. New Hodge classes appear when the Hodge structure has extra symmetry, most visibly endomorphisms. Endomorphisms usually only create new *divisor* classes and products of them, for which the conjecture is again automatic. Weil's observation is that an imaginary quadratic field acting with balanced multiplicities creates Hodge classes that divisors do not explain.

Definition 1.1 (Weil type and Weil classes; [Mil22, §2.1], [MZ98, §1]). Let A be a complex abelian variety of dimension $2n$ and $\iota\colon K \to \mathrm{End}(A) \otimes \mathbb{Q}$ an action of an imaginary quadratic field. The pair (A, ι) is of *Weil type* if both eigenspaces of K on $H^{1,0}(A)$ have dimension n. Then $H^1(A, \mathbb{Q})$ is a K-vector space of dimension $2n$, and the *Weil classes* are

$$\mathrm{HW}(A, \iota) = \textstyle\bigwedge_K^{2n} H^1(A, \mathbb{Q}) \;\subset\; \bigwedge_{\mathbb{Q}}^{2n} H^1(A, \mathbb{Q}) = H^{2n}(A, \mathbb{Q}).$$

Over $\mathbb{C}$ the space $H^1(A, \mathbb{C})$ splits into the subspace V_ι on which $a \in K$ acts by a and the subspace $V_{\bar\iota}$ on which it acts by $\bar a$, each of dimension $2n$, and $\mathrm{HW}(A, \iota) \otimes \mathbb{C} = \bigwedge^{2n} V_\iota \oplus \bigwedge^{2n} V_{\bar\iota}$. Weil type means that V_ι contains n vectors of type $(1, 0)$ and n of type $(0, 1)$, so the line $\bigwedge^{2n} V_\iota$ has type (n, n), and so does its conjugate: the Weil classes are Hodge classes [Wei79], [Del82, Prop. 4.4]. If $\mathrm{End}^0(A) = K$, no non-zero Weil class is a combination of products of divisors [MZ98, Criterion 13], and for the very general member of a family of Weil type $\mathrm{Hdg}^n(A) = \mathbb{Q}\Theta^n \oplus \mathrm{HW}(A, \iota)$ has dimension 3 [Gor99, Thm. 4.11]. So the Weil classes are the first Hodge classes on abelian varieties for which the conjecture has real content; Weil suggested looking among them for counterexamples [Gor99, §4].

Example 1.2 (the smallest place to see a Weil class). For $n = 3$, $\dim_{\mathbb{Q}} H^1(A, \mathbb{Q}) = 12$ and $H^6(A, \mathbb{Q}) = \bigwedge^6 H^1$ has dimension $\binom{12}{6} = 924$, with Hodge numbers $h^{p,6-p} = \binom{6}{p}\binom{6}{6-p} = 1, 36, 225, 400, 225, 36, 1$; the Weil classes form a 2-dimensional rational subspace of the 400-dimensional $H^{3,3}$, and for a very general Weil sixfold Hdg^3 is only 3-dimensional. To see one concretely, let $E_\omega = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\omega)$, on which K acts by multiplication, and let $A = E_\omega^6$ with K acting by multiplication on the factors $1, 2, 6$ and through the conjugate on the factors $3, 4, 5$. On $H^1(E_\omega, \mathbb{C}) = \mathbb{C}\,\mathrm{d}z \oplus \mathbb{C}\,\mathrm{d}\bar z$, multiplication by ω pulls $\mathrm{d}z$ back to $\omega\,\mathrm{d}z$, and the conjugate action pulls $\mathrm{d}\bar z$ back to $\omega\,\mathrm{d}\bar z$. So V_ι is spanned by $\mathrm{d}z_1, \mathrm{d}z_2, \mathrm{d}\bar z_3, \mathrm{d}\bar z_4, \mathrm{d}\bar z_5, \mathrm{d}z_6$, and the Weil classes are the rational classes in the span of

$$\mathrm{d}z_1 \wedge \mathrm{d}z_2 \wedge \mathrm{d}\bar z_3 \wedge \mathrm{d}\bar z_4 \wedge \mathrm{d}\bar z_5 \wedge \mathrm{d}z_6 \qquad \text{and its complex conjugate,}$$

visibly of type $(3,3)$. This A is a product of CM elliptic curves, very far from general (its Hdg^3 is much larger than 3-dimensional), so it shows the classes, not the difficulty; the difficulty is to find cycles for them on every A of a given type at once, including the very general one. The same A reappears in Example 1.7.

Exercise 1.1. Let (A,ι) be of Weil type of dimension 6. (a) Show that $\dim_{\mathbb{Q}} H^6(A,\mathbb{Q}) = 924$ and $h^{3,3}(A) = 400$. (b) Show that $\dim_K \bigwedge^6_K H^1(A,\mathbb{Q}) = 1$, hence $\dim_{\mathbb{Q}} \mathrm{HW}(A,\iota) = 2$. (c) For (A,ι) of Weil type of dimension $2n$, compute $\dim_{\mathbb{Q}} H^{2n}(A,\mathbb{Q})$ and the ratio $2/\dim H^{2n}$ for $n = 1,2,3,4$.

Exercise 1.2. In Example 1.2, (a) check that V_ι is the span claimed, and (b) show directly that $\bigwedge^6 V_\iota$ has type $(3,3)$. (c) Now let K act on E^6_ω by multiplication on four factors and through the conjugate on two. Show that $\bigwedge^6_K H^1$ is still a 2-dimensional rational subspace of H^6, but of type $(4,2)+(2,4)$, so it contains no non-zero Hodge class: Weil type is exactly what makes these classes Hodge [Del82, Prop. 4.4].

Physics aside 1.3 (Weil classes as baryons). Hodge classes are the invariants of the Hodge group, a symmetry group of the Hodge structure [MZ99, (1.4)]. For a very general Weil sixfold the Hodge group is $\mathrm{SU}(V,h)$; over $\mathbb{C}$ it is $\mathrm{SL}_6(\mathbb{C})$, acting on V_ι as the fundamental representation and on $V_{\bar\iota}$ as its dual. Invariants in $\bigwedge^6(V_\iota \oplus V_{\bar\iota})$ are of two kinds: contractions of a fundamental index with an anti-fundamental one, which give Θ^3, and the totally antisymmetric ϵ-tensor, $\bigwedge^6 V_\iota$ and its conjugate, which give the Weil classes. This is the familiar statement that in an SU(6) gauge theory a mesonic bilinear $\bar q q$ and a baryonic $\epsilon\, q^6$ are both singlets, while for U(6) only the mesons are: the extra U(1) acts on $\epsilon\, q^6$ with charge 6. Products of divisor classes are invariant under the larger Lefschetz group [Mil99, Cor. 4.5], which for $\mathrm{End}^0(A) = K$ is the centralizer of K in $\mathrm{Sp}(H^1,E)$ [Gor99, p. 2.14], that is $\mathrm{U}(V,h)$ (a K-linear map preserves $E = \mathrm{Tr}(\theta h)$ if and only if it preserves h, by the uniqueness in Definition 2.1); it contains that U(1), so products of divisors cannot produce the Weil classes. The analogy stops here: being a singlet is being a Hodge class, but the Hodge conjecture asks whether the singlet is the class of a subvariety, and nothing in the gauge theory corresponds to that.

1.3. Split and non-split: the class 2

A polarization λ of (A,ι) is *compatible* if its Rosati involution restricts to complex conjugation on $\iota(K)$. Its alternating form E_λ on $H^1(A,\mathbb{Q})$ is then the trace of a unique Hermitian form, $E_\lambda = \mathrm{Tr}_{K/\mathbb{Q}}(\sqrt{-3}\, h_\lambda)$, of signature $(3,3)$ for a sixfold (Definition 2.1; a Hermitian form on $H_1(A,\mathbb{Q})$ instead, as in much of the literature, has the same determinant class, Remark 2.2). Its determinant is well defined only up to norms: a change of K-basis by a matrix S multiplies a Gram determinant by $\det S \cdot \overline{\det S} = \mathrm{Nm}(\det S)$. So the invariant is the class

$$\det h_\lambda \in \mathbb{Q}^\times / \mathrm{Nm}\, K^\times, \qquad \mathrm{Nm}\, K^\times = \{x^2 + 3y^2 \neq 0 : x,y \in \mathbb{Q}\},$$

and a rational number is a norm from K exactly when it is positive and every prime $p \equiv 2 \pmod 3$ divides it to an even power (Lemma 2.4). By Landherr's theorem [Lan35], [Del82, Prop. 4.1], two Hermitian spaces over K of the same dimension are isometric if and only if they have the same signature and the same determinant class. For signature $(3,3)$ the determinant is negative, so the classes are $-1,-2,-5,-10,-11,\dots$, infinitely many, and the class -1 is the *split* one: exactly for it $H^1(A,\mathbb{Q})$ contains a three-dimensional K-subspace on which h_λ vanishes [Del82, Cor. 4.2]. Schoen's theorem on sixfolds and the result on sixfolds claimed in Markman's preprint concern the split class (Section 1.5). Markman's preprint states its result for sixfolds with $\det h \equiv -1$ [Mar25, Thm. 1.5.1].

Example 1.4 (the model of the class -2). The diagonal form $h_2 = \langle 2,1,1,-1,-1,-1\rangle$ on K^6 has signature $(3,3)$ and $\det h_2 = -2$. It is not isometric to the split form $\langle 1,1,1,-1,-1,-1\rangle$, because $-2/-1 = 2$ is not a norm: $2 \equiv 2 \pmod 3$ divides 2 to the first

power. By Landherr, every Hermitian space of signature $(3,3)$ with $\det \equiv -2$ is isometric to h_2; for instance $\langle 1,1,1,-1,-1,-2\rangle$ is, and so is the invariant form of the G_{32} family (Example 2.13).

Conventions. The *discriminant* is $\operatorname{disc} h = (-1)^n \det h$ for a form of dimension $2n$, so $\operatorname{disc} h = -\det h$ for sixfolds, and the class of the theorem is $\operatorname{disc} h = 2$; we index the Shimura variety of these sixfolds by the discriminant, $\mathrm{Sh}_3^{(2)}$. Deligne and Markman call the unsigned class $\det h$ the discriminant [Del82, §4], [Mar25, §1.1], so that the theorem's class is their "discriminant -2", while Voisin's "discriminant", which she takes positive, agrees with our disc [Voi26, §2.2]. Section 2.1 tabulates the conventions. Since neither 2 nor -2 is a norm, the verdict "not split" is the same in all of them. The moduli space $\mathrm{Sh}_3^{(2)}$ is a quotient of the domain $X^+ = \mathrm{U}(3,3)/\mathrm{U}(3)\times\mathrm{U}(3)$, of complex dimension $3^2 = 9$ (Proposition 2.15); for comparison, all principally polarized abelian sixfolds form a space of dimension $6\cdot 7/2 = 21$.

Exercise 1.3. (a) Show that $x^2 + 3y^2 = 2$ has no solution with $x, y \in \mathbb{Q}$. (Clear denominators and reduce modulo 4 and modulo 8.) (b) Deduce that neither 2 nor -2 is a norm from K, and that the class of -2 in $\mathbb{Q}^\times/\operatorname{Nm} K^\times$ is different from that of -1. (c) Show that 2 stays prime in $\mathbb{Z}[\omega]$ by showing that $x^2 + x + 1$ has no root modulo 2, and relate this to (a). (d) Which of $3, 5, 7, 13$ are norms from K? Give x, y where they are.

Exercise 1.4. Using only the determinant class and the signature, decide which of the following Hermitian forms on K^6 are isometric to $h_2 = \langle 2,1,1,-1,-1,-1\rangle$, which to the split form, and which to neither: $\langle 1,1,1,-1,-1,-2\rangle$, $\langle 2,1,1,-2,-2,-1\rangle$, $\langle 3,1,1,-1,-1,-1\rangle$, $\langle 5,1,1,-1,-1,-1\rangle$, $\langle 7,2,1,-1,-1,-1\rangle$.

1.4. The Main Theorem

We state the theorem with $K = \mathbb{Q}(\zeta)$, $\zeta = e^{2\pi \mathrm{i}/6}$, so $K = \mathbb{Q}(\sqrt{-3})$, and $\theta = \sqrt{-3}$; h_λ is the Hermitian form of λ normalized by $E_\lambda = \operatorname{Tr}_{K/\mathbb{Q}}(\theta h_\lambda)$ (Definition 2.1).

Theorem 1.5 (Main Theorem, computer-assisted). *Let A be a complex abelian variety of dimension 6 and $\iota\colon K \to \operatorname{End}(A)\otimes\mathbb{Q}$ a homomorphism. Suppose that A admits a polarization λ such that*

(a) the Rosati involution of λ restricts to complex conjugation on $\iota(K)$,
(b) both eigenspaces of $\iota(\theta)$ on $H^{1,0}(A)$ have dimension 3, and
(c) $\det h_\lambda \equiv -2 \bmod \operatorname{Nm} K^\times$, that is, $\operatorname{disc} h_\lambda = 2$; in particular h_λ is not split.

Then every class in $\mathrm{HW}(A,\iota) = \bigwedge_K^6 H^1(A,\mathbb{Q})$ is algebraic.

Dependencies. Besides its own arguments and standard facts, the proof uses three published results: Schoen's theorem on Hodge classes of generalized Prym varieties for simple branching data [Sch88, Thm. 2.0, Cor. 3.1], Shioda's results on Fermat varieties [Shi79], and the Lefschetz standard conjecture for abelian varieties [Lie68; Kle68]. The proof uses Schoen's theorem for the étale pieces $5, 7, 8$ of step (ii) and the cyclic-cover theorem (Theorem 8.1) for the eight ramified pieces. For the étale pieces Patel and Zhang [PZ27] give an alternative reference. The computer-assisted parts of the proof are exact computations in GAP (the tuple: product 1, membership in G_{32}, generation, classes; the invariant Hermitian form from braid monodromy, of determinant -128; the characters, subgroups and subcovers and the integral relation; the Dehn-twist relation on M_ρ; and the fixed spaces on the non-abelian component) and one ball-arithmetic certificate (the 9×9 minor, with the two 18×18 pivot minors before it). The tuple is Section 4.5 and Chapter C, the form Example 2.13, the group data and the relation Proposition 4.10 and Section 7.5, the Dehn-twist relation and the fixed spaces Section 6.5, and the certificate, computed with Arb [Joh17], Section 6.6. It uses no unrefereed result.

Remark 1.6 (what the hypotheses mean). Hypotheses (a) and (b) say that (A,ι,λ) is a polarized abelian sixfold of Weil type of signature $(3,3)$; (c) singles out the Hermitian class. No condition is imposed on the type of λ (it need not be principal), on which order of K acts

by genuine endomorphisms, or on $\mathrm{End}^0(A)$: abelian varieties with complex multiplication are included, and the conclusion concerns $\mathrm{HW}(A,\iota)$ only, not other Hodge classes. Condition (c) is a property of the triple (A,ι,λ), not of (A,ι): the same (A,ι) can carry compatible polarizations of different classes (Example 1.7). This is why the hypothesis is existential, "admits a polarization λ such that": if one compatible polarization has class -2, the Weil classes are algebraic, whatever the classes of the other polarizations. Rescaling λ by $c\in\mathbb{Q}_{>0}$ multiplies $\det h_\lambda$ by $c^6=\mathrm{Nm}(c^3)$ (Exercise 2.1), so (c) depends only on the ray $\mathbb{Q}_{>0}\lambda$. The theorem concerns only the class -2, the field $\mathbb{Q}(\sqrt{-3})$ and dimension 6.

Example 1.7 (one abelian sixfold, two classes). Let B_1 be an abelian threefold with an action ι_1 of K of signature $(2,1)$ and a compatible polarization λ_1, and put $A=B_1\times B_1$, with K acting by ι_1 on the first factor and by the conjugate action $\bar\iota_1$ on the second, so that the signature becomes $(2,1)+(1,2)=(3,3)$. For every $c\in\mathbb{Z}_{>0}$ the product $\lambda_1\times c\lambda_1$ is compatible, and its determinant class is $-c$. So A is split for $c=1$ and of class 2 for $c=2$, and the Main Theorem applies to (A,ι) because of the polarization with $c=2$. In the concrete model $B_1=E_\omega^3$, with K acting by multiplication on two factors and through the conjugate on the third and the product polarization, this is a computation with 2×2 integral matrices. With the form on H^1 built from the dual of the Riemann form, as in Example 2.3, the Hermitian form of one factor E_ω is $\pm\frac{1}{3}$ (the sign recording the action), on a factor polarized by $c\lambda_0$ it is $\pm\frac{1}{3c}$, and in total

$$\det h_{\lambda_1\times c\lambda_1}=-\frac{1}{3^6c^3}\equiv -c,\qquad \text{since } 3^6c^4=\mathrm{Nm}(27c^2),$$

with signature $(3,3)$ for every c; the form on H_1 gives $-c^3/3^6$, the same class (Remark 2.2). The sixfold A here is E_ω^6 of Example 1.2, and Example 2.3 works out the same phenomenon with independent multipliers on all six factors.

Exercise 1.5. (a) In Example 1.7, show without coordinates that $\det h_{\lambda_1\times c\lambda_1}\equiv -c$. (Use that, for the conjugate action and the same λ_1, the form $(x,y)\mapsto -\overline{h_{\lambda_1}(x,y)}$ is Hermitian with the same trace form; that the determinant of a 3×3 form scales by $c^{\pm 3}$ when λ_1 is scaled by c, the sign of the exponent depending on the convention; and that c^2 and $(\det h_{\lambda_1})^2$ are norms.) (b) For which $c\in\{1,\dots,12\}$ is $\lambda_1\times c\lambda_1$ split, and for which is it of class 2? (c) Why does the answer to (b) not contradict the fact that the Weil classes of (A,ι) do not depend on λ?

1.5. What was known, and what is new

Known cases. For abelian fourfolds Schoen proved the Weil classes algebraic for $K=\mathbb{Q}(\sqrt{-3})$ and every discriminant [Sch98], and Markman's preprint states the result for every K, as a consequence of its claim for sixfolds combined with Schoen's degeneration argument [Mar25, Cor. 1.6.1], [Voi26, Lemme 2.9, Cor. 2.10]. For sixfolds, Schoen proved algebraicity on the split sixfolds over $\mathbb{Q}(\sqrt{-3})$ [Sch98], [Mos26, p. 2], Koike proved it on the split sixfolds over $\mathbb{Q}(\mathrm{i})$ [Koi04], and Markman's preprint states the result for the split sixfolds for every K [Mar25, Thm. 1.5.1].

OpenAI's preprints. In October 2026 OpenAI posted a preprint [Ope26a] whose abstract states that it proves the Hodge conjecture for every self-power of the Jacobian at the tensor-Hodge-generic points of families of abelian covers of curves; its proof uses results claimed in two further preprints [Ope26b; Ope26c]. Its Lemma 3.4 uses the construction behind the cyclic-cover theorem (Chapters 8 and 9), and contractions of determinant lines of the kind used in step (vi) are among the tools collected in [Ope26a, §2]. We have not assessed its claims.

Severn's preprint. On 14 October 2026, before the note and this companion were published, Severn posted a preprint [Sev26b] that claims the Main Theorem: in the convention of Deligne and Markman, the algebraicity of the Weil classes on every abelian sixfold of Weil type over $\mathbb{Q}(\sqrt{-3})$ of discriminant -2. It uses the ρ-isotypic factors (Prym–Tyurin varieties) of

G_{32}-covers of $\mathbb{P}^1$ with the branching type of Section 4.5, by a different route. Its Weil classes come from Schoen's algebraic classes on fifty-one cyclic intermediate quotients, combined by correspondences given by Lefschetz classes through a theorem of Milne, with the quotients chosen by an integral identity among characters of a torus. Here eleven pieces are combined through an integral relation among multiplicity vectors and correspondences in $\mathbb{C}[G]$, and the cyclic-cover theorem supplies the Weil lines of the ramified pieces (Chapters 7 and 11). Its dominance step certifies rank nine at a point of the family near an explicit cover, where here rank nine is obtained by a degeneration to a boundary point (Chapter 6). Its proof has not been checked here. In two later preprints the same author claims, through a general transfer theorem [Sev26a] applied to covers with group G_{34}, the algebraicity of the Weil classes on every abelian sixfold of Weil type over $\mathbb{Q}(\sqrt{-3})$, of every discriminant [Sev26c].

What is new. The Main Theorem was claimed first in Severn's preprint, and the two routes are compared above. The ingredients specific to the proof explained here are the step (vi) that assembles a single isotypic Weil line from eleven cyclic pieces through a signed integral relation among multiplicity vectors and correspondences in $\mathbb{C}[G]$ (Chapters 7 and 11), whose plan, Weil classes obtained from determinant lines of cyclic subcovers combined by a contraction under an integrality condition on multiplicity vectors, was formulated while reading the statements of OpenAI's preprint [Ope26a]; a proof of the cyclic-cover theorem whose construction is Schoen's and whose injectivity step, by extension by zero, takes the place of his Lemma 2.8 (Chapters 8 and 9); and the proof of dominance by a degeneration to a boundary point with a ball-arithmetic certificate (Chapter 6).

1.6. A map of the proof

The proof has eight steps, numbered (i)–(viii) throughout this companion; Figure 1.1 shows them with the chapter that explains each, and the arrows mean "is used by".

Step (i): the family (Chapters 2 and 4 to 6). The group $G = G_{32} \subset \mathrm{GL}_4(K)$ is generated by complex reflections of order 3; ρ is its four-dimensional reflection representation, which is defined over K, and V_ρ is ρ viewed as an 8-dimensional $\mathbb{Q}[G]$-module with $\mathrm{End}_{\mathbb{Q}[G]}(V_\rho) = K$ (Propositions 4.10 and 3.27). A G-cover $C \to \mathbb{P}^1$ branched at 13 points, with local monodromies six reflections of each of the two kinds and one involution (Section 4.5), has genus 505441, and

$$\mathcal{B}_C = \mathrm{Hom}_{\mathbb{Q}[G]}(V_\rho, H^1(C, \mathbb{Q}))$$

is a six-dimensional K-vector space with Hodge numbers $(3, 3)$, the first cohomology $H^1(A_C, \mathbb{Q})$ of a Weil sixfold A_C, defined up to isogeny (Definition 5.17, Example 5.13, and Proposition 5.18). The covers in one Hurwitz component $\mathcal{H}$ form a family; $\mathcal{H}$ is a finite étale cover of the 13-dimensional space $\mathcal{U}_{13}$ of 13-point subsets of $\mathbb{C}$ (Definition 5.6), and modulo the action of $\mathrm{PGL}_2(\mathbb{C})$ on the branch points the curves have $13 - 3 = 10$ moduli. (a) On every member $\det h \equiv -2$, an exact computation of the invariant Hermitian form from 67 braid monodromy matrices (Example 2.13). (b) The period map $\mathcal{H} \to \mathrm{Sh}_3^{(2)}$ has rank 9, proved by a degeneration to a boundary point of the family whose last step is a ball-arithmetic certificate, so the family dominates a connected component, and the curves with a given generic period point form, modulo $\mathrm{PGL}_2(\mathbb{C})$, a one-dimensional family (Theorem 6.6; $10 - 9 = 1$).

Step (ii): eleven pieces (Chapters 3 and 7). For a subgroup $H \leq G$ containing the centre and a character ψ of H that restricts on the centre to the central character of ρ, the subspace $W_{H,\psi} \subset H^1(C, \mathbb{C})$ on which H acts by ψ is a character eigenspace of the cyclic cover $X = C/\ker\psi \to Y = C/H$, and its multiplicity vector $v_{H,\psi}$ is computed by Frobenius reciprocity; such a subspace is a *piece* (Definition 7.1). An integral relation $e_\rho = \sum_{j=1}^{11} c_j v_j$ with $c_j = \pm 1$, seven plus signs and four minus signs, holds as an identity of integer vectors (Proposition 7.8, Table 1.1).

Steps (iii)–(iv): the Weil lines of the pieces (Chapters 8 to 10). For each piece the top exterior power Λ_{ψ_j} of the eigenspace is in the complex span of algebraic classes on $J(X_j)$.

For the eight ramified pieces this is the cyclic-cover theorem (Theorem 8.1); for the three étale pieces it is Schoen's theorem with $r = 0$, or Patel–Zhang (Theorems 10.10 and 10.11).

Step (v): pull-back to $J(C)$ (Chapters 7 and 10). The norm map $J(C) \to J(X_j)$ pulls Λ_{ψ_j} back to the line $\Lambda_j = \bigwedge^{\mathrm{top}} W_j$, again in the complex span of algebraic classes (Proposition 7.10).

Step (vi): assembly (Chapter 11). The relation says that the seven pieces with $c_j = +1$ have the same multiplicities as the four with $c_j = -1$ plus one copy of $u \otimes M_\rho$, $u \in \rho$. A matrix Ψ over $\mathbb{C}[G]$ realizes the isomorphism; its top exterior power preserves the complex span of algebraic classes because $\mathbb{Q}[G]$ is Zariski dense in $\mathbb{C}[G]$, and a contraction with the conjugate of the four negative pieces leaves the line $\bigwedge^6(u \otimes M_\rho)$ in the complex span of algebraic classes. For a rational $u \in V_\rho$ with component u' in ρ, the line of u' and its complex conjugate span the complexified Weil classes of a quotient of $J(C)$ that is K-isogenous to A_C, and a rational class in the complex span of algebraic classes is algebraic; so the Weil classes of A_C are algebraic, on every member of $\mathcal{H}$.

Steps (vii)–(viii): every sixfold (Chapter 12). The locus where a flat Weil class is algebraic is a countable union of closed algebraic subsets of (a finite cover of) the Shimura variety; it contains the image of the dominating family, which contains a dense Zariski-open subset, so by Baire's theorem it is everything. Finally, every (A, ι, λ) of the theorem has a Hermitian space isometric to the family's by Landherr, hence is K-isogenous to a fibre, and algebraicity of Weil classes is invariant under K-isogeny (Theorem 12.13).

piece j	1	2	3	4	5	6	7	8	9	10	11
c_j	-1	$+1$	-1	$+1$	$+1$	$+1$	-1	$+1$	$+1$	-1	$+1$
m	6	6	6	12	24	6	24	12	6	12	6
r	544	384	448	654	0	336	0	0	160	672	64
$g(Y_j)$	989	913	1361	1537	2341	3397	4213	4681	77	6793	449
d_j	2520	2208	3168	3726	4680	7128	8424	9360	312	14256	960
Schoen 1988					✓		✓	✓			
Theorem 8.1	✓	✓	✓	✓		✓			✓	✓	✓
Patel–Zhang					✓		✓	✓			

TABLE 1.1. The eleven pieces in miniature: m the degree of the cyclic cover $X_j \to Y_j$, r its number of branch points ($r = 0$: étale), $d_j = \dim W_j = 2g(Y_j) - 2 + r$. The last three rows mark which result gives the Weil line of each piece in the proof: Schoen's theorem [Sch88, Thm. 2.0, Cor. 3.1], the cyclic-cover theorem, and Patel–Zhang [PZ27] as an alternative for the étale pieces. The full table, with subgroups and exponents, is Table 7.1. The signed sum of the row d_j is $\sum_j c_j d_j = 28374 - 28368 = 6 = \dim M_\rho$.

EXERCISE 1.6. (a) Thirteen points on $\mathbb{P}^1$ modulo $\mathrm{PGL}_2(\mathbb{C})$ have $13 - 3 = 10$ moduli, and $\dim \mathrm{Sh}_3^{(2)} = 9$. The period map is constant on the orbits of $\mathrm{PGL}_2(\mathbb{C})$ on the branch points. If it has rank 9, what is the dimension of a generic fibre of the induced map on the 10 moduli, and of a generic fibre of the period map on $\mathcal{H}$ itself, which has dimension 13? (b) Why does step (vii) need rank exactly 9, and not merely an infinite family? (Compare with a family whose image is a curve in $\mathrm{Sh}_3^{(2)}$.) (c) The genus of a cover of the family is $1 + \frac{1}{2}|G|(-2 + \sum_i (1 - 1/e_i))$ with $|G| = 155520$ and ramification indices $e_i = 3$ (12 times) and 2 (once). Check that it is 505441.

EXERCISE 1.7. (a) From Table 1.1, compute $D_+ = \sum_{c_j=+1} d_j$ and $D_- = \sum_{c_j=-1} d_j$, and check $D_+ - D_- = 6$. (b) Explain why $D_+ - D_- = \dim M_\rho$ is forced by the relation $e_\rho = \sum_j c_j v_j$ together with $d_j = \sum_{\rho'} v_j(\rho') \dim M_{\rho'}$. (c) Which of the eleven pieces fail the prime condition of Section 7.5? What is the smallest multiple of e_ρ in the integral span of the étale pieces and the ramified pieces among the 334 of Section 7.5 that satisfy it?

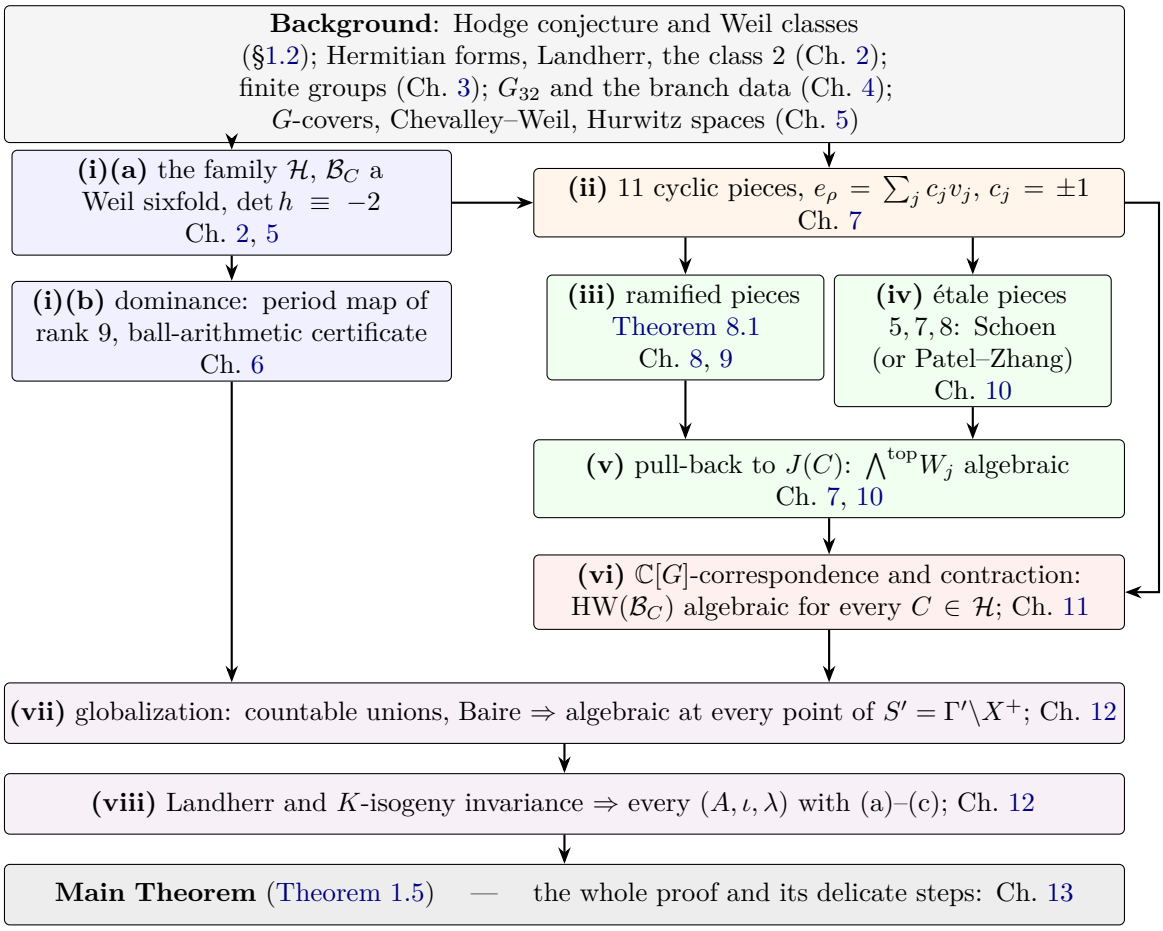


FIGURE 1.1. The eight steps of the proof of Theorem 1.5; arrows mean "is used by". Chapter numbers refer to this companion. Schoen: [Sch88, Thm. 2.0, Cor. 3.1]; Theorem 8.1: the cyclic-cover theorem; Patel–Zhang: [PZ27]. Landherr's theorem (Ch. 2) also enters (viii). Steps (ii)–(vi) use no genericity and hold on every member of $\mathcal{H}$; only (vii) needs the dominance (i)(b).

1.7. Where the ingredients come from

Schoen. The cyclic ingredients are largely Schoen's. His 1988 theorem [Sch88, Thm. 2.0, Cor. 3.1] states that the Weil Hodge structure of the generalized Prym variety of a cyclic cover with fully paired ("simple") branching data (Definition 7.6) is spanned by algebraic classes, for every degree $m > 2$ and every base genus; the proof explained here takes the Weil lines of the étale pieces $5, 7, 8$ from this theorem, and those of the eight ramified pieces from the cyclic-cover theorem (Table 9.1). The mechanism behind the cyclic-cover theorem, a quotient of a self-product, the special Abel–Jacobi fibre and its cone over a Fermat quotient (the *Fermat cone*), is that of Schoen's 1988 and 1989 papers [Sch88; Sch89]. The isogeny and specialization pattern of steps (vii)–(viii) also follows Schoen [Sch98], and the argument of step (vii) that the locus where a flat Weil class is algebraic is a countable union of closed algebraic subsets is that of Charles and Schnell [CS14].

Others. Patel and Zhang [PZ27] prove algebraicity of the Weil classes for étale covers with finite abelian group over a base of genus at least 2, an alternative reference for the étale pieces. Shioda's results on Fermat varieties [Shi79] supply the algebraic classes on the Fermat variety in the proof of the cyclic-cover theorem, through linear subspaces for fully paired characters (Chapter 9). Van Geemen and Verra applied Schoen's cyclic result to a cyclic

subcover of a non-abelian étale cover [GV03]. The relation to OpenAI's preprints and to Severn's preprint is described in Section 1.5.

1.8. What the proof relies on

Table 1.2 lists, for each step, the kind of argument and the computations it uses. Three kinds of input occur: published theorems, cited with locations; arguments written out in this companion, with standard facts recalled and cited; and exact or certified computations, whose results the companion explains and illustrates on small cases but does not reproduce.

step	chapters	argument	computation
(i)(a) $\det \equiv -2$	2, 5	invariant forms of a family (Lemma 2.10)	exact, in GAP: the tuple (product 1, membership in G_{32}, generation, classes); the invariant form from 67 braid matrices, determinant -128
(i)(b) rank 9	6	degeneration to a boundary point	exact, in GAP: the Dehn-twist relation on M_ρ, the fixed spaces on the non-abelian component; Arb ball arithmetic: a 9×9 minor, two 18×18 pivot minors
(ii) relation	3, 7	Frobenius reciprocity	exact, in GAP: characters, subgroups and subcovers of G_{32}; the integral relation
(iii) ramified	8, 9	the cyclic-cover theorem	none
(iv) étale	10	Schoen 1988 with $r = 0$; alternatively Patel–Zhang	the data of the three pieces
(v)–(vi)	7, 11	correspondences in $\mathbb{C}[G]$, density, contraction	the multiplicity data of (ii)
(vii)–(viii)	12	countable unions, Landherr, isogenies; standard facts recalled	none

TABLE 1.2. The kind of argument and the computations behind each step. Chapter 13 discusses the steps in more detail.

The steps that a reader checking the proof may want to examine first are: (1) the cyclic-cover theorem at the scale of the pieces (r up to 672, base genus up to 6793), on which the eight ramified pieces rest; (2) the dominance certificate (Section 6.6); (3) the assembly step (vi); (4) the dependence on GAP's character table and subgroup enumeration of G_{32}; (5) the standard facts that are recalled rather than proved. Chapter 13 discusses them in turn.

REMARK 1.8 (Where this is used). The step numbering (i)–(viii) of Figure 1.1 is used in every chapter ("step (iii)" in the text and the box under each chapter title), Theorem 1.5 is the statement that Chapter 12 completes, and Table 1.2 is expanded in Chapter 13. The convention for the class 2 (Section 1.3) is fixed in detail in Chapter 2.

CHAPTER 2

Weil sixfolds, Hermitian forms and the Shimura variety of class 2

Proof steps: (i)(a); setting of (vii); Landherr in (viii)

Where we are. This chapter contains one computed input of the proof and the linear algebra behind three of its steps. The computed input is the determinant class of the G_{32} family, step (i)(a): every member of the family is a Weil sixfold whose Hermitian form has $\det h \equiv -2$, so it is not split. The linear algebra is the dictionary between a polarization compatible with K and a Hermitian form over K, Landherr's classification of those forms, which is what step (viii) uses to reach every sixfold of the theorem, and the nine-dimensional period domain X^+ on which step (vii) takes place. The chapter also fixes, once for the whole companion, the conventions for Hermitian forms (Remark 2.2). Nothing here is new mathematics: the classification is due to Landherr, and we quote it in the form given by Deligne [Del82, §4]; the determinant of the G_{32} family rests on an exact computation. What the chapter adds is worked examples with actual numbers and the one argument that turns a computation with braid matrices into a statement about every member of the family (Lemma 2.10).

Background and notation. The chapter needs Definition 1.1, and norms and traces in number fields [Mil20, Ch. 2]. Throughout, $K = \mathbb{Q}(\sqrt{-3})$, $\theta = \sqrt{-3}$, $\zeta = e^{2\pi \mathrm{i}/6}$ and $\omega = \zeta^2 = e^{2\pi \mathrm{i}/3}$, so $\theta = 2\zeta - 1 = 2\omega + 1$, as in Chapter 3.

2.1. The Hermitian form and the class 2 in two pages

A complex abelian variety A of dimension 6 with an action ι of K and a compatible polarization λ carries more structure than its polarization alone. The polarization gives an alternating form E_λ on $H^1(A, \mathbb{Q})$; the action of K makes $H^1(A, \mathbb{Q})$ a six-dimensional vector space over K; and compatibility (the Rosati involution induces complex conjugation on K) says exactly that E_λ is the trace of a Hermitian form,

$$E_\lambda = \mathrm{Tr}_{K/\mathbb{Q}}(\theta\, h_\lambda)$$

(Definition 2.1). Over $\mathbb{R}$ the Hermitian form is just a form of signature $(3,3)$, the same for every Weil sixfold, and the space of compatible complex structures is the same nine-dimensional domain $X^+ = \mathrm{U}(3,3)/\mathrm{U}(3) \times \mathrm{U}(3)$ (Definition 2.14 and Proposition 2.15). Over $\mathbb{Q}$ the forms of signature $(3,3)$ fall into infinitely many isometry classes. By Landherr's theorem the class is decided by one invariant, the determinant of a Gram matrix taken modulo norms from K,

$$\det h \;\in\; \mathbb{Q}^\times / \mathrm{Nm}\, K^\times,$$

which for signature $(3,3)$ is the class of a negative number (Proposition 2.6). A rational number is a norm from K when it is positive and every prime $p \equiv 2 \pmod 3$ divides it to an even power, so the possible classes are $-1, -2, -5, -10, -11, \ldots$, one for each finite set of such primes. The class -1 is the *split* one: exactly then $H^1(A, \mathbb{Q})$ contains a three-dimensional K-subspace on which h vanishes. The theorem of this companion concerns the class $\det h \equiv -2$, which is not split because 2 is inert in K: $2 \neq x^2 + 3y^2$ for rational x, y.

The proof needs three things from this picture. First, in step (i)(a), the Weil sixfolds A_C of the G_{32} family, with $H^1(A_C, \mathbb{Q}) = \mathcal{B}_C$, have $\det h \equiv -2$. An exact computation

gives a Hermitian form invariant under 67 braid monodromy matrices, with determinant $-128 = -2 \cdot 8^2$ (Example 2.13). This form is not a priori the polarization; the link is that the space of invariant forms is one-dimensional, so the polarization is a rational multiple of the computed form, and a rational multiple changes the determinant of a 6×6 Gram matrix by a sixth power, which is a norm (Lemma 2.10). Second, in step (vii), the period map of the family lands in a quotient $\Gamma' \backslash X^+$, and X^+ is a connected complex manifold of dimension 9, a bounded convex domain in $\mathbb{C}^9$ (Proposition 2.15). Third, in step (viii), every sixfold of the theorem has a Hermitian form of the same dimension, signature and determinant class as the family's, so by Landherr the two Hermitian spaces are isometric, and this is the first step in identifying the sixfold up to isogeny with a member of the family (Section 2.7).

Two naming conventions make this harder to read than it is, and the table below fixes them once. The *determinant* $\det h$ is the unsigned class; the *discriminant* is $\operatorname{disc} h = (-1)^n \det h$ for a form of rank $2n$, so $\operatorname{disc} h = -\det h$ for sixfolds (Definition 2.1). We index the Shimura varieties by the discriminant. Deligne calls the unsigned class $f = \prod_i h(v_i, v_i)$, for an orthogonal basis (v_i), the discriminant, notes that its sign is $(-1)^b$ for signature (a, b), and proves that a form of signature (n, n) is split if and only if $f \equiv (-1)^n$ [Del82, §4, Cor. 4.2]; Milne uses the same name [Mil22, §2.1]. Markman's "discriminant $\det H$" is also unsigned, and his split sixfolds have $\det H \equiv -1$ [Mar25, §1.1, Lemma 3.1.3, Thm. 1.5.1]. Voisin defines her discriminant as the determinant of the matrix of a Hermitian form on $H_1(A, \mathbb{Q})$ and calls it positive, with the split class 1 [Voi26, §2.2, Thm. 1.4]; read in this way, it agrees with our disc.

convention	the class of the theorem	the split class
this companion	$\det h \equiv -2$, $\operatorname{disc} h = 2$, $\mathrm{Sh}_3^{(2)}$	$\det h \equiv -1$, $\operatorname{disc} h = 1$
Deligne, Milne, Markman	"discriminant -2"	"discriminant -1"
Voisin	"discriminant 2"	"discriminant 1"
skew-Hermitian matrix θh	$\det(\theta h) \equiv 2$	$\det(\theta h) \equiv 1$

Whatever the convention, the class of the theorem is ± 2, and since neither 2 nor -2 is a norm, the verdict "not split" does not depend on it.

Sources. Landherr's theorem is [Lan35]; we use it in the form of [Del82, Prop. 4.1, Cor. 4.2], together with the dictionary between polarizations and Hermitian forms [Del82, Lemma 4.6]. The period domain is standard, and its description in Proposition 2.15 is proved in full. The determinant -2 of the G_{32} family rests on an exact computation (Example 2.13).

2.2. From a polarization to a Hermitian form

We put the Hermitian form on $H^1(A, \mathbb{Q})$, because that is where the Weil classes and the cohomology of curves live. Much of the literature puts it on $H_1(A, \mathbb{Q})$; both conventions give the same invariants (Remark 2.2).

Definition 2.1 (the Hermitian form of (A, ι, λ)). Let A be a complex abelian variety of dimension $2n$, $\iota\colon K \to \operatorname{End}(A) \otimes \mathbb{Q}$ an action of K, and λ a polarization whose Rosati involution restricts to complex conjugation on $\iota(K)$ [Mil08, I, §14], [Mil22, §2.1]. Let E_λ be the alternating form of λ on $V = H^1(A, \mathbb{Q})$ (Remark 2.2), a K-vector space through $a \mapsto \iota(a)^*$. The *Hermitian form* of (A, ι, λ) is the unique Hermitian form h_λ on V, K-linear in the first variable, with

$$E_\lambda(x, y) = \operatorname{Tr}_{K/\mathbb{Q}}(\theta\, h_\lambda(x, y)) \qquad (x, y \in V).$$

Its *determinant* $\det h_\lambda$ is the class of $\det(h_\lambda(e_i, e_j))$ in $\mathbb{Q}^\times / \operatorname{Nm} K^\times$ for any K-basis (e_i), and its *discriminant* is $\operatorname{disc} h_\lambda = (-1)^n \det h_\lambda$. The triple is *split* if V contains an n-dimensional K-subspace on which h_λ vanishes.

Existence and uniqueness: compatibility gives $E_\lambda(ax, y) = E_\lambda(x, \bar{a}y)$ for $a \in K$, and for every alternating $\mathbb{Q}$-bilinear form E on a K-vector space with this property there is a unique Hermitian form h, K-linear in the first variable, with $E = \operatorname{Tr}_{K/\mathbb{Q}}(\theta h)$ [Del82,

Lemma 4.6, Sublemma 4.7] (stated there for a Riemann form on $H_1(A,\mathbb{Q})$; the proof uses only the two properties of E). Explicitly, writing $h(x,y) = u + w\theta$ with $u, w \in \mathbb{Q}$ gives $E(x,y) = \operatorname{Tr}(u\theta - 3w) = -6w$ and $E(x,\theta y) = \operatorname{Tr}(\theta\bar{\theta}\, h(x,y)) = 6u$, so

$$h(x,y) = \tfrac{1}{6}E(x,\theta y) + \tfrac{1}{2\theta}E(x,y). \tag{2.1}$$

Conversely $\operatorname{Tr}(\theta h)$ is alternating for every Hermitian h, because $\overline{\theta h(x,y)} = -\theta h(y,x)$. The determinant is well defined because a change of basis by S multiplies it by $\det S \cdot \overline{\det S} = \operatorname{Nm}(\det S)$, and it is rational because a Hermitian matrix has real determinant.

Remark 2.2 (the conventions of this companion). (1) *Space and linearity.* The form h_λ lives on $V = H^1(A,\mathbb{Q})$, with K acting by pull-back; it is K-linear in the first variable, $h(y,x) = \overline{h(x,y)}$, and $E_\lambda = \operatorname{Tr}_{K/\mathbb{Q}}(\theta h_\lambda)$ with $\theta = \sqrt{-3}$.

(2) *The sign of the Riemann form.* The polarization λ gives an alternating form Q_λ on $H_1(A,\mathbb{Q})$, invariant under the complex structure J of $\operatorname{Lie} A = H_1(A,\mathbb{R})$. We fix its sign by $Q_\lambda(u,Ju) > 0$ for $u \neq 0$. For the elliptic curve $E_\tau = \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$ with its principal polarization this means $Q(1,\tau) = +1$, that is, $Q = -\operatorname{Im} H$ with $H(z,w) = z\bar{w}/\operatorname{Im}\tau$; the classical Riemann form $\operatorname{Im} H$ of [Mil08, I, Lemma 2.4, Ex. 2.5] is $-Q$. The form E_λ on V is the dual alternating form, $E_\lambda(D_\lambda u, D_\lambda v) = Q_\lambda(u,v)$ for the isomorphism $D_\lambda\colon H_1(A,\mathbb{Q}) \to V$, $D_\lambda(u) = Q_\lambda(u,\cdot)$. Compatibility of λ means $Q_\lambda(au,v) = Q_\lambda(u,\bar{a}v)$ for $a \in K$ acting on H_1 [Del82, §4]; hence D_λ is K-antilinear and $E_\lambda(ax,y) = E_\lambda(x,\bar{a}y)$.

(3) *The complex structure on $V_\mathbb{R}$.* Let J^* be the complex structure $(J^*x)(u) = x(Ju)$ on $V_\mathbb{R} = H^1(A,\mathbb{R})$. For $x = D_\lambda(u)$, invariance of Q_λ under J gives $D_\lambda(Ju) = -J^*D_\lambda(u)$, hence $E_\lambda(x,J^*x) = -Q_\lambda(u,Ju) < 0$. So $J_A := -J^*$ is the complex structure on $V_\mathbb{R}$ with $E_\lambda(x,J_Ax) > 0$ for $x \neq 0$; it commutes with K and preserves E_λ, and it is the one used in Definition 2.14 and Section 2.7. With the opposite sign in (2), J_A would be J^*; nothing below depends on which.

(4) *Forms on H_1.* Deligne and Markman put the Hermitian form on $H_1(A,\mathbb{Q})$, K-linear in the first variable, with Riemann form $\operatorname{Tr}_{K/\mathbb{Q}}(f\psi_\lambda)$ for some $f \in K^\times$ with $\bar{f} = -f$ [Del82, Lemma 4.6]. Take $f = \theta$ (another choice $u\theta$, $u \in \mathbb{Q}^\times$, multiplies ψ_λ by u^{-1}). Then

$$h_\lambda(D_\lambda u, D_\lambda v) = -\overline{\psi_\lambda(u,v)},$$

because the right side is Hermitian, K-linear in $D_\lambda u$ (as D_λ is antilinear), and has trace form $\operatorname{Tr}(-\theta\,\overline{\psi_\lambda(u,v)}) = \operatorname{Tr}(\theta\psi_\lambda(u,v)) = Q_\lambda(u,v) = E_\lambda(D_\lambda u, D_\lambda v)$, so uniqueness applies. In a K-basis (e_i) of H_1 and the basis $(D_\lambda e_i)$ of V the Gram matrices are G and $-\overline{G}$: for even rank the determinants agree, and the signature is swapped. So the unordered signature and the determinant class, which are all the proof uses, are the same in both conventions.

(5) *Operations that keep the class.* Let (W,k) be a nondegenerate Hermitian space of even rank $2n$ with Gram matrix G, determinant $\delta \in \mathbb{Q}^\times$ and signature (p,q). A change of basis multiplies δ by a norm. Rescaling k by $r \in \mathbb{Q}^\times$ multiplies δ by $r^{2n} = \operatorname{Nm}(r^n)$. The dual Hermitian space on $\operatorname{Hom}_K(W,K)$ has Gram matrix $\overline{G}^{-1}$, the same signature, and determinant $\delta^{-1} = \delta \cdot \operatorname{Nm}(\delta^{-1}) \equiv \delta$. Giving W the conjugate K-structure replaces k by $\bar{k}$, with the same signature (p,q) and determinant δ (it is $-\bar{k}$, the form with the same trace form, that has signature (q,p)). Only replacing k by the skew-Hermitian θk changes the class, by $\theta^{2n} = (-3)^n \equiv (-1)^n$.

Three normalizations do not change the class. Replacing λ by $c\lambda$ with $c \in \mathbb{Q}_{>0}$, or θ by $u\theta$ with $u \in \mathbb{Q}^\times$, multiplies h_λ by a rational number r, and therefore its determinant by $r^{2n} = \operatorname{Nm}(r^n)$, a norm; replacing h by its conjugate or transpose (linearity in the second variable) does not change the rational number $\det h$. A skew-Hermitian matrix θh, however, has determinant $\theta^{2n}\det h = (-3)^n \det h \equiv (-1)^n \det h$: this convention does swap the sign, and it is the last row of the table in Section 2.1.

Exercise 2.1. (a) Show that replacing λ by $c\lambda$, $c \in \mathbb{Q}_{>0}$, replaces h_λ by ch_λ and $\det h_\lambda$ by $c^6 \det h_\lambda$ for a sixfold, and that $c^6 = \operatorname{Nm}(c^3)$. (b) Show that using $\theta' = u\theta$, $u \in \mathbb{Q}^\times$, in

Definition 2.1 replaces h_λ by $u^{-1}h_\lambda$. (c) For a Hermitian form of odd rank m, show that $c \mapsto c^m$ can change the class (take $c = 2$), so that the determinant class of an odd-rank form is not an invariant of the ray $\mathbb{Q}_{>0}\lambda$. Why is this harmless for the forms in this companion?

Example 2.3 (one abelian variety, two classes). The simplest Weil sixfold is a product of CM elliptic curves. Let $E_\omega = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\omega)$ with its principal polarization λ_0, for which $Q(1,\omega) = 1$ on the lattice basis $(1,\omega)$ of H_1 (the sign of Remark 2.2(2)), and let K act on E_ω either by multiplication, $a \mapsto a$, or through the conjugate, $a \mapsto \bar{a}$. Let $A = E_\omega^6$, with K acting by multiplication on the first three factors and through the conjugate on the last three; then K acts on $\operatorname{Lie} A = \mathbb{C}^6$ with multiplicities $(3,3)$. For positive integers $c_1, \dots, c_6$ the product polarization $\lambda_c = c_1\lambda_0 \times \cdots \times c_6\lambda_0$ is compatible, since each factor is. On one factor the formula (2.1), applied to Q on H_1, gives the form ψ of Remark 2.2(4) with $\psi(1,1) = Q(1,\theta)/6 = Q(1,2\omega+1)/6 = \frac{1}{3}$ for the standard action and $-\frac{1}{3}$ for the conjugate one, so in the K-basis of H_1 formed by the six vectors 1,

$$\psi_c = \left\langle \tfrac{c_1}{3}, \tfrac{c_2}{3}, \tfrac{c_3}{3}, -\tfrac{c_4}{3}, -\tfrac{c_5}{3}, -\tfrac{c_6}{3} \right\rangle, \qquad \det \psi_c = -\frac{c_1 \cdots c_6}{3^6} \equiv -c_1 \cdots c_6.$$

Computing with the 12×12 integral matrices of Q and of $\iota(\theta)$, the form h of Definition 2.1 on H^1, built from the dual alternating form, has signature $(3,3)$ and the same class as ψ_c: for $c = (1,1,1,1,1,1)$ the determinant is $-\frac{1}{729} \equiv -1$, split; for $c = (2,1,1,1,1,1)$ it is $-\frac{2}{729}$ on H_1 and $-\frac{1}{1458}$ on H^1, both $\equiv -2$, the class of the theorem; for $c = (2,2,1,1,1,1)$ and $(3,1,1,1,1,1)$ it is split again ($4 = \operatorname{Nm} 2$, $3 = \operatorname{Nm}\theta$), and $c = (5,1,1,1,1,1)$ gives the class -5. So the same (A,ι) carries compatible polarizations of several classes, which is why the Main Theorem asks for the *existence* of one polarization with $\det h_\lambda \equiv -2$ (Example 1.7 makes the same point with $B_1 \times B_1$). For this particular A the conclusion of the theorem is already known: every Hodge class on a power of an elliptic curve is a combination of products of divisors [Gor99, §3]. The example illustrates the statement; it is not an application of it.

Exercise 2.2. In Example 2.3: (a) show that λ_c has $\det \equiv -2$ exactly when the product $c_1 \cdots c_6$ is twice a norm, and list the vectors c with entries in $\{1,2\}$ and $c_1 \geq c_2 \geq c_3$, $c_4 \geq c_5 \geq c_6$ that give the class -2; (b) check that ψ_c is positive on the σ-part of $\operatorname{Lie} A$, i.e. on the first three factors, as the computation in the proof of Proposition 2.15(a), applied on H_1 to Q and the complex structure of $\operatorname{Lie} A$, predicts.

The hyperbolic plane is the building block of split forms, and its determinant is the first thing to compute.

Exercise 2.3. Let $P = Ke \oplus Kf$ with $h(e,e) = h(f,f) = 0$ and $h(e,f) = 1$ (the *hyperbolic plane*). (a) Show $\det h = -1$ and that $e \pm \frac{1}{2}f$ are orthogonal with h-values ± 1, so $P \cong \langle 1, -1 \rangle$. (b) Deduce that the orthogonal sum of three hyperbolic planes has signature $(3,3)$, determinant -1 and discriminant 1, and that it is split. (c) Conversely, show that a split six-dimensional space has $\det \equiv -1$, using that a basis e_1, e_2, e_3 of a three-dimensional totally isotropic subspace extends to a basis $e_1, \dots, e_6$ with $h(e_i, e_{3+i}) = 1$ and $h(e_i, e_j) = 0$ for $j \neq i \pm 3$ [Del82, proof of Cor. 4.2], and conclude without Landherr's theorem that a form with $\det h \equiv -2$ is not split.

2.3. Landherr's theorem and the determinant classes

Over $\mathbb{R}$, Hermitian forms are classified by their signature (Sylvester's law of inertia). Over K there is one more invariant, and Landherr proved in 1935 that there is no other [Lan35]. We state the case the proof uses; for CM fields the statement is [Del82, Prop. 4.1], and for any quadratic extension of number fields, with its local-global form, see [Gro21, Thm. 3.1]. Two elementary lemmas come first.

Lemma 2.4 (norms from K). *A number $c \in \mathbb{Q}^\times$ lies in $\operatorname{Nm} K^\times$ if and only if $c > 0$ and $v_p(c)$ is even for every prime $p \equiv 2 \pmod 3$.*

PROOF. $\mathbb{Z}[\omega]$ is a principal ideal domain [Mil20, p. 54], and by [Mil20, Ex. 3.44] with $m=-3$ and $\left(\frac{-3}{p}\right)=\left(\frac{p}{3}\right)$, a prime $p\equiv 2 \pmod 3$ (including $p=2$) is inert, a prime $p\equiv 1 \pmod 3$ splits, and $3=\operatorname{Nm}(\theta)$ ramifies. If $c=\operatorname{Nm}(\alpha)$, then $c>0$, and $v_p(c)=2v_{(p)}(\alpha)$ for inert p, because (p) is the only prime above p and it is stable under conjugation. Conversely, a split p equals $\operatorname{Nm}(\pi)$ for a generator π of a prime above p, $3=\operatorname{Nm}(\theta)$, and $p^2=\operatorname{Nm}(p)$; so c is a product of norms and of inverses of norms. □

LEMMA 2.5 (orthogonal bases and isotropic vectors). *Let (W,k) be a nondegenerate Hermitian space over K.*

(a) W has an orthogonal basis. In such a basis $k\cong\langle a_1,\dots,a_m\rangle$ with $a_i\in\mathbb{Q}^\times$, $\det k\equiv a_1\cdots a_m$, and the signature counts the positive and the negative a_i.

(b) For $a,b\in\mathbb{Q}^\times$ the plane $\langle a,b\rangle$ is isotropic if and only if $-ab\in\operatorname{Nm}K^\times$. So a plane of determinant δ has no isotropic vector exactly when $-\delta\notin\operatorname{Nm}K^\times$.

(c) Every indefinite Hermitian form of rank $m\geq 3$ over K is isotropic.

PROOF. (a) If $k(x,x)=0$ for all x, then $k(x,y)+\overline{k(x,y)}=0$ for all x,y (expand $k(x+y,x+y)$), and replacing x by θx gives $\theta\big(k(x,y)-\overline{k(x,y)}\big)=0$, so $k=0$. Hence some e_1 has $k(e_1,e_1)\neq 0$, $W=Ke_1\oplus e_1^\perp$, and induction applies. The diagonal entries $k(e_i,e_i)$ are rational, and the statements on the determinant and the signature are read off from the diagonal matrix [Del82, §4]. (b) An isotropic vector $(x,y)\neq 0$ has $a\operatorname{Nm}(x)+b\operatorname{Nm}(y)=0$, which forces $x,y\neq 0$ and $-b/a=\operatorname{Nm}(x/y)$, and $-ab=a^2(-b/a)$. Conversely, if $-b/a=\operatorname{Nm}(t)$ then $(t,1)$ is isotropic. (c) Write $k\cong\langle a_1,\dots,a_m\rangle$ by (a). The rational quadratic form $x\mapsto k(x,x)$ on the $2m$-dimensional $\mathbb{Q}$-vector space W is $\bigoplus_i a_i\operatorname{Nm}$, with $\operatorname{Nm}(x+y\theta)=x^2+3y^2$. It is indefinite of rank $2m\geq 6$, so it has a nonzero rational zero by Meyer's theorem [Ser73, Ch. IV], and a zero of $x\mapsto k(x,x)$ is an isotropic vector of k. □

PROPOSITION 2.6 (Landherr for sixfolds over K). *Let $K=\mathbb{Q}(\sqrt{-3})$.*

(a) Two nondegenerate Hermitian forms of rank 6 and signature $(3,3)$ over K are isometric if and only if their determinants agree in $\mathbb{Q}^\times/\operatorname{Nm}K^\times$.

(b) The determinant of such a form is the class of a negative rational number, so $\operatorname{disc}h=-\det h$ is a class in $\mathbb{Q}^\times_{>0}/\operatorname{Nm}K^\times$, and every class occurs: $\langle\delta,1,1,-1,-1,-1\rangle$ has $\operatorname{disc}=\delta$.

(c) A form of rank 6 and signature $(3,3)$ is split if and only if $\operatorname{disc}h=1$, i.e. $\det h\equiv -1$.

(d) A positive rational number is a norm from K if and only if every prime $p\equiv 2 \pmod 3$ occurs in it to an even power. Hence $\mathbb{Q}^\times_{>0}/\operatorname{Nm}K^\times$ is a vector space over $\mathbb{Z}/2$ with basis the classes of the primes $2,5,11,17,23,29,\dots$, and the class of the theorem, $\operatorname{disc}h=2$, is not split.

PROOF. (a) Over a CM field, a nondegenerate Hermitian form is determined up to isometry by its signatures at the real places and its determinant modulo norms [Del82, Prop. 4.1]; here the totally real subfield is $\mathbb{Q}$, with one real place. (b) By Lemma 2.5(a) the form is isometric to $\langle a_1,\dots,a_6\rangle$ with exactly three negative a_i, so $\det=a_1\cdots a_6<0$; every class occurs because the diagonal form $\langle\delta,1,1,-1,-1,-1\rangle$ has determinant $-\delta$. (c) A form of signature (n,n) is split if and only if its determinant is $\equiv(-1)^n$ [Del82, Cor. 4.2]; take $n=3$. (d) The first sentence is Lemma 2.4; the classes of distinct primes $p\equiv 2 \pmod 3$ are independent because a product of distinct such primes has odd valuation at each of them. Since $2\equiv 2 \pmod 3$ occurs in 2 to the first power, $2\notin\operatorname{Nm}K^\times$, and by (c) the class $\operatorname{disc}=2$ is not split. □

Deligne attributes (a) to Landherr. Part (d) can be checked by hand: if $2c^2=a^2+3b^2$ with integers not all divisible by 3, reducing modulo 3 forces $3\mid a$ and $3\mid c$, then $3\mid b$, a contradiction (another descent, modulo 4 and 8, is Exercise 1.3).

Intuition 2.7 (one bit per inert prime). A Hermitian form of signature $(3,3)$ looks the same over $\mathbb{R}$ and over every completion $\mathbb{Q}_p$ at which nothing interesting happens. What

it remembers is, for each prime $p \equiv 2 \pmod 3$ (the primes that stay prime in $\mathbb{Z}[\omega]$), one bit: the parity of the exponent of p in $\det h$. The split class has all bits zero; the class of the theorem has exactly the bit of $p = 2$ set. Abelian varieties do not see these bits locally (they all live on the same domain X^+), but the arithmetic group acting on X^+ does, and so do the constructions of cycles for Weil classes on sixfolds in Schoen's work and in Markman's preprint, which concern the split class (Section 1.3; [Sch98], [Mar25, Thm. 1.5.1], [Voi26, Thm. 1.4]).

For example, the positive integer norms up to 30 are $1, 3, 4, 7, 9, 12, 13, 16, 19, 21, 25, 27, 28$, for instance $7 = \mathrm{Nm}(2+\theta)$, $13 = \mathrm{Nm}(1+2\theta)$, $19 = \mathrm{Nm}(4+\theta)$, $21 = \mathrm{Nm}(3+2\theta)$, and the primes below 40 that are norms are $3, 7, 13, 19, 31, 37$, those that are not are $2, 5, 11, 17, 23, 29$.

Exercise 2.4. (a) Write 3, 7 and 13 as norms $x^2 + 3y^2$ of elements $x + y\theta \in K$. (b) Show by descent that 5 is not a norm (reduce $5c^2 = a^2 + 3b^2$ modulo 5, using that -3 is not a square modulo 5). (c) Which classes in $\mathbb{Q}^\times_{>0}/\,\mathrm{Nm}\,K^\times$ do 6, 24, 108 and 128 represent? (d) Show that a prime $p \equiv 1 \pmod 3$ is a norm, using Lemma 2.4, and find the representation for $p = 31$.

Example 2.8 (a model of the class 2). The form $h_2 = \langle 2, 1, 1, -1, -1, -1\rangle$ on K^6 has signature $(3, 3)$ and $\det h_2 = -2$, so $\mathrm{disc}\, h_2 = 2$, and by Proposition 2.6(a) every Hermitian form of the theorem is isometric to it, including the form of the G_{32} family (Example 2.13) and $\psi_{(2,1,1,1,1,1)}$ of Example 2.3. It is not split, but it is far from anisotropic: the vectors $w_1 = e_2 + e_4$ and $w_2 = e_3 + e_5$ satisfy $h_2(w_i, w_j) = 0$ for all i, j, so they span a two-dimensional totally isotropic subspace. Its orthogonal complement modulo itself is the plane $\langle 2, -1\rangle$ on e_1, e_6, and by Lemma 2.5(b) a binary form $\langle a, b\rangle$ is isotropic exactly when $-ab$ is a norm; here $-ab = 2$, so $\langle 2, -1\rangle$ has no isotropic vector. In the language of Witt decomposition, $h_2 \cong H \perp H \perp \langle 2, -1\rangle$ with H the hyperbolic plane: two hyperbolic planes and an anisotropic plane, Witt index 2, one short of split. Every indefinite Hermitian form of rank at least 3 over K is isotropic (Lemma 2.5(c)), so the obstruction to being split always sits in a last plane of this kind.

Exercise 2.5. For h_2 of Example 2.8: (a) verify $h_2(w_i, w_j) = 0$; (b) show that h_2 has no three-dimensional totally isotropic subspace, using Exercise 2.3(c); (c) find a vector $x \in K^6$ with $h_2(x, x) = 0$ that is not in $\mathrm{span}_K(w_1, w_2)$, and check that x, w_1, w_2 do not span a totally isotropic subspace.

2.4. The class 2 in small families: cyclic covers

The class 2 already occurs for cyclic covers of the line, and one of them is the smallest worked example of everything in this chapter. For a μ_6-cover $C\colon y^6 = \prod_i (x - t_i)^{a_i}$ with integers $a_i \not\equiv 0 \pmod 6$, let ζ act by the automorphism $(x, y) \mapsto (x, \zeta y)$ and take eigenspaces for its pull-back action. The part $H^1(C, \mathbb{Q})_{\Phi_6}$ on which it acts with eigenvalues $\zeta^{\pm 1}$ is a K-vector space whose dimension is the number of branch points minus 2, the polarization of the Jacobian makes it a Hermitian space, and it is H^1 of an abelian subvariety of the Jacobian, up to isogeny [Moo10, pp. 4.2, 4.5]. It is of Weil type when the Hodge numbers d_1, d_5 of the two eigenspaces agree; by the Chevalley–Weil formula, $d_k = -1 + \sum_i \langle -k a_i/6\rangle$ with $\langle\cdot\rangle$ the fractional part [Moo10, Lemma 2.7, (3.2.2), 4.5].

Example 2.9 (a Weil surface of class 2). Take $a = (3, 1, 4, 4)$. Riemann–Hurwitz gives $2g - 2 = 6 \cdot (-2) + \sum_i (6 - \gcd(a_i, 6)) = -12 + 3 + 5 + 4 + 4 = 4$, so $g = 3$; the eigenspaces of ζ^k have dimensions $2, 1, 0, 1, 2$ for $k = 1, \dots, 5$, and $d_1 = -1 + \frac{1}{6}(3 + 5 + 2 + 2) = 1 = d_5$. So $H^1(C, \mathbb{Q})_{\Phi_6}$ is a two-dimensional K-space with Hodge numbers $(1, 1)$: a Weil abelian surface. Its Hermitian form can be computed from twisted cycles. Let γ_1, γ_2 be the twisted cycles joining consecutive branch points, δ_1, δ_2 the corresponding cycles for the dual local system, and $c_j = \zeta^{a_j}$ the local monodromies. The intersection formula of Kita and Yoshida [Mat22,

Thm. 3] gives

$$\langle\gamma_j,\delta_j\rangle=\frac{1-c_jc_{j+1}}{(1-c_j)(1-c_{j+1})},\qquad \langle\gamma_j,\delta_{j+1}\rangle=\frac{-1}{1-c_{j+1}},\qquad \langle\gamma_j,\delta_{j-1}\rangle=\frac{-c_j}{1-c_j},$$

and reading δ_k as the complex conjugate of γ_k makes $H=(\langle\gamma_j,\delta_k\rangle)$ skew-Hermitian. Here $\langle\gamma_1,\delta_1\rangle=\theta/2$, $\langle\gamma_2,\delta_2\rangle=\theta/3$ and $\langle\gamma_1,\delta_2\rangle=-(1+\theta)/2$, so

$$\psi=\theta H=\begin{pmatrix}-3/2 & 3/2-\theta/2\\ 3/2+\theta/2 & -1\end{pmatrix},\qquad \det\psi=-\tfrac{3}{2}.$$

We use without proof the standard identification of this intersection form with the cup product on the eigenspace, up to a rational factor. Since $\frac{3}{2}=2\cdot\frac{3}{4}$ and $\frac{3}{4}=\mathrm{Nm}(\theta/2)$, the determinant is -2 times a norm. For a surface, $n=1$, so $\mathrm{disc}=(-1)^1\det\equiv 2$, not split. In rank two this has a stronger meaning: the plane has *no* isotropic vector at all ($-\det=\frac{3}{2}$ is not a norm, Lemma 2.5(b)), so by Remark 2.16 the arithmetic quotients of its one-dimensional domain are compact Shimura curves. The Weil classes of a surface lie in H^2 and are divisor classes by the Lefschetz $(1,1)$ theorem; the surface is only the smallest place where the class 2 can be seen.

A cyclic family of class-2 sixfolds would not be enough unless its base had dimension at least $9=\dim \mathrm{Sh}_3^{(2)}$ (Proposition 2.15), because the theorem is about every point of $\mathrm{Sh}_3^{(2)}$. The G_{32} family of the following chapters has a dominant period map, which is what steps (i)(b) and (vii) use (Chapters 6 and 12).

Exercise 2.6. (a) Recompute $g=3$, the eigenspace dimensions and $d_1=d_5=1$ for $a=(3,1,4,4)$. (b) Show that $\det\psi=-\frac{3}{2}$ lies in the class -2, and that this class is not -1. (c) Verify directly that the matrix $\psi=\left(\begin{smallmatrix}-3/2 & 3/2-\theta/2\\ 3/2+\theta/2 & -1\end{smallmatrix}\right)$ of Example 2.9 has no nonzero isotropic vector in K^2.

2.5. Invariant forms of a family and the determinant of the G_{32} family

For the G_{32} family no cell model of the curve is feasible: the curves have genus 505441 (Chapter 5). What can be computed is the monodromy of the family on $\mathcal{B}_C=\mathrm{Hom}_{\mathbb{Q}[G]}(V_\rho,H^1(C,\mathbb{Q}))$, through explicit braid matrices on parabolic cohomology, a six-dimensional K-vector space (Chapter 5). The polarization of $\mathcal{B}_C$ is invariant under this monodromy, because monodromy preserves the cup product on $H^1(C,\mathbb{Q})$ and commutes with G. The question is how to get its determinant class from forms one can compute, and the answer is that one does not need the polarization itself.

Lemma 2.10 (determinant classes of invariant forms). *Let V be a K-vector space of dimension m and Γ a group acting K-linearly on V. Let S_Γ be the K-vector space of Γ-invariant sesquilinear forms on V (K-linear in the first variable, conjugate-linear in the second).*

(a) If $\dim_K S_\Gamma=1$ and S_Γ contains a nonzero Hermitian form h_0, then the Γ-invariant Hermitian forms are exactly the rational multiples rh_0, $r\in\mathbb{Q}$.

(b) If moreover m is even, all nonzero Γ-invariant Hermitian forms have the same determinant class.

(c) If $V\otimes_{K,\sigma}\mathbb{C}$ is an irreducible representation of Γ, then $\dim_K S_\Gamma\le 1$.

(d) If $\Gamma_0\subseteq\Gamma$ is a subgroup (for instance the group generated by finitely many monodromy matrices), then $S_\Gamma\subseteq S_{\Gamma_0}$; so $\dim_K S_{\Gamma_0}=1$ together with one nonzero Γ-invariant Hermitian form gives $\dim_K S_\Gamma=1$.

Proof. (a) If h is invariant and Hermitian, then $h=ah_0$ for some $a\in K$, and $h(y,x)=\overline{h(x,y)}$ gives $a\,\overline{h_0(x,y)}=\bar a\,\overline{h_0(x,y)}$ for all x,y; as $h_0\neq 0$, $a=\bar a\in\mathbb{Q}$. (b) $\det(rh_0)=r^m\det h_0$, and $r^m=\mathrm{Nm}(r^{m/2})$ for even m. (c) A sesquilinear form s is the same as the K-linear map $V\to\overline{V}^\vee$, $x\mapsto s(x,\cdot)$, into the space of conjugate-linear functionals, and s is invariant exactly when this map commutes with Γ. After $\otimes_{K,\sigma}\mathbb{C}$ both sides are irreducible of the same

dimension (the second is the conjugate dual of the first), and by Schur's lemma (Lemma 3.1; the proof there works for any group acting on a finite-dimensional complex space) the space of Γ-maps between two irreducible representations is at most one-dimensional. Since $\mathrm{Hom}_\Gamma(V,W)\otimes_K\mathbb{C}=\mathrm{Hom}_\Gamma(V\otimes\mathbb{C},W\otimes\mathbb{C})$ for K-linear representations, $\dim_K S_\Gamma\le 1$. (d) Invariance under Γ implies invariance under Γ_0; the Hermitian form shows $\dim_K S_\Gamma\ge 1$. □

Part (d) is the logical shape of the computation: one never needs the whole monodromy group, only enough elements of it to cut the invariant forms down to a line, plus the knowledge that the polarization is invariant under all of it.

Remark 2.11 (conventions do not matter here either). The braid matrices act on parabolic cohomology of the local system of ρ, which is identified with $\mathcal{B}_C$, or with its dual, or with its conjugate, depending on the convention chosen for the monodromy action. Each of these identifications is a monodromy-equivariant isomorphism that is K-linear or K-conjugate-linear, possibly composed with dualization. By Remark 2.2(5), a conjugate-linear isomorphism transports a Hermitian form to one whose Gram matrix is the complex conjugate, with the same rational determinant, and dualization inverts the determinant, which keeps the class. So the class computed on either side is the class of the other.

Remark 2.12 (an invariant polarization of $\mathcal{B}_C$). For $u\in V_\rho\setminus\{0\}$, $E_u(f,f')=E_C(f(u),f'(u))$, with E_C the cup-product polarization of $H^1(C,\mathbb{Q})$, is a polarization of $\mathcal{B}_C$ that is invariant under monodromy and compatible with K (Proposition 5.18). With Lemma 2.10 this is all that step (i)(a) needs: one invariant compatible polarization exists, and its Hermitian form is a nonzero invariant Hermitian form.

Example 2.13 (the determinant of the G_{32} family). For the tuple $\boldsymbol{\gamma}$ of Section 4.5 and its Hurwitz component $\mathcal{H}$ (Chapter 5), an exact computation in GAP, from 67 explicit braid monodromy matrices acting on the six-dimensional parabolic cohomology over K and with no cup-product formula, shows that the space of invariant sesquilinear forms is one-dimensional over K, spanned by a Hermitian form of signature $(3,3)$ and determinant -128, which is -2 times a norm:

$$-128=-2\cdot 8^2=-2\cdot\mathrm{Nm}(8).$$

By Lemma 2.10(d), (a), (b) and Remark 2.11, the Hermitian form h of the polarization of A_C is a rational multiple of the computed form, and since $m=6$ is even, $\det h\equiv -2$, i.e. $\mathrm{disc}\,h=2$. The Hermitian space $(\mathcal{B}_C,h)$ is locally constant over the connected base $\mathcal{H}$, so this holds for every member, not only for the base point of the computation. With Proposition 2.6: every member of $\mathcal{H}$ is a non-split Weil sixfold of class 2, and its Hermitian space is isometric to $h_2=\langle 2,1,1,-1,-1,-1\rangle$. The determinant is negative, as it must be for a Hermitian form of signature $(3,3)$ (Proposition 2.6(b)); the skew-Hermitian matrix θh would have positive determinant.

Exercise 2.7. (a) Show that $-128\equiv -2$ modulo $\mathrm{Nm}\,K^\times$, by factoring it and applying Proposition 2.6(d), and that $-128\cdot r^6$ lies in the same class for every $r\in\mathbb{Q}^\times$. (b) Suppose the space of invariant sesquilinear forms had come out two-dimensional. Give an example (any K-space with a group action, for instance $V=K^2$ with Γ trivial) where two invariant Hermitian forms of the same rank 2 have different determinant classes, and explain why the computation of Example 2.13 would then not have determined $\det h$.

2.6. The period domain X^+ and the Shimura variety of class 2

Fix the Hermitian space of the family. In step (vii) one fixes a base point $t_0\in\mathcal{H}$ and an isometry $(\mathcal{B}_{C_{t_0}},h)\cong(V,h)$; by Example 2.13 we may think of (V,h) as K^6 with the form h_2. All Hodge structures of the members of the family, and of every sixfold of the theorem, are then points of one space.

Definition 2.14 (the domain X^+). Let (V,h) be a nondegenerate Hermitian space over K of dimension 6 and signature $(3,3)$, and $E=\mathrm{Tr}_{K/\mathbb{Q}}(\theta h)$. Let $V_\mathbb{R}=V\otimes_\mathbb{Q}\mathbb{R}$, a complex

vector space of dimension 6 through $K \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{C}$, $\theta \mapsto \mathrm{i}\sqrt{3}$. Then X^+ is the set of $\mathbb{R}$-linear maps $J\colon V_{\mathbb{R}} \to V_{\mathbb{R}}$ with $J^2 = -1$ that commute with the action of K, preserve E, and satisfy $E(x, Jx) > 0$ for all $x \neq 0$.

Each $J \in X^+$ is a complex structure on the torus $V_{\mathbb{R}}/V_{\mathbb{Z}}$ (for any lattice $V_{\mathbb{Z}}$) for which E is a polarization and K acts by endomorphisms: a polarized abelian sixfold with K-action whose H^1 is V with the Hodge structure given by J. For a sixfold (A, ι, λ) the complex structure J_A of Remark 2.2(3) satisfies the three conditions. The definition says nothing about the multiplicities $(3,3)$ or about Hodge numbers; part (a) below shows that they follow from the signature of h.

Proposition 2.15 (the domain X^+). *In the situation of Definition 2.14:*

(a) For $J \in X^+$ let $V_\pm = \{x \in V_{\mathbb{R}} : Jx = \pm \mathrm{i}x\}$, where i is the complex structure from $K \otimes \mathbb{R} = \mathbb{C}$. Then $V_{\mathbb{R}} = V_+ \oplus V_-$, the two summands are h-orthogonal, h is positive definite on V_+ and negative definite on V_-, and $\dim_{\mathbb{C}} V_+ = \dim_{\mathbb{C}} V_- = 3$.

(b) $J \mapsto V_+$ is a bijection from X^+ onto the set of three-dimensional complex subspaces of $V_{\mathbb{R}}$ on which h is positive definite.

(c) In a basis of $V_{\mathbb{R}}$ in which $h = \langle 1,1,1,-1,-1,-1\rangle$, these subspaces are the graphs $\{(u, Zu) : u \in \mathbb{C}^3\}$ of the matrices $Z \in \mathrm{Mat}_3(\mathbb{C})$ with $1 - Z^*Z$ positive definite. So X^+ is a bounded convex open subset of $\mathrm{Mat}_3(\mathbb{C}) = \mathbb{C}^9$: a connected, contractible complex manifold of dimension 9.

(d) The group $\mathrm{U}(V,h)(\mathbb{R}) \cong \mathrm{U}(3,3)$ acts transitively on X^+ by $J \mapsto gJg^{-1}$, with stabilizer $\mathrm{U}(V_+) \times \mathrm{U}(V_-) \cong \mathrm{U}(3) \times \mathrm{U}(3)$, so $X^+ \cong \mathrm{U}(3,3)/\mathrm{U}(3) \times \mathrm{U}(3)$, and $\dim_{\mathbb{R}} \mathrm{U}(3,3) - \dim_{\mathbb{R}}(\mathrm{U}(3) \times \mathrm{U}(3)) = 36 - 18 = 18 = 2 \cdot 9$.

Proof. On $V_{\mathbb{R}}$, $\mathrm{Tr}_{K/\mathbb{Q}}$ becomes $2\,\mathrm{Re}$, so $E(x,y) = 2\,\mathrm{Re}(\mathrm{i}\sqrt{3}\,h(x,y)) = -2\sqrt{3}\,\mathrm{Im}\,h(x,y)$. (a) J is $\mathbb{C}$-linear and $J^2 = -1$, so $V_{\mathbb{R}} = V_+ \oplus V_-$. For $x \in V_+$, $h(x,Jx) = h(x,\mathrm{i}x) = -\mathrm{i}\,h(x,x)$ (conjugate-linear in the second variable), so $E(x,Jx) = 2\sqrt{3}\,h(x,x)$, and positivity gives $h(x,x) > 0$; for $x \in V_-$ the same computation gives $E(x,Jx) = -2\sqrt{3}\,h(x,x)$, so $h(x,x) < 0$. Since J preserves E and commutes with K, it preserves h (by the uniqueness in Definition 2.1); for $x \in V_+$, $y \in V_-$ this gives $h(x,y) = h(Jx,Jy) = h(\mathrm{i}x, -\mathrm{i}y) = \mathrm{i}\cdot\overline{(-\mathrm{i})}\,h(x,y) = -h(x,y)$, so $h(x,y) = 0$. A positive definite and a negative definite subspace that together span a space of signature $(3,3)$ have dimensions 3 and 3. (b) Given a positive three-dimensional V_+, put $V_- = V_+^{\perp_h}$ (negative definite, since h has signature $(3,3)$) and $J = \mathrm{i}$ on V_+, $-\mathrm{i}$ on V_-. Then $J^2 = -1$, J commutes with K and preserves h, hence E, and for $x = x_+ + x_-$ the computation of (a) gives $E(x,Jx) = 2\sqrt{3}(h(x_+,x_+) - h(x_-,x_-)) > 0$. The two constructions are inverse. (c) Let V_+^0, V_-^0 be the spans of the first and last three basis vectors. A positive V_+ meets the negative V_-^0 only in 0, so it projects isomorphically onto V_+^0 and is the graph of a linear map $Z\colon V_+^0 \to V_-^0$; then $h((u,Zu),(u,Zu)) = |u|^2 - |Zu|^2$, which is positive for all $u \neq 0$ exactly when the operator norm of Z is less than 1. The set of such Z is the open unit ball of the operator norm, convex and bounded. (d) Transitivity: given positive V_+, V_+', orthonormal bases of $V_\pm$ and of $V_\pm'$ are both bases in which $h = \langle 1,1,1,-1,-1,-1\rangle$, and the linear map between them is in $\mathrm{U}(V,h)(\mathbb{R})$ and carries V_+ to V_+'. The stabilizer preserves V_+ and $V_- = V_+^\perp$ and is unitary on each. The Lie algebra of $\mathrm{U}(p,q)$ has real dimension $(p+q)^2$, so the dimensions are 36 and $9+9$. □

Deligne describes the same domain as the set of maximal positive subspaces [Del82, proof of Thm. 4.8]; for Weil $2n$-folds it is the n^2-dimensional bounded symmetric domain of $\mathrm{SU}(n,n)$ [Gor99, §4]. The domain depends only on $(V_{\mathbb{R}}, h)$, so it is the same for every determinant class; only the arithmetic group acting on it sees the class.

Exercise 2.8. Let (V,h) be a K-Hermitian space of dimension $2n$, now of unknown signature (p,q), and suppose some J as in Definition 2.14 exists with K acting on $(V_{\mathbb{R}}, J)$ with multiplicities (n,n). Show that $(p,q) = (n,n)$. Then show conversely that for signature

(p,q) every J in the analogue of X^+ gives multiplicities (p,q), so that the space is empty unless the multiplicities are prescribed as (p,q).

Exercise 2.9. (a) Show that $\mathfrak{u}(p,q)=\{X\in \mathrm{Mat}_{p+q}(\mathbb{C}) : X^*I_{p,q}+I_{p,q}X=0\}$ has real dimension $(p+q)^2$, and deduce $\dim_{\mathbb{C}} \mathrm{U}(n,n)/\mathrm{U}(n)\times\mathrm{U}(n)=n^2$. (b) Check that the chart of Proposition 2.15(c) gives the same number for $n=3$. (c) For $n=1$ and $h=\langle 1,-\delta\rangle$ with $\delta\in\mathbb{Q}_{>0}$, show that the positive complex lines in $V_{\mathbb{R}}=\mathbb{C}^2$ are spanned by the vectors e_1+we_2 with $|w|<\delta^{-1/2}$, and identify this disc with $\{|z|<1\}$ for every δ.

Remark 2.16 (isotropic vectors and compactness). If $0\neq e\in V$ with $h(e,e)=0$ and $0\neq a\in K$ with $\bar a=-a$, then $u(x)=x+a\,h(x,e)\,e$ is K-linear, preserves h (the cross terms add up to $(a+\bar a)h(x,e)h(e,y)=0$), and satisfies $(u-1)^2=0$: a unipotent $u\neq 1$ in $\mathrm{SU}(V,h)(\mathbb{Q})$. Conversely, if $u\neq 1$ in $\mathrm{U}(V,h)(\mathbb{Q})$ is unipotent, pick w with $v=(u-1)w\neq 0$ and $(u-1)v=0$; then $h(v,v)=h(uw,uv)-h(w,v)=0$. So $\mathrm{SU}(V,h)$ is anisotropic over $\mathbb{Q}$ exactly when h has no isotropic vector, and then the quotient of the domain by an arithmetic subgroup is compact; otherwise it is not [Mil05, Thm. 3.3(b)].

The *Shimura variety of class* 2 is a quotient of X^+ by an arithmetic group. In step (vii) one takes a lattice $V_{\mathbb{Z}}\subset V$ coming from integral cohomology and the torsion-free group $\Gamma'=\mathrm{U}(V_{\mathbb{Z}},h)\cap\Gamma_{\mathrm{Sp}}(N)$, $N\geq 3$, and works on $S'=\Gamma'\backslash X^+$; in general one writes $\mathrm{Sh}_3^{(2)}$ for a connected component of the Shimura variety of the group of unitary similitudes $\mathrm{GU}(V,h)$ [Mil05, App. B, case (A)]. These are algebraic varieties of dimension 9; they are not compact, since a form of rank $6\geq 3$ is isotropic (Lemma 2.5(c), Remark 2.16), so the period map of a family of Weil sixfolds can run off to infinity, which is what Chapter 6 exploits at a boundary point. Every Weil sixfold carries the two-dimensional space of Weil classes $\mathrm{HW}(A)=\bigwedge^6_K H^1(A,\mathbb{Q})\subset H^6(A,\mathbb{Q})$, a $\mathbb{Q}$-subspace of dimension $2\binom{6}{6}=2$ inside a space of dimension $\binom{12}{6}=924$ (Definition 1.1). Over X^+ these classes form a local system of rank 2, and the content of step (vii) is that algebraicity of the classes on an open subset of a quotient of X^+ spreads to all of it (Chapter 12).

2.7. How Landherr enters step (viii)

Step (vii) proves the Weil classes algebraic on the fibres of one universal family over $S'=\Gamma'\backslash X^+$. Step (viii) must reach an arbitrary (A,ι) satisfying the hypotheses of the Main Theorem, and it starts with Landherr. Let λ be a polarization of A with the properties (a)–(c) of the theorem. By (a) and Definition 2.1, h_λ is a Hermitian form on the six-dimensional K-space $H^1(A,\mathbb{Q})$. By (b), its signature is $(3,3)$: the complex structure J_A of Remark 2.2(3) commutes with K, preserves E_λ and is positive for it, so the proof of Proposition 2.15(a), which uses the signature only in its last sentence, shows that h_λ is positive definite on $V_+=\{J_Ax=\mathrm{i}x\}$ and negative definite on V_-; and $\dim_{\mathbb{C}}V_\pm$ are the multiplicities of K on $(H^1(A,\mathbb{R}),J_A)$, which are $(3,3)$ by (b) (Exercise 2.8). By (c), $\det h_\lambda\equiv -2$. The form h on V has the same three invariants (Example 2.13). By Proposition 2.6(a) there is a K-linear isometry

$$f\colon (H^1(A,\mathbb{Q}),h_\lambda)\xrightarrow{\sim}(V,h),$$

which therefore carries $E_\lambda=\mathrm{Tr}(\theta h_\lambda)$ to $E=\mathrm{Tr}(\theta h)$. Since J_A commutes with K, preserves E_λ and is positive for it, $x_A=fJ_Af^{-1}$ satisfies the three conditions of Definition 2.14, i.e. $x_A\in X^+$. Nothing about the sign of the multiplier of a similitude enters, because f is an isometry of h itself and positivity is transported. The remaining steps, that the fibre of the universal family over the image of x_A in S' is K-isogenous to A and that algebraicity of Weil classes is invariant under K-isogeny, are Chapter 12.

Two remarks on what this step uses and what it does not. It uses Landherr only as a statement about *rational* Hermitian spaces: the isometry f need not respect any lattice, and neither the type of λ nor the order of K acting integrally on A plays a role, since a change of lattice is absorbed by an isogeny. And it uses the hypothesis (c) for one polarization only: if λ is replaced by $c\lambda$, $\det h_\lambda$ changes by c^6, a norm (Exercise 2.1), while a polarization of a

different shape can change the class, as Example 2.3 shows; for the step it suffices that one λ with $\det h_\lambda \equiv -2$ exists.

What just happened. A compatible polarization is the trace of a Hermitian form h over K; over $\mathbb{R}$ it is the form of signature $(3,3)$ whose positive three-planes make up the nine-dimensional bounded domain X^+, and over $\mathbb{Q}$ it is classified by $\det h$ modulo norms (Landherr). The theorem's class $\det h \equiv -2$, $\operatorname{disc} h = 2$, is not split because 2 is inert in $\mathbb{Q}(\sqrt{-3})$. The G_{32} family has this class on every member because its space of monodromy-invariant forms is a line, so the invariant form computed from braid matrices, of determinant -128, has the class of the polarization. And Landherr's theorem, applied to rational Hermitian spaces, is what puts every sixfold of the theorem on the same domain X^+ as the family.

Remark 2.17 (Where this is used). Example 2.13 with Lemma 2.10 is step (i)(a), the non-split class of the G_{32} family, used in Chapter 5 and in the statement of Chapter 1. Definition 2.14 and Proposition 2.15 are the target of the period map whose rank nine is certified in Chapter 6 (step (i)(b)) and the space on which step (vii) runs in Chapter 12. Proposition 2.6 and Section 2.7 are the first half of step (viii) in Chapter 12. The conventions of Section 2.1 and Remark 2.2 are the ones used everywhere in this companion.

Part 2

The group, the covers and the family

CHAPTER 3

Representations of finite groups over $\mathbb{Q}$, K and $\mathbb{C}$

Proof steps: supports (ii) Frobenius, (vi) Wedderburn and density, V_ρ

Where we are. This chapter contains no step of the proof. It supplies the finite-group representation theory that two steps use: step (ii), which turns isotypic pieces of $H^1(C)$ into integer vectors by Frobenius reciprocity and finds an integral relation among them, and step (vi), which builds a correspondence with complex coefficients from Wedderburn's decomposition of $\mathbb{C}[G]$ and makes it algebraic by a density argument. It also explains why the rational Hodge structure $\mathcal{B}_C = \mathrm{Hom}_{\mathbb{Q}[G]}(V_\rho, H^1(C,\mathbb{Q}))$ is a vector space over $K = \mathbb{Q}(\sqrt{-3})$ of the right dimension. Everything here is textbook material except Proposition 3.18, a short consequence of Brauer's theorem proved here. Needs: linear algebra and the definition of a group representation, and norms and traces in number fields [Mil20, Ch. 2]; no finite-group theory is assumed. Throughout, $\omega = e^{2\pi\mathrm{i}/3}$, so that $K = \mathbb{Q}(\omega)$; we also write $\zeta = e^{2\pi\mathrm{i}/6}$, so that $\omega = \zeta^2$.

3.1. What the proof needs from representation theory

A finite group G acts on a curve C, hence linearly on the $2g$-dimensional space $H^1(C,\mathbb{C})$. As for any finite-dimensional representation, this space splits into isotypic parts, $H^1(C,\mathbb{C}) = \bigoplus_{\rho'} \rho' \otimes M_{\rho'}$, a sum over the irreducible representations ρ' of G, where $M_{\rho'} = \mathrm{Hom}_G(\rho', H^1(C,\mathbb{C}))$ is the *multiplicity space*: G acts on the first factor only, and the Hodge structure lives on the second (Proposition 3.5). The Weil classes of the theorem live on M_ρ for one particular four-dimensional ρ, and the whole difficulty is that M_ρ is not visibly the cohomology of anything simpler. The proof handles it with four facts from this chapter.

Frobenius reciprocity (Theorem 3.10). For a subgroup $H \le G$ and a character $\psi\colon H \to \mathbb{C}^\times$, the subspace of $H^1(C,\mathbb{C})$ on which H acts through ψ is $\bigoplus_{\rho'} \rho'^{(H,\psi)} \otimes M_{\rho'}$, and $\dim \rho'^{(H,\psi)} = \langle \mathrm{Res}_H \rho', \psi\rangle = \langle \rho', \mathrm{Ind}_H^G \psi\rangle$. So the piece $W_{H,\psi}$ of step (ii) is recorded by an integer vector $v_{H,\psi} = (\dim \rho'^{(H,\psi)})_{\rho'}$, which is nothing but the list of multiplicities of the irreducible constituents of the induced representation $\mathrm{Ind}_H^G \psi$. The geometric point (Chapter 7) is that $W_{H,\psi}$ is an eigenspace of a cyclic cover of curves, about which much is known.

Brauer's induction theorem (Theorem 3.15). Every irreducible character is an integral combination of characters $\mathrm{Ind}_H^G \psi$ with ψ one-dimensional. In vector form, the indicator vector e_ρ is an integral combination $\sum_j c_j v_{H_j,\psi_j}$. For G_{32} such a relation with eleven pieces and coefficients ± 1 exists (Chapter 7); Proposition 3.18 explains why one must exist inside the family the proof uses.

Wedderburn's theorem and matrix units (Theorem 3.20 and Definition 3.21). The group algebra $\mathbb{C}[G]$ is the product of the full matrix algebras $\mathrm{End}(\rho')$, one for each irreducible ρ'. So any collection of linear maps, one on each irreducible, is realized by a single element of $\mathbb{C}[G]$, and explicit formulas (matrix units) say which one. Step (vi) uses this to build an element Ψ of $\mathrm{Mat}_{5\times 7}(\mathbb{C}[G])$ that moves the eleven pieces onto one copy of M_ρ.

Density (Proposition 3.26). Elements of $\mathbb{Q}[G]$, after clearing denominators, act on $J(C)$ by homomorphisms of abelian varieties, which preserve algebraic classes; elements of $\mathbb{C}[G]$ do not obviously act geometrically at all. Since $\mathbb{Q}[G]$ is Zariski dense in $\mathbb{C}[G]$ and the operation that

matters ($\Psi \mapsto \bigwedge^D \Psi$) is polynomial, complex combinations of algebraic classes are preserved as well.

Finally, *rationality* (Proposition 3.27): ρ is defined over K and has non-real character, so viewed as a $\mathbb{Q}$-vector space it is an irreducible $\mathbb{Q}[G]$-module V_ρ with $\mathrm{End}_{\mathbb{Q}[G]}(V_\rho) = K$. That is why $\mathcal{B}_C$ is a K-vector space, with $\dim_K \mathcal{B}_C = \dim M_\rho$.

The smallest group on which all of this is visible is $\mathrm{SL}(2,3)$, realized as the complex reflection group $G_4 \subset \mathrm{GL}_2(\mathbb{Z}[\omega])$ (Example 3.14), the little brother of G_{32} (Chapter 4). Its two-dimensional reflection representation ρ is not induced from any one-dimensional character, because $\mathrm{SL}(2,3)$ has no subgroup of index 2, but

$$\rho = \mathrm{Ind}_{C_4}^G \psi_4 - \mathrm{Ind}_{C_6}^G \psi_6 \ , \qquad \text{i.e.} \qquad e_\rho = v_{C_4,\psi_4} - v_{C_6,\psi_6} \ ,$$

for suitable characters ψ_4, ψ_6 of cyclic subgroups of orders 4 and 6 (Example 3.19). The minus sign is what forces the contraction of step (vi): a negative coefficient cannot be realized by a map *from* a piece, only by pairing a piece away. The G_{32} relation has the same shape, with seven plus signs and four minus signs.

Sources. All statements are standard [Ser77; Eti+11] except Proposition 3.18, which is proved here.

3.2. Characters, isotypic parts and central characters

A *representation* of a finite group G is a homomorphism $D\colon G \to \mathrm{GL}(V)$ for a finite-dimensional complex vector space V; we often write gv for $D(g)v$ and call V a G-module. Maps commuting with the action are *G-maps*, and $\mathrm{Hom}_G(V,W)$ is the space of them. Averaging any Hermitian inner product over G gives an invariant one, so every representation is unitary in a suitable basis and every invariant subspace has an invariant complement: every representation is a direct sum of irreducible ones (Maschke's theorem, [Eti+11, §3.1]). This is the finite-group version of the statement, familiar from compact Lie groups, that unitary representations are completely reducible. The basic tool for decomposing is Schur's lemma.

Lemma 3.1 (Schur). *If ρ, ρ' are irreducible, every G-map $\rho \to \rho'$ is either 0 or an isomorphism, and* $\mathrm{End}_G(\rho) = \mathbb{C} \cdot \mathrm{id}$.

Proof. Kernel and image of a G-map are invariant subspaces, which gives the first part. For the second, a G-endomorphism f of ρ has an eigenvalue $\lambda \in \mathbb{C}$, and $f - \lambda$ is a G-map with nonzero kernel, hence zero. □

Decomposing a representation is a question about traces.

Definition 3.2. The *character* of $D\colon G \to \mathrm{GL}(V)$ is $\chi_V(g) = \mathrm{Tr}\, D(g)$. It is constant on conjugacy classes. For class functions ϕ, ϕ' put $\langle \phi, \phi' \rangle = \frac{1}{|G|} \sum_{g \in G} \phi(g)\overline{\phi'(g)}$.

Since $D(g)$ has finite order, its eigenvalues are roots of unity, so $\chi_V(g^{-1}) = \overline{\chi_V(g)}$, and the character values lie in a cyclotomic field. The *character field* $\mathbb{Q}(\chi)$ is the field generated by the values.

Theorem 3.3 (orthogonality). *The characters of the irreducible representations form an orthonormal basis of the space of class functions. Consequently the number of irreducible representations equals the number of conjugacy classes, a representation V contains ρ' exactly $\langle \chi_V, \chi_{\rho'} \rangle$ times, V is irreducible if and only if $\langle \chi_V, \chi_V \rangle = 1$, and $\sum_{\rho'} (\dim \rho')^2 = |G|$.*

This is [Ser77, Ch. 2] and [Eti+11, §§3.5, 3.8]; the last identity comes from decomposing the regular representation $\mathbb{C}[G]$, whose character is $|G|$ at 1 and 0 elsewhere, so that it contains each ρ' exactly $\dim \rho'$ times.

EXAMPLE 3.4 (S_3). The symmetric group S_3 has three conjugacy classes: the identity, the three transpositions and the two 3-cycles. Its three irreducible characters are

	1	(12)	(123)
class size	1	3	2
$\mathbf{1}$	1	1	1
sgn	1	-1	1
ρ_2	2	0	-1

where ρ_2 is the two-dimensional representation on $\{x \in \mathbb{C}^3 : x_1+x_2+x_3 = 0\}$ by permutation of coordinates. Orthogonality can be read off: $\langle \rho_2, \rho_2\rangle = \frac{1}{6}(4+0+2\cdot 1) = 1$ and $\langle \rho_2, \mathrm{sgn}\rangle = \frac{1}{6}(2+0-2) = 0$; and $1+1+4 = 6$. The permutation representation $\mathbb{C}^3$ has character $(3,1,0)$, which is $\mathbf{1}+\rho_2$.

The decomposition into irreducibles is not unique as a decomposition into subspaces when an irreducible occurs more than once, but the grouping by type is canonical, and it comes with a natural tensor structure.

PROPOSITION 3.5 (isotypic decomposition). *For every representation V of G, the evaluation map*

$$\bigoplus_{\rho'} \rho' \otimes \mathrm{Hom}_G(\rho', V) \longrightarrow V\ , \qquad u \otimes f \longmapsto f(u)\ ,$$

a sum over representatives ρ' of the isomorphism classes of irreducibles, is an isomorphism of G-modules, with G acting on the first factors. If V carries a further linear structure commuting with G (for $V = H^1(C,\mathbb{C})$: the Hodge decomposition), it acts on the second factors $M_{\rho'} = \mathrm{Hom}_G(\rho', V)$.

PROOF. Both sides are additive in V, so by Maschke it suffices to treat V irreducible, $V \cong \rho''$. Then by Schur's lemma $\mathrm{Hom}_G(\rho', V) = 0$ for $\rho' \ncong \rho''$ and is spanned by one isomorphism f_0 for $\rho' = \rho''$, and $u \otimes f_0 \mapsto f_0(u)$ is an isomorphism. A linear map T of V commuting with G acts on $\mathrm{Hom}_G(\rho', V)$ by $f \mapsto T \circ f$, and $T(f(u)) = (T\circ f)(u)$. □

Physics aside 3.6 (degeneracy spaces). This is the decomposition a physicist writes for a Hilbert space with a symmetry group, $\mathcal{H} = \bigoplus_j V_j \otimes \mathcal{D}_j$, where V_j is the irreducible multiplet and $\mathcal{D}_j$ the space of "other quantum numbers"; an operator commuting with the symmetry acts on $\mathcal{D}_j$ alone, as the Hamiltonian does, and this is how the Hodge structure acts on $M_{\rho'}$. The analogy stops at rationality: in physics only $\mathbb{C}$ matters, while the proof needs to know which of these subspaces are defined over $\mathbb{Q}$ or over K, because only rational subspaces of $H^1(C,\mathbb{Q})$ correspond to abelian subvarieties of the Jacobian (Section 3.7).

A special case of Schur's lemma is used constantly. If z is central in G, then $D(z)$ commutes with $D(G)$, so on an irreducible ρ' it is a scalar $\kappa_{\rho'}(z)$.

DEFINITION 3.7. The *central character* of an irreducible ρ' is the homomorphism $\kappa_{\rho'} \colon Z(G) \to \mathbb{C}^\times$ with $D(z) = \kappa_{\rho'}(z)\,\mathrm{id}$. Then $\chi_{\rho'}(zg) = \kappa_{\rho'}(z)\chi_{\rho'}(g)$ for $z \in Z(G)$, $g \in G$.

For G_{32} the centre is cyclic of order 6 and the central character κ_0 of the reflection representation ρ is faithful; exactly 14 irreducibles share it, and only these contribute to the pieces used in step (ii) (Chapters 4 and 7). Since κ_0 takes non-real values, the part of $H^1(C,\mathbb{C})$ with central character κ_0 meets its complex conjugate only in 0; this isotropy is used in the contraction of step (vi).

Intuition 3.8 (characters as fingerprints). A character is the cheapest invariant of a representation, one number per conjugacy class, and orthogonality says it is a complete one: two representations with the same character are isomorphic. Every multiplicity in

this companion, from $\dim M_\rho = 6$ for G_{32} to the vectors $v_{H,\psi}$ of the eleven pieces, is computed as an inner product of characters; no matrices of size 155520 are ever needed.

3.3. Induced representations and Frobenius reciprocity

Step (ii) needs the multiplicities of each ρ' in the subspace of $H^1(C,\mathbb{C})$ on which a subgroup H acts through a given one-dimensional character ψ. Frobenius reciprocity answers this with a character computation on G.

Definition 3.9. Let $H \le G$ and let W be a representation of H. The *induced representation* is $\operatorname{Ind}_H^G W = \mathbb{C}[G] \otimes_{\mathbb{C}[H]} W$. Concretely, if $g_1, \dots, g_k$ are representatives of the cosets G/H, then $\operatorname{Ind}_H^G W = \bigoplus_i g_i \otimes W$, of dimension $[G:H]\dim W$, and g sends $g_i \otimes w$ to $g_j \otimes hw$ where $gg_i = g_j h$. Its character is

$$\chi_{\operatorname{Ind} W}(g) = \frac{1}{|H|} \sum_{\substack{x \in G \\ x^{-1}gx \in H}} \chi_W(x^{-1}gx) \ .$$

For a representation V of G and a one-dimensional character ψ of H, write $V^{(H,\psi)} = \{v \in V : hv = \psi(h)v \text{ for all } h \in H\}$.

Theorem 3.10 (Frobenius reciprocity). *For $H \le G$, a representation W of H and a representation V of G,*

$$\langle \operatorname{Res}_H V, W\rangle_H = \langle V, \operatorname{Ind}_H^G W\rangle_G \ .$$

In particular, for a one-dimensional $\psi\colon H \to \mathbb{C}^\times$ and an irreducible ρ' of G,

$$\dim \rho'^{(H,\psi)} = \langle \operatorname{Res}_H \rho', \psi\rangle_H = \langle \rho', \operatorname{Ind}_H^G \psi\rangle_G \ ,$$

and for every representation $V = \bigoplus_{\rho'} \rho' \otimes M_{\rho'}$ one has $V^{(H,\psi)} = \bigoplus_{\rho'} \rho'^{(H,\psi)} \otimes M_{\rho'}$.

Proof. Insert the character formula:

$$\langle V, \operatorname{Ind} W\rangle_G = \frac{1}{|G||H|} \sum_{g \in G} \sum_{x\colon x^{-1}gx \in H} \chi_V(g)\overline{\chi_W(x^{-1}gx)} \ .$$

Substitute $h = x^{-1}gx$; since $\chi_V(g) = \chi_V(h)$, each pair $(x,h) \in G \times H$ occurs once, and the sum becomes $\frac{1}{|G||H|}\sum_x \sum_{h\in H} \chi_V(h)\overline{\chi_W(h)} = \langle \operatorname{Res}_H V, W\rangle_H$. For one-dimensional ψ, $\langle \operatorname{Res}_H \rho', \psi\rangle_H$ is the multiplicity of ψ in $\operatorname{Res}_H \rho'$ (Theorem 3.3 for H), which is $\dim \rho'^{(H,\psi)}$. The last statement holds because H acts on $\rho' \otimes M_{\rho'}$ through the first factor. (Alternatively, $\operatorname{Hom}_H(W, \operatorname{Res} V) = \operatorname{Hom}_G(\operatorname{Ind} W, V)$ by the universal property of the tensor product, [Eti+11, §4.10], [Ser77, Ch. 3, Ch. 7].) □

Example 3.11 (Frobenius reciprocity on S_3). Let $C_3 = \langle c\rangle$, $c = (123)$, and let ψ_ω be the character of C_3 with $\psi_\omega(c) = \omega$. The character formula gives $\operatorname{Ind}_{C_3}^{S_3} \psi_\omega = (2, 0, \omega + \omega^2) = (2, 0, -1) = \rho_2$, so ρ_2 is induced from a one-dimensional character. In the basis $e_1 = 1 \otimes 1$, $e_2 = s \otimes 1$ with $s = (12)$, the matrices are

$$D(c) = \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}, \qquad D(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Running over all subgroups up to conjugacy and all one-dimensional characters, the vectors $v_{H,\psi} = (\langle \rho', \operatorname{Ind}_H^{S_3} \psi\rangle)_{\rho'}$ with $\rho' = (\mathbf{1}, \operatorname{sgn}, \rho_2)$ are

H	1	C_2	C_2	C_3	C_3	C_3	S_3	S_3
ψ	$\mathbf{1}$	$\mathbf{1}$	sgn	$\mathbf{1}$	ψ_ω	ψ_{ω^2}	$\mathbf{1}$	sgn
$v_{H,\psi}$	$(1,1,2)$	$(1,0,1)$	$(0,1,1)$	$(1,1,0)$	$(0,0,1)$	$(0,0,1)$	$(1,0,0)$	$(0,1,0)$

For instance $\operatorname{Ind}_{C_2}^{S_3} \mathbf{1}$ is the permutation representation on the three cosets, $\mathbf{1} + \rho_2$. Every irreducible is a single Ind here; for $\mathrm{SL}(2,3)$ this will fail.

EXERCISE 3.1. Check the column $(H,\psi) = (C_2, \mathrm{sgn})$ of the table in Example 3.11 twice: once by restricting each irreducible character of S_3 to $C_2 = \langle(12)\rangle$ and computing $\langle \mathrm{Res}\,\rho', \mathrm{sgn}\rangle$, and once by computing $\mathrm{Ind}_{C_2}^{S_3}\mathrm{sgn}$ from the character formula and decomposing it with Theorem 3.3.

DEFINITION 3.12. A representation, or its character, is *monomial* if it is induced from a one-dimensional character of some subgroup. An irreducible monomial $\rho' = \mathrm{Ind}_H^G \psi$ has $\dim \rho' = [G:H]$.

If the reflection representation ρ of G_{32} were monomial, $\rho = \mathrm{Ind}_H^G \psi$, then $e_\rho = v_{H,\psi}$ and one piece would suffice; the proof would need neither the integral relation nor the contraction of step (vi). It is not monomial for G_{32}: e_ρ is none of the vectors $v_{H,\psi}$ of the 334 pieces of Section 7.5, and by the first step of the proof of Proposition 3.18 a monomial $\rho = \mathrm{Ind}_H^G \psi$ would also be induced from a pair in that family. The two-dimensional example below shows how non-monomial irreducibles arise in a small group.

The central character interacts simply with induction, and this is what makes the family of pieces in step (ii) natural.

EXERCISE 3.2. Let $Z \le Z(G)$ be a central subgroup, $H \supseteq Z$ a subgroup and ψ a one-dimensional character of H. Show that every $z \in Z$ acts on $\mathrm{Ind}_H^G \psi$ as the scalar $\psi(z)$. Deduce that every irreducible constituent ρ' of $\mathrm{Ind}_H^G \psi$ has $\kappa_{\rho'}|_Z = \psi|_Z$, so that $v_{H,\psi}(\rho') = 0$ unless ρ' has central character $\psi|_Z$ on Z.

Physics aside 3.13 (Wigner's little groups). Induced representations are familiar from Wigner's construction of the unitary irreducible representations of the Poincaré group: fix a momentum, take an irreducible representation of its little group, and induce. Mackey's theory generalizes this to semidirect products, and for finite groups [Ser77, Ch. 8] it shows, for example, that an irreducible representation either restricts to a normal subgroup as a multiple of a single irreducible, or is induced from a proper subgroup. What has no counterpart in the Poincaré case is the arithmetic question of the next section: which *integral* combinations of induced representations produce a given irreducible one.

3.4. The group $\mathrm{SL}(2,3)$ as the reflection group G_4

The proof uses the reflection group G_{32} of order 155520, whose character table no one computes by hand. The reflection group G_4 of order 24 has every feature that matters (a reflection representation over K with non-real character, a centre, non-monomial irreducibles), and it is small enough to do everything explicitly.

EXAMPLE 3.14 ($G_4 \cong \mathrm{SL}(2,3)$). Let

$$r_1 = \begin{pmatrix} 1 & 1 \\ 0 & \omega \end{pmatrix}, \qquad r_2 = \begin{pmatrix} \omega & 0 \\ -\omega & 1 \end{pmatrix} \qquad \in \mathrm{GL}_2(\mathbb{Z}[\omega])\ .$$

Each is a *complex reflection* of order 3: it fixes a line pointwise ($r_i - 1$ has rank one) and has the eigenvalue ω on a complementary line. They satisfy the braid relation $r_1 r_2 r_1 = r_2 r_1 r_2$ and generate a group $G = G_4$ of order 24, isomorphic to $\mathrm{SL}(2,3)$. (Any conjugate of these matrices in $\mathrm{GL}_2(K)$ would do.) Its centre is $\{\pm 1\}$. The seven conjugacy classes, listed with the order of an element, the class size, and the trace and determinant of the defining representation ρ, are

class	1	-1	$6a$	$3a$	4	$3b$	$6b$
order, size	$1,1$	$2,1$	$6,4$	$3,4$	$4,6$	$3,4$	$6,4$
Tr, det	$2,\ 1$	$-2,\ 1$	$\omega,\ \omega^2$	$-\omega,\ \omega^2$	$0,\ 1$	$-\omega^2,\ \omega$	$\omega^2,\ \omega$

Class $3b$ consists of the four reflections with eigenvalue ω (it contains r_1, r_2; their trace is $1+\omega = -\omega^2$), and class $3a$ of their squares; so G_4 contains 8 reflections. The character table,

with the Frobenius–Schur indicator $\nu(\chi) = \frac{1}{|G|}\sum_g \chi(g^2)$ and the character field, is

	1	-1	$6a$	$3a$	4	$3b$	$6b$	ν	$\mathbb{Q}(\chi)$
χ_1	1	1	1	1	1	1	1	1	$\mathbb{Q}$
χ_2	1	1	ω^2	ω^2	1	ω	ω	0	K
χ_3	1	1	ω	ω	1	ω^2	ω^2	0	K
χ_4	2	-2	1	-1	0	-1	1	-1	$\mathbb{Q}$
$\chi_5 = \rho$	2	-2	ω	$-\omega$	0	$-\omega^2$	ω^2	0	K
χ_6	2	-2	ω^2	$-\omega^2$	0	$-\omega$	ω	0	K
χ_7	3	3	0	0	-1	0	0	1	$\mathbb{Q}$

The defining representation is χ_5: irreducible, with character field K, and non-real (its complex conjugate is χ_6). The three one-dimensional characters factor through $G/[G,G] \cong C_3$, the three-dimensional one is the rotation representation of the tetrahedral group $G/\{\pm 1\} \cong A_4$, and χ_4, χ_6 are χ_5 times the two non-trivial one-dimensional characters. Every subgroup has order $1, 2, 3, 4, 6, 8$ or 24; there is no subgroup of order 12.

Exercise 3.3. Verify from the table of Example 3.14 that $24 = 1+1+1+4+4+4+9$, that $\langle \chi_5, \chi_5 \rangle = 1$ and $\langle \chi_5, \chi_4 \rangle = 0$, and that the central character of χ_4, χ_5, χ_6 is the sign on $\{\pm 1\}$ while that of $\chi_1, \chi_2, \chi_3, \chi_7$ is trivial.

Exercise 3.4. Show that $\mathrm{SL}(2,3)$ has no subgroup of index 2, in two ways: (a) a subgroup of index 2 contains every element of odd order and every square, and count; (b) $G/[G,G] \cong C_3$ (read off from the number of one-dimensional characters). Conclude that none of χ_4, χ_5, χ_6 is monomial.

So the two-dimensional irreducibles of $\mathrm{SL}(2,3)$ are not induced from anything one-dimensional. The question of the next section is what replaces induction for them.

3.5. Brauer's and Artin's induction theorems

Characters of G span a ring under pointwise addition and multiplication; integral combinations of characters, $\sum n_i \chi_{\rho_i}$ with $n_i \in \mathbb{Z}$, are *virtual characters*, and they form the representation ring $R(G)$, a free abelian group with basis the irreducible characters. In this basis a virtual character is an integer vector, and $\mathrm{Ind}_H^G \psi$ becomes $v_{H,\psi}$ by Theorem 3.10. A subgroup is *elementary* if it is the direct product of a cyclic group and a p-group for some prime p.

Theorem 3.15 (Brauer). *Every character of a finite group G is an integral linear combination of characters $\mathrm{Ind}_E^G \phi$ induced from one-dimensional characters ϕ of elementary subgroups $E \leq G$. In particular the vectors $v_{H,\psi}$, over all subgroups H and one-dimensional ψ, span $\mathbb{Z}^{\mathrm{Irr}(G)}$.*

This is [Ser77, Ch. 10] (Brauer, 1947); we do not reproduce the proof, which occupies that chapter. The older theorem of Artin [Ser77, Ch. 9], [Eti+11, §4.25] uses only cyclic subgroups and pays with denominators:

Theorem 3.16 (Artin). *Every character of G is a linear combination with rational coefficients of characters induced from one-dimensional characters of cyclic subgroups.*

The difference between rational and integral coefficients is the difference that matters in step (ii), because a relation $e_\rho = \sum_j c_j v_j$ is turned into geometry by taking one copy of the Jacobian for each unit of $|c_j|$, and a coefficient $\frac{1}{2}$ has no such meaning. A multiple ke_ρ can be realized, but then the construction produces k copies of M_ρ rather than one; for the K-rational version of the construction this changes which Weil classes are reached (Chapter 11). This motivates the following bookkeeping.

DEFINITION 3.17. For a family $\mathcal{F}$ of pairs (H,ψ) and an irreducible ρ, let $k_{\min}(\mathcal{F},\rho)$ be the least integer $k\geq 1$ with ke_ρ in the $\mathbb{Z}$-span of $\{v_{H,\psi}:(H,\psi)\in\mathcal{F}\}$, and ∞ if e_ρ is not even in their $\mathbb{Q}$-span.

Brauer's theorem says $k_{\min}=1$ for the family of all pairs. For G_{32} one finds $k_{\min}=\infty$ for the 265 ramified pieces (their vectors span a lattice of rank $13<14$), $k_{\min}=1$ for all 334 pieces (the eleven-term relation), $k_{\min}=2$ for the ramified pieces that satisfy the prime condition of Section 7.5 together with the étale pieces, and $k_{\min}=4$ for the K-rational combinations discussed in Chapter 11; the first three are computed in Section 7.5, the last by the Smith-form computation of Chapter 11.

EXERCISE 3.5. Let $G=C_2\times C_2$ with its four one-dimensional characters. List the seven characters induced from one-dimensional characters of cyclic subgroups (the trivial subgroup and the three subgroups of order 2). Show that each has an even sum of coefficients in the basis of irreducibles, so the trivial character is not an integral combination of them, and write it as a combination with coefficients in $\frac{1}{2}\mathbb{Z}$. Why does this not contradict Brauer's theorem?

In step (ii) the pieces are not arbitrary pairs: the subgroup must contain the centre $Z=Z(G)$ and $\psi|_Z$ must equal the central character κ_0 of ρ, because only then is $C/\ker\psi\to C/H$ a cyclic cover whose ψ-eigenspace lies in the part of $H^1(C)$ with central character κ_0. An integral relation is what Brauer's theorem leads one to expect in this family, and the following refinement, a short consequence of Theorem 3.15, shows that one exists.

PROPOSITION 3.18 (Brauer with a prescribed central character). *Let $Z\leq Z(G)$ be a central subgroup and χ an irreducible character whose central character restricts to κ on Z. Then $\chi=\sum_i n_i\operatorname{Ind}_{H_i}^G\psi_i$ with $n_i\in\mathbb{Z}$, subgroups $H_i\supseteq Z$, and one-dimensional characters ψ_i of H_i with $\psi_i|_Z=\kappa$. Equivalently, e_χ lies in the $\mathbb{Z}$-span of the vectors $v_{H,\psi}$ with $H\supseteq Z$ and $\psi|_Z=\kappa$.*

PROOF. By Theorem 3.15, $\chi=\sum_j m_j\operatorname{Ind}_{E_j}^G\phi_j$ with $m_j\in\mathbb{Z}$ and ϕ_j one-dimensional. Fix j, write $E=E_j$, $\phi=\phi_j$, and put $H=EZ$, a subgroup because Z is central. The map $E\times Z\to H$, $(e,z)\mapsto ez$, is surjective with kernel $\{(z,z^{-1}):z\in E\cap Z\}$. Hence for every character κ' of Z that agrees with ϕ on $E\cap Z$, the formula $\tilde\phi_{\kappa'}(ez)=\phi(e)\kappa'(z)$ defines a one-dimensional character of H extending ϕ, and there are $[Z:E\cap Z]=[H:E]$ of them. By Theorem 3.10 each occurs in $\operatorname{Ind}_E^H\phi$ with multiplicity $\langle\phi,\operatorname{Res}_E\tilde\phi_{\kappa'}\rangle=1$; comparing dimensions, $\operatorname{Ind}_E^H\phi=\sum_{\kappa'}\tilde\phi_{\kappa'}$. Induction is transitive ($\mathbb{C}[G]\otimes_{\mathbb{C}[H]}\mathbb{C}[H]\otimes_{\mathbb{C}[E]}W=\mathbb{C}[G]\otimes_{\mathbb{C}[E]}W$), so

$$\operatorname{Ind}_E^G\phi=\sum_{\kappa'|_{E\cap Z}=\phi|_{E\cap Z}}\operatorname{Ind}_H^G\tilde\phi_{\kappa'}\ .$$

By Exercise 3.2, every irreducible constituent of $\operatorname{Ind}_H^G\tilde\phi_{\kappa'}$ has central character κ' on Z. Let $P_\kappa\colon R(G)\to R(G)$ be the projection that keeps the irreducible characters with central character κ on Z and kills the others; it is additive and fixes χ. Applying it,

$$\chi=P_\kappa\chi=\sum_j m_j\sum_{\kappa'}P_\kappa\operatorname{Ind}_{E_jZ}^G\tilde\phi_{j,\kappa'}=\sum_{j:\ \phi_j|_{E_j\cap Z}=\kappa|_{E_j\cap Z}}m_j\operatorname{Ind}_{E_jZ}^G\tilde\phi_{j,\kappa}\ ,$$

which is of the required form with $H_j=E_jZ$ and $\psi_j=\tilde\phi_{j,\kappa}$, since $\tilde\phi_{j,\kappa}|_Z=\kappa$. □

Remark on the proof 3.1: Brauer and the family of pieces. Proposition 3.18 shows that Brauer's theorem applies to the family $\{(H,\psi):H\supseteq Z(G),\ \psi|_{Z(G)}=\kappa_0\}$, so an integral relation $e_\rho=\sum c_jv_j$ among the vectors of the 334 pieces exists by theory (they contain every pair of the family up to conjugacy). The proof does not use this: the relation of Proposition 7.8 is verified directly, and what the proof needs beyond existence (coefficients ±1, and every piece used satisfying the hypotheses of step (iii) or step (iv)) does not follow from Brauer's theorem at all.

EXAMPLE 3.19 (a Brauer relation for $\mathrm{SL}(2,3)$). Take $G = G_4$, $Z = \{\pm 1\}$ and the target $\rho = \chi_5$, with central character $\kappa(-1) = -1$. With H taken up to conjugacy, the pairs (H, ψ) with $H \supseteq Z$ and $\psi(-1) = -1$ are six, and up to conjugacy of pairs five: (Z, sgn); one on C_4, the characters with $\psi_4(-1) = -1$, i.e. $\psi_4(x) = \pm\mathrm{i}$ on a generator x, being conjugate under the normalizer of C_4 in Q_8 (an element of Q_8 inverts x), so that they induce the same character; and three on $C_6 = \langle g \rangle$, $g \in 6a$, with $\psi_6(g) \in \{-1, -\omega, -\omega^2\}$. None exist on Q_8 or on G, whose one-dimensional characters are trivial on -1. The character formula gives

	1	-1	$6a$	$3a$	4	$3b$	$6b$	$v = (\langle \chi_i, \mathrm{Ind} \rangle)_{i=1..7}$
$\mathrm{Ind}_Z^G \mathrm{sgn}$	12	-12	0	0	0	0	0	$(0,0,0,2,2,2,0)$
$\mathrm{Ind}_{C_4}^G \psi_4$	6	-6	0	0	0	0	0	$(0,0,0,1,1,1,0)$
$\mathrm{Ind}_{C_6}^G \psi_6$ $(\psi_6(g) = -\omega)$	4	-4	$-\omega$	ω	0	ω^2	$-\omega^2$	$(0,0,0,1,0,1,0)$

so that $\mathrm{Ind}_{C_4} \psi_4 = \chi_4 + \chi_5 + \chi_6$ and $\mathrm{Ind}_{C_6} \psi_6 = \chi_4 + \chi_6$. Hence

$$\chi_5 = \mathrm{Ind}_{C_4}^G \psi_4 - \mathrm{Ind}_{C_6}^G \psi_6 = \mathrm{Ind}_Z^G \mathrm{sgn} - \mathrm{Ind}_{C_4}^G \psi_4 - \mathrm{Ind}_{C_6}^G \psi_6 \ .$$

The second, three-term form is the relation used in the exact model of step (vi) for $\mathrm{SL}(2,3)$ (Chapter 11). Frobenius reciprocity can be seen directly in the second line: ρ restricted to C_4 has eigenvalues $\pm\mathrm{i}$ (trace 0, determinant 1), so it contains ψ_4 once, and $v_{C_4,\psi_4}(\chi_5) = 1$. On C_6, $\rho(g)$ has eigenvalues -1 and $-\omega^2$ (trace ω, determinant ω^2), so ρ does not contain the character $\psi_6(g) = -\omega$, and $v_{C_6,\psi_6}(\chi_5) = 0$. The minus sign cannot be avoided: e_{χ_5} is no single $v_{H,\psi}$, since χ_5 is not monomial (Exercise 3.4).

What just happened. Frobenius reciprocity converts each pair (H, ψ) into an integer vector, the multiplicities of $\mathrm{Ind}_H^G \psi$. Brauer's theorem, refined in Proposition 3.18 to pairs with a prescribed central character, says that each indicator vector e_ρ is an integral combination of these vectors; for non-monomial ρ some coefficients are negative. Artin's theorem gives only rational combinations, and the least common denominator ($k_{\min}$) matters once combinations are turned into products of Jacobians.

3.6. The group algebra: Wedderburn's theorem and matrix units

Step (vi) needs, for each irreducible ρ', a linear map $A_{\rho'}$ between direct sums of copies of ρ', chosen freely, and then one element of $\mathrm{Mat}(\mathbb{C}[G])$ that acts as $A_{\rho'}$ on the ρ'-parts of every representation simultaneously. The group algebra provides exactly this freedom. For a field F of characteristic 0, $F[G]$ is the vector space with basis G and multiplication extending that of G; a representation of G over F is the same thing as an $F[G]$-module.

THEOREM 3.20 (Wedderburn decomposition of $\mathbb{C}[G]$). *Let $\rho_1, \dots, \rho_s$ represent the irreducible representations of G, realized by matrices $D_i \colon G \to \mathrm{GL}_{d_i}(\mathbb{C})$. Then*

$$\mathbb{C}[G] \longrightarrow \prod_{i=1}^{s} \mathrm{End}(\rho_i) = \prod_{i=1}^{s} \mathrm{Mat}_{d_i}(\mathbb{C}) \ , \qquad \sum_g a_g g \longmapsto \Big(\sum_g a_g D_i(g) \Big)_i \ ,$$

is an isomorphism of algebras.

PROOF. The map is an algebra homomorphism because each D_i is multiplicative. It is injective: an element x in the kernel acts as 0 on every irreducible, hence on every representation (Maschke), in particular on the regular representation $\mathbb{C}[G]$, where $x \cdot 1 = x$. Both sides have dimension $|G| = \sum_i d_i^2$ (Theorem 3.3). ([Ser77, Ch. 6]; [Eti+11, §§2.3, 3.1].) □

Because the isomorphism is explicit, its inverse can be written down. The elements of $\mathbb{C}[G]$ that map to the elementary matrices are the matrix units.

Definition 3.21 (matrix units). For an irreducible ρ of dimension d, realized by $D\colon G \to \mathrm{GL}_d(\mathbb{C})$, and $1 \le a, b \le d$, put

$$E^{\rho}_{ab} = \frac{d}{|G|} \sum_{g \in G} D(g^{-1})_{ba}\, g \;\in \mathbb{C}[G] \,.$$

Proposition 3.22. *Under the isomorphism of Theorem 3.20, E^{ρ}_{ab} maps to the elementary matrix e_{ab} in the factor $\mathrm{End}(\rho)$ and to 0 in every other factor. Consequently $E^{\rho}_{ab}E^{\rho}_{cd} = \delta_{bc}E^{\rho}_{ad}$, $E^{\rho}_{ab}E^{\rho'}_{cd} = 0$ for $\rho \not\cong \rho'$, and $\sum_a E^{\rho}_{aa}$ is the central idempotent $\frac{d}{|G|}\sum_g \chi_\rho(g^{-1})g$ that projects every representation onto its ρ-isotypic part. On $V = \bigoplus_{\rho'} \rho' \otimes M_{\rho'}$, the element E^{ρ}_{ab} acts as $e_{ab} \otimes \mathrm{id}_{M_\rho}$ on $\rho \otimes M_\rho$ and as 0 elsewhere; in particular $E^{\rho}_{aa}V = e_a \otimes M_\rho$, a copy of the multiplicity space.*

Proof. The image of E^{ρ}_{ab} in the factor of ρ' has (i,j) entry $\frac{d}{|G|}\sum_g D(g^{-1})_{ba}D'(g)_{ij}$. The orthogonality relations for matrix coefficients [Ser77, Ch. 2], [Eti+11, §3.7] state that $\frac{1}{|G|}\sum_g D(g^{-1})_{ba}D'(g)_{ij} = \frac{1}{d}\delta_{bj}\delta_{ai}$ if $D' = D$, and 0 if $\rho' \not\cong \rho$. (They follow from Schur's lemma applied to the averaged map $\sum_g D'(g)\,X\,D(g^{-1})$ for an elementary matrix X.) So the image is e_{ab}, respectively 0. The product rules hold in $\prod \mathrm{Mat}_{d_i}(\mathbb{C})$, hence in $\mathbb{C}[G]$. Summing over $a = b$ gives $\frac{d}{|G|}\sum_g \mathrm{Tr}\, D(g^{-1})\, g$. The action on $\rho' \otimes M_{\rho'}$ is through the first factor, where E^{ρ}_{ab} acts by its image in $\mathrm{End}(\rho')$. □

Example 3.23 (matrix units of S_3). In the basis of Example 3.11, with $c = (123)$ and $s = (12)$ (products read right to left, so sc means c first), the elements of S_3 are $1, c, c^2, s, sc, sc^2$, and $D(sc^k) = D(s)D(c)^k$. The formula of Definition 3.21 gives, with $d/|G| = \frac{1}{3}$,

$$E_{11} = \tfrac{1}{3}(1 + \omega^2 c + \omega c^2), \qquad E_{12} = \tfrac{1}{3}(s + \omega\, sc + \omega^2 sc^2),$$
$$E_{21} = \tfrac{1}{3}(s + \omega^2 sc + \omega\, sc^2), \qquad E_{22} = \tfrac{1}{3}(1 + \omega c + \omega^2 c^2).$$

For instance E_{11} involves $D(c^{-1})_{11} = D(c^2)_{11} = \omega^2$ and $D(c^{-2})_{11} = D(c)_{11} = \omega$, and $D(g^{-1})_{11} = 0$ for the transpositions. The sum $E_{11} + E_{22} = \frac{1}{3}(2 - c - c^2)$ is the central idempotent of ρ_2, with coefficients $\frac{2}{6}\chi_{\rho_2}(g^{-1})$. A direct computation in the six-dimensional regular representation confirms that $E_{ab}E_{cd} = \delta_{bc}E_{ad}$, that the trivial and sign characters kill every E_{ab}, and that E_{11} has rank 2 there, the multiplicity of ρ_2 in $\mathbb{C}[S_3]$ (Proposition 3.22). The complex units depend on the basis: realizing ρ_2 over $\mathbb{Q}$ on $f_1 = x_1 - x_2$, $f_2 = x_2 - x_3$ instead gives rational matrix units such as $E^{\mathbb{Q}}_{11} = \frac{1}{3}(1 - c - s + sc)$.

Exercise 3.6. Deduce $E_{ab}E_{cd} = \delta_{bc}E_{ad}$ from Theorem 3.20 without computing in $\mathbb{C}[G]$. Then verify $E_{12}E_{21} = E_{11}$ for S_3 by multiplying out the expressions of Example 3.23.

Intuition 3.24 (a switchboard). Wedderburn's theorem says that $\mathbb{C}[G]$ is a block-diagonal matrix algebra with one full block per irreducible, and the matrix units are its standard basis. In step (vi) the eleven pieces are, for each ρ', subspaces $\mathfrak{u}_+(\rho')$ of a direct sum $\rho'^{\oplus 7}$, and the target is a subspace $\mathfrak{t}(\rho')$ of $\rho'^{\oplus 5}$ of the same dimension; a linear map $A_{\rho'}$ between them is a matrix of numbers, and $\sum_{a,b}(A_{\rho'})_{ab}E^{\rho'}_{ab}$, summed over ρ' and arranged as a 5×7 matrix, is the single element $\Psi \in \mathrm{Mat}_{5\times 7}(\mathbb{C}[G])$ that does all the switching at once. The multiplicity spaces $M_{\rho'}$, which carry the geometry, are untouched.

Over $\mathbb{Q}$ the picture is the same with the matrix algebras replaced by matrix algebras over division algebras. The simple factors of $\mathbb{Q}[G]$ correspond to the orbits of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the irreducible characters; the factor of the orbit of χ has centre $\mathbb{Q}(\chi)$ and is $\mathrm{Mat}_n(D)$ for a division algebra D with centre $\mathbb{Q}(\chi)$, whose degree m ($\dim_{\mathbb{Q}(\chi)} D = m^2$) is the *Schur index* of χ over $\mathbb{Q}$; χ is realisable over a field $F \supseteq \mathbb{Q}(\chi)$ exactly when $D \otimes_{\mathbb{Q}(\chi)} F$ splits, and over $\mathbb{Q}(\chi)$ itself exactly when $m = 1$ [Ser77, Ch. 12]. We only need examples.

Example 3.25 ($\mathbb{Q}[S_3]$ and $\mathbb{Q}[\mathrm{SL}(2,3)]$). All characters of S_3 are rational and ρ_2 is realized over $\mathbb{Q}$ (Example 3.23), so $\mathbb{Q}[S_3] \cong \mathbb{Q}\times\mathbb{Q}\times \mathrm{Mat}_2(\mathbb{Q})$, of dimension $1+1+4=6$. For G_4 the Galois orbits in the table of Example 3.14 are $\{\chi_1\}$, $\{\chi_2,\chi_3\}$, $\{\chi_4\}$, $\{\chi_5,\chi_6\}$, $\{\chi_7\}$, and

$$\mathbb{Q}[G_4] \cong \mathbb{Q}\times K\times D\times \mathrm{Mat}_2(K)\times \mathrm{Mat}_3(\mathbb{Q})\ , \qquad 1+2+4+8+9=24\ .$$

The factor $\mathrm{Mat}_2(K)$ is there because χ_5 is realized over K by the matrices r_1, r_2. The factor D belongs to the rational character χ_4: it is 4-dimensional over $\mathbb{Q}$, and it is not $\mathrm{Mat}_2(\mathbb{Q})$, because χ_4 has Frobenius–Schur indicator -1, so it is not realisable over $\mathbb{R}$ [Eti+11, §4.1], let alone over $\mathbb{Q}$; hence D is a quaternion division algebra and χ_4 has Schur index 2 over $\mathbb{Q}$. (It is the algebra of rational Hamilton quaternions, since the quaternion group $Q_8 \le \mathrm{SL}(2,3)$ maps onto its units $\pm1,\pm i,\pm j,\pm k$.) Over K, however, $\chi_4 = \chi_5\chi_2$ is realized by the twisted matrices $\chi_2(r_i)\, r_i = \omega r_i$, so its Schur index over K is 1.

Exercise 3.7. (a) Compute $\nu(\chi_4) = -1$ from the table of Example 3.14, using that squaring sends the classes $1, -1, 6a, 3a, 4, 3b, 6b$ to $1, 1, 3b, 3b, -1, 3a, 3a$. (b) Check that $g \mapsto \chi_2(g)\rho(g)$ is a representation with character χ_4, and that $\chi_2(r_1) = \chi_2(r_2) = \omega$ (use the class of r_i). Conclude that χ_4 is realized over K although not over $\mathbb{R}$.

The last fact needed from the group algebra is the bridge from $\mathbb{C}[G]$, where Wedderburn's theorem lives, to $\mathbb{Q}[G]$, which acts on Jacobians.

Proposition 3.26 (density). *Let $N = ab|G|$ and identify $\mathrm{Mat}_{b\times a}(\mathbb{C}[G])$ with $\mathbb{C}^N$ by the coefficients of the group elements, so that $\mathrm{Mat}_{b\times a}(\mathbb{Q}[G]) = \mathbb{Q}^N$ and $\mathrm{Mat}_{b\times a}(\mathbb{Q}[G])\otimes_{\mathbb{Q}}\mathbb{C} = \mathrm{Mat}_{b\times a}(\mathbb{C}[G])$.*

(a) A polynomial $f \in \mathbb{C}[x_1,\dots,x_N]$ that vanishes on $\mathbb{Q}^N$ is zero; that is, $\mathbb{Q}^N$ is Zariski dense in $\mathbb{C}^N$.

(b) If $\Phi\colon \mathbb{C}^N \to \mathbb{C}^M$ is a polynomial map, then $\Phi(\mathbb{C}^N)$ lies in the $\mathbb{C}$-linear span of $\Phi(\mathbb{Q}^N)$. In particular, if $\Phi(x)$ maps a subspace $A \subseteq \mathbb{C}^{M'}$ into a subspace B for all rational x, where $\mathbb{C}^M = \mathrm{Hom}(\mathbb{C}^{M'}, \mathbb{C}^{M''})$, then so does $\Phi(\Psi)$ for every $\Psi \in \mathrm{Mat}_{b\times a}(\mathbb{C}[G])$.

Proof. (a) Induction on N. For $N=1$, a nonzero polynomial has finitely many roots, but $\mathbb{Q}$ is infinite. For $N>1$ write $f = \sum_k f_k(x_1,\dots,x_{N-1})x_N^k$. For fixed $q \in \mathbb{Q}^{N-1}$ the one-variable polynomial $f(q, x_N)$ vanishes on $\mathbb{Q}$, so all $f_k(q)=0$; by induction all $f_k = 0$. (b) Let $\ell\colon \mathbb{C}^M \to \mathbb{C}$ be a linear form vanishing on $\Phi(\mathbb{Q}^N)$. Then $\ell\circ\Phi$ is a polynomial vanishing on $\mathbb{Q}^N$, hence zero by (a), so ℓ vanishes on $\Phi(\mathbb{C}^N)$. Since a subspace is the intersection of the kernels of the linear forms vanishing on it, $\Phi(\mathbb{C}^N) \subseteq \mathrm{span}_{\mathbb{C}}\,\Phi(\mathbb{Q}^N)$. For the last claim, the set of $T \in \mathrm{Hom}(\mathbb{C}^{M'},\mathbb{C}^{M''})$ with $T(A)\subseteq B$ is a linear subspace containing $\Phi(\mathbb{Q}^N)$. □

In step (vi), $\Phi(\Psi) = \bigwedge^D \Psi^*$ is the map induced on H^D by the action of Ψ on H^1; it is a homogeneous polynomial of degree D in the coefficients of Ψ. For rational x with nx integral, $n \in \mathbb{Z}_{>0}$, it is n^{-D} times pullback by a homomorphism of abelian varieties, which preserves algebraic classes; (b) with A, B the complexified spaces of algebraic classes gives the same for Ψ (Chapter 11).

Exercise 3.8. Show that (a) of Proposition 3.26 fails over a finite field: find a nonzero polynomial over $\mathbb{F}_3$ vanishing on all of $\mathbb{F}_3$. Where exactly does the proof use that $\mathbb{Q}$ is infinite? Show also that $\mathbb{Z}^N$ is Zariski dense in $\mathbb{C}^N$, so that integral matrices alone would suffice in (b).

3.7. Rational representations: V_ρ and End $= K$

Cohomology of a variety has a rational structure, $H^1(C,\mathbb{C}) = H^1(C,\mathbb{Q})\otimes\mathbb{C}$, and G acts on $H^1(C,\mathbb{Q})$. The isotypic part of a complex irreducible ρ is usually not defined over $\mathbb{Q}$: complex conjugation (which fixes $H^1(C,\mathbb{Q})$) carries the ρ-part to the $\bar\rho$-part. The rational object that contains it is V_ρ, and the multiplicity space M_ρ acquires a rational form as a K-vector space. The field $K = \mathbb{Q}(\sqrt{-3})$ is an imaginary quadratic, hence CM, field [Mil20, Ch. 5, p. 90]; fix the embedding $\iota\colon K \hookrightarrow \mathbb{C}$ with $\iota(\omega) = e^{2\pi\mathrm{i}/3}$ and write $\bar\iota$ for its conjugate.

PROPOSITION 3.27. *Let $\rho\colon G \to \mathrm{GL}_n(K)$ be a representation over K which is absolutely irreducible (irreducible over $\mathbb{C}$ via ι) and whose character is not real-valued. Let V_ρ be K^n regarded as a $\mathbb{Q}$-vector space of dimension $2n$ with its $\mathbb{Q}[G]$-action. Then:*
(a) $V_\rho \otimes_{\mathbb{Q}} \mathbb{C} \cong \rho \oplus \bar\rho$, where ρ and $\bar\rho$ are $K^n \otimes_{K,\iota} \mathbb{C}$ and $K^n \otimes_{K,\bar\iota} \mathbb{C}$, and $\bar\rho \not\cong \rho$;
(b) $\mathrm{End}_{\mathbb{Q}[G]}(V_\rho) = K$, acting by scalar multiplication on K^n;
(c) V_ρ is an irreducible $\mathbb{Q}[G]$-module;
(d) for every $\mathbb{Q}[G]$-module H, $\mathrm{Hom}_{\mathbb{Q}[G]}(V_\rho, H)$ is a K-vector space (by $(af)(v) = f(av)$) with $\dim_K \mathrm{Hom}_{\mathbb{Q}[G]}(V_\rho, H) = \dim_{\mathbb{C}} \mathrm{Hom}_G(\rho, H \otimes \mathbb{C})$;
(e) for every $u \in V_\rho \setminus \{0\}$ the evaluation $\mathrm{Hom}_{\mathbb{Q}[G]}(V_\rho, H) \to H$, $f \mapsto f(u)$, is injective.

PROOF. (a) The map $K \otimes_{\mathbb{Q}} \mathbb{C} \to \mathbb{C} \times \mathbb{C}$, $a \otimes z \mapsto (\iota(a)z, \bar\iota(a)z)$, is an isomorphism of $\mathbb{C}$-algebras ($K = \mathbb{Q}[x]/(x^2+x+1)$ and x^2+x+1 has two distinct roots in $\mathbb{C}$). Hence $V_\rho \otimes_{\mathbb{Q}} \mathbb{C} = K^n \otimes_K (K \otimes_{\mathbb{Q}} \mathbb{C}) = (K^n \otimes_{K,\iota} \mathbb{C}) \oplus (K^n \otimes_{K,\bar\iota} \mathbb{C})$, G-equivariantly. The characters of the two summands are $\iota \circ \chi$ and $\bar\iota \circ \chi = \overline{\iota \circ \chi}$, which differ because the character is not real; so $\rho \not\cong \bar\rho$ by Theorem 3.3, and $\bar\rho$ is irreducible because ρ is. (b) Extension of scalars gives $\mathrm{End}_{\mathbb{Q}[G]}(V_\rho) \otimes_{\mathbb{Q}} \mathbb{C} = \mathrm{End}_{\mathbb{C}[G]}(\rho \oplus \bar\rho)$ (a G-map is the solution of linear equations with rational coefficients), which is $\mathbb{C} \times \mathbb{C}$ by Schur's lemma, since $\rho \not\cong \bar\rho$. So $\dim_{\mathbb{Q}} \mathrm{End}_{\mathbb{Q}[G]}(V_\rho) = 2$. Scalar multiplication by $a \in K$ commutes with $\rho(G) \subset \mathrm{GL}_n(K)$, so $K \subseteq \mathrm{End}_{\mathbb{Q}[G]}(V_\rho)$, and comparing dimensions gives equality. (c) If $V_\rho = V' \oplus V''$ with nonzero $\mathbb{Q}[G]$-submodules (Maschke holds over $\mathbb{Q}$ with the same averaging proof), the projection onto V' along V'' is an idempotent in $\mathrm{End}_{\mathbb{Q}[G]}(V_\rho) = K$ other than $0, 1$, impossible in a field. (d) The K-action is well defined because K commutes with G on V_ρ. As in (b), $\mathrm{Hom}_{\mathbb{Q}[G]}(V_\rho, H) \otimes_{\mathbb{Q}} \mathbb{C} = \mathrm{Hom}_G(\rho \oplus \bar\rho, H_{\mathbb{C}}) = \mathrm{Hom}_G(\rho, H_{\mathbb{C}}) \oplus \mathrm{Hom}_G(\bar\rho, H_{\mathbb{C}})$, and $a \in K$ acts on the two summands by $\iota(a)$ and $\bar\iota(a)$. Complex conjugation of $H_{\mathbb{C}} = H \otimes \mathbb{C}$ is G-equivariant and antilinear and maps the ρ-isotypic part onto the $\bar\rho$-isotypic part, so the two summands have the same dimension $m = \dim \mathrm{Hom}_G(\rho, H_{\mathbb{C}})$. Hence $\dim_{\mathbb{Q}} \mathrm{Hom}_{\mathbb{Q}[G]}(V_\rho, H) = 2m$ and $\dim_K = m$. (e) If $f(u) = 0$, then f vanishes on the $\mathbb{Q}[G]$-submodule generated by u, which is V_ρ by (c). □

For the reflection representation ρ of $G_{32} \subset \mathrm{GL}_4(K)$, absolutely irreducible with character field K (Chapter 4), this gives the facts the proof uses: V_ρ is an 8-dimensional irreducible $\mathbb{Q}[G]$-module with $\mathrm{End}_{\mathbb{Q}[G]}(V_\rho) = K$, and by (d), $\mathcal{B}_C = \mathrm{Hom}_{\mathbb{Q}[G]}(V_\rho, H^1(C, \mathbb{Q}))$ is a K-vector space of dimension $\dim M_\rho$ ($= 6$, by the Chevalley–Weil formula, Chapter 5). Part (e) is the embedding $f \mapsto f(u)$ of $\mathcal{B}_C$ into $H^1(C, \mathbb{Q})$ at the end of step (vi); the complexification of its image is $u' \otimes M_\rho \oplus u'' \otimes M_{\bar\rho}$, where $u = u' + u''$ in $\rho \oplus \bar\rho$.

EXAMPLE 3.28 (G_4). For $G_4 \subset \mathrm{GL}_2(K)$ of Example 3.14, $V_\rho = K^2 \cong \mathbb{Q}^4$. Writing each entry $x + y\omega$ of r_1, r_2 as the 2×2 rational matrix $\left(\begin{smallmatrix} x & -y \\ y & x-y \end{smallmatrix}\right)$ of multiplication by $x + y\omega$ in the basis $(1, \omega)$, the commutant of r_1, r_2 in $\mathrm{Mat}_4(\mathbb{Q})$ is 2-dimensional, as (b) predicts. For the twisted representation $\chi_4 = \chi_2\chi_5$, realized over K by $\omega r_1, \omega r_2$ (Example 3.25), the same computation gives a 4-dimensional commutant: its character is real, $\chi_4 \otimes \mathbb{C}$ appears twice in $V \otimes \mathbb{C}$, and $\mathrm{End}_{\mathbb{Q}[G]}$ is the quaternion algebra D. The hypothesis "not real-valued" in Proposition 3.27 cannot be dropped.

EXERCISE 3.9. (a) Let $C_3 = \langle c \rangle$ act on $V = K$ by multiplication by ω. Write the action on $\mathbb{Q}^2$ in the basis $(1, \omega)$, compute $\mathrm{End}_{\mathbb{Q}[C_3]}(V)$ directly as the commutant of one 2×2 matrix, and identify it with K. (b) Explain, using (a) of Proposition 3.27 and Schur's lemma, why for the representation χ_4 of G_4 realized over K the analogous commutant has dimension 4 over $\mathbb{Q}$ rather than 2.

Intuition 3.29 (real, complex, quaternionic over $\mathbb{Q}$). Over $\mathbb{R}$ an irreducible complex representation is of real, complex or quaternionic type according to its Frobenius–Schur indicator $1, 0, -1$, and the commutant of its realification is $\mathbb{R}$, $\mathbb{C}$ or the quaternions. Proposition 3.27 is the arithmetic version of the complex case, with K in the role of $\mathbb{C}$:

the non-real character of ρ makes V_ρ a rational representation whose commutant is the field K, and this K is the field that acts on the Weil sixfold A_C, through its action on $H^1(A_C, \mathbb{Q}) = \mathcal{B}_C$ by way of V_ρ (Chapter 2, Definition 1.1).

REMARK 3.30 (Where this is used). Proposition 3.5 and Theorem 3.10 give the description of the pieces $W_{H,\psi} = \bigoplus \rho'^{(H,\psi)} \otimes M_{\rho'}$ and their vectors $v_{H,\psi}$ in Chapter 7 (step (ii)), with Exercise 3.2 explaining why only the 14 irreducibles with central character κ_0 enter. Theorem 3.15 and Proposition 3.18 and Definition 3.17 are the background for the eleven-piece relation and for the counts $k_{\min} = 1, 2, 4$ quoted in Chapters 7, 11 and 13. Theorem 3.20, Definition 3.21, and Propositions 3.22 and 3.26 are used in Chapter 11 to build $\Psi \in \mathrm{Mat}_{5\times 7}(\mathbb{C}[G])$ and to show that $\bigwedge^D \Psi^*$ preserves complexified algebraic classes (step (vi)). Proposition 3.27 is used in Chapter 4 ($\mathrm{End}_{\mathbb{Q}[G]}(V_\rho) = K$ for G_{32}), in Chapter 5 ($\dim_K \mathcal{B}_C = \dim M_\rho$) and at the end of step (vi) (the embedding of $\mathcal{B}_C$ by evaluation). The group $G_4 = \mathrm{SL}(2, 3)$ of Example 3.14 returns in Chapter 4 as the smallest reflection group of the same kind as G_{32}, and its Brauer relation of Example 3.19 in the exact toy model of the contraction in Chapter 11.

CHAPTER 4

Complex reflection groups and the group G_{32}

Proof steps: supports (i)

Where we are. This chapter contains no step of the proof; it introduces the group with which step (i) starts. The family of curves of the proof is a family of G-covers of $\mathbb{P}^1$ for one particular finite group, $G = G_{32} \subset \mathrm{GL}_4(K)$, the complex reflection group number 32 in the list of Shephard and Todd, and the abelian sixfolds are cut out of the Jacobians by its four-dimensional reflection representation ρ. What the later chapters use is a short list of facts about G_{32}: its order, its centre, the representation ρ and why it is defined over K, the three conjugacy classes from which the branch data of the family are drawn, and the fourteen irreducible representations that share the central character of ρ. They are collected in Proposition 4.10 and Table 4.2. To make them intelligible we explain what a complex reflection group is, why such groups have polynomial rings of invariants, and how the degrees of the invariants control the order, the number of reflections and the centre, first on the group $G_4 \cong \mathrm{SL}(2,3)$ of Example 3.14, then on G_{32}. Covers are Chapter 5. The general theory is standard or in the literature. Needs: Chapter 3. Notation as in Chapter 3: $\omega = e^{2\pi \mathrm{i}/3}$, $K = \mathbb{Q}(\omega)$, $\zeta = e^{2\pi \mathrm{i}/6}$, so that $\omega = \zeta^2$, and κ_0 is the central character of ρ.

4.1. What the proof needs from G_{32}

A *complex reflection* is a linear map of finite order that fixes a hyperplane pointwise; on the complementary line it multiplies by a root of unity $\lambda \neq 1$, so its eigenvalues are $(1, \dots, 1, \lambda)$ (Definition 4.1). Real reflections are the case $\lambda = -1$. A finite group generated by complex reflections is a *complex reflection group*, and Shephard and Todd classified the irreducible ones in 1954: three infinite families and 34 exceptional groups $G_4, \dots, G_{37}$ (Theorem 4.2). The exceptional group G_{32} acts on $\mathbb{C}^4$, can be written with entries in $K = \mathbb{Q}(\omega)$, and is generated by four reflections of order 3,

$$r_i = 1 + (\omega - 1)\frac{a_i a_i^*}{a_i^* a_i}\,, \qquad a_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},\ a_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix},\ a_3 = \begin{pmatrix} 0 \\ 1 \\ \sqrt{-3} \\ 0 \end{pmatrix},\ a_4 = \begin{pmatrix} 0 \\ -1 \\ 1 \\ -1 \end{pmatrix},$$

each of which multiplies its root vector a_i by ω and fixes the orthogonal hyperplane $a_i^{\perp}$. They satisfy the braid relations $r_i r_{i+1} r_i = r_{i+1} r_i r_{i+1}$, and r_i, r_j commute for $|i - j| \geq 2$, so G_{32} is drawn as a chain of four nodes (Figure 4.1). The facts the proof uses are these (Proposition 4.10).

- $|G_{32}| = 155520 = 2^7 \cdot 3^5 \cdot 5 = 12 \cdot 18 \cdot 24 \cdot 30$, the product of the degrees of the basic invariants. The group contains $80 = 11 + 17 + 23 + 29$ reflections, all of order 3, with 40 reflecting hyperplanes, and its centre is $\mu_6 = \{\zeta^k \cdot 1\}$, of order $6 = \gcd(12, 18, 24, 30)$. As an abstract group $G_{32} \cong C_3 \times \mathrm{Sp}_4(\mathbb{F}_3)$.
- The defining representation $\rho\colon G_{32} \hookrightarrow \mathrm{GL}_4(K)$ is absolutely irreducible with character field K and non-real character, so (Proposition 3.27) V_ρ, which is K^4 seen as an 8-dimensional $\mathbb{Q}[G]$-module, is irreducible with $\mathrm{End}_{\mathbb{Q}[G]}(V_\rho) = K$. This is why $\mathcal{B}_C = \mathrm{Hom}_{\mathbb{Q}[G]}(V_\rho, H^1(C, \mathbb{Q}))$ is a K-vector space.

- The branch data of the family are taken from three of the 102 conjugacy classes: the 40 reflections with eigenvalue ω (determinant ζ^2), their 40 squares (determinant ζ^4), and a class of 90 involutions with eigenvalues $(1, 1, -1, -1)$, the only class of involutions with a two-dimensional fixed space (Table 4.2).
- Exactly 14 of the 102 irreducible representations have the central character $\kappa_0(\zeta^k) = \zeta^k$ of ρ; their degrees are $4, 4, 20, 20, 20, 20, 20, 36, 36, 60, 60, 60, 64, 80$. Only these 14 enter the integer vectors $v_{H,\psi}$ of step (ii) (Exercise 3.2).

The family is built from one explicit tuple $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_{13})$ of six reflections of each kind and one involution of the class of size 90, with product 1, which generates G_{32} (Section 4.5). How such a tuple defines a curve with a G_{32}-action, and why these classes give a Weil sixfold of signature $(3, 3)$, is the subject of Chapter 5; the only input from this chapter is the list of eigenvalues in Table 4.2.

Why G_{32}, and not a smaller group? The proof needs a reflection group whose reflection representation is defined over K, and a family of covers with a non-split Weil sixfold whose curves have at least $9 = \dim \mathrm{Sh}_3^{(2)}$ moduli. For G_{25} and G_{26} of rank 3, twelve-point families have $12 - 3 = 9$ moduli, but we know of no non-split member of them. The thirteen-point family of G_{32} used here is non-split by Chapter 2 and dominant by Chapter 6; nothing in this chapter explains either fact.

Sources. The definitions and the theorems of Chevalley, Shephard–Todd, Molien and Springer are standard or in the literature [ST54; Che55; Spr74], with the survey [GM03] as an open reference for statements and tables. The data of G_{32} in Proposition 4.10 are exact computations in GAP from the four reflections above; the numbers about G_4 can be checked by hand.

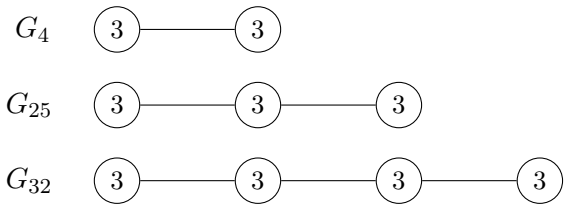


FIGURE 4.1. Diagrams of G_4, G_{25}, G_{32}: one node for each generating reflection of order 3, an edge for the braid relation $rr'r = r'rr'$, no edge for commuting generators. In each case these relations together with $r_i^3 = 1$ present a group of the right order (24, 648, 155520; coset enumeration in GAP).

4.2. Reflections and reflection groups

Every finite subgroup of $\mathrm{GL}_n(\mathbb{C})$ preserves a positive definite Hermitian form (average any one over the group), so one may think of its elements as unitary matrices. In this picture a reflection is the simplest possible non-trivial element.

DEFINITION 4.1. A *(complex) reflection* on a finite-dimensional complex vector space V is an element $s \in \mathrm{GL}(V)$ of finite order such that $\mathrm{rk}(s - 1) = 1$. Its fixed space $H_s = \ker(s - 1)$ is the *reflecting hyperplane*. A finite subgroup $G \subset \mathrm{GL}(V)$ is a *(complex) reflection group* if it is generated by reflections, and *irreducible* if V is an irreducible representation of G.

Since s has finite order it is diagonalisable, and $\mathrm{rk}(s - 1) = 1$ means that the eigenvalue 1 has multiplicity $n - 1$; the remaining eigenvalue is $\lambda = \det s$, a root of unity $\neq 1$. If $\langle\,,\rangle$ is an invariant Hermitian form and a spans the λ-eigenline (a *root* of s), then $H_s = a^\perp$ and

$$s(v) = v + (\lambda - 1)\frac{\langle v, a\rangle}{\langle a, a\rangle}\, a\ , \tag{4.1}$$

which for the standard form is the matrix $1+(\lambda-1)aa^*/(a^*a)$ used for G_{32} above. The powers s^k with $\lambda^k \neq 1$ are reflections in the same hyperplane, with eigenvalue λ^k; so reflections of order 3 come in pairs $\{s, s^2\}$ sharing a hyperplane.

EXERCISE 4.1. (a) Show that the matrix $s = 1 + (\lambda - 1)aa^*/(a^*a)$ satisfies $sa = \lambda a$ and $sv = v$ for $v \perp a$; deduce $\det s = \lambda$, $\operatorname{Tr} s = n - 1 + \lambda$, and that s is unitary if $|\lambda| = 1$. (b) For the matrix $r_1 = \left(\begin{smallmatrix} 1 & 1 \\ 0 & \omega \end{smallmatrix}\right)$ of Example 3.14, find the fixed line and the ω-eigenline, and check $\operatorname{Tr} r_1 = 1 + \omega = -\omega^2$ and $\det r_1 = \omega$. Why is r_1 not of the form $1 + (\omega - 1)aa^*/(a^*a)$ for any a?

The structure theory begins with a reduction: a reflection group is the direct product of irreducible reflection groups acting on the summands of an orthogonal decomposition of V [GM03, §1.13]. An irreducible G is *imprimitive* if it permutes the summands of some decomposition $V = V_1 \oplus \cdots \oplus V_k$ with $k > 1$, and *primitive* otherwise.

THEOREM 4.2 (Shephard–Todd). *Every irreducible complex reflection group is conjugate in* $\mathrm{GL}(V)$ *to one of: the groups* $G(de, e, n)$ *of* $n \times n$ *monomial matrices whose non-zero entries are* de*-th roots of unity with product a* d*-th root of unity* ($de \geq 2$, $(de, e, n) \neq (2, 2, 2)$)*; the symmetric group* S_{n+1} *acting on* $\{x \in \mathbb{C}^{n+1} : \sum x_i = 0\}$*; or one of* 34 *primitive exceptional groups* $G_4, \ldots, G_{37}$*, of ranks* 2 *to* 8.

This is [ST54], as stated in [GM03, Thm. 1.14]; the numbering $G_4, \ldots, G_{37}$ is Shephard and Todd's, who reserved G_1, G_2, G_3 for the symmetric groups, the imprimitive family and the cyclic groups. For the imprimitive groups $G(de, e, n)$ the representation V is monomial in the sense of Definition 3.12: G permutes the coordinate lines transitively, so $V = \operatorname{Ind}_H^G \psi$, where H is the stabilizer of the first line and ψ the character by which H acts on it. For such a group the indicator vector e_ρ would be a single vector $v_{H,\psi}$, and the Brauer-type relation of step (ii) would be trivial. The interesting groups for the proof are therefore the primitive ones, and among them those whose reflection representation has character field K.

group	rank	degrees	order	reflections	source
G_4	2	$4, 6$	24	8 of order 3	Example 4.3
G_5	2	$6, 12$	72		[GM03]
G_{25}	3	$6, 9, 12$	648	24 of order 3	[GM03]
G_{26}	3	$6, 12, 18$	1296	$24 + 9$ of orders 3, 2	[GM03]
G_{32}	4	$12, 18, 24, 30$	155520	80 of order 3	Proposition 4.10
G_{33}	5	$4, 6, 10, 12, 18$	51840		[GM03]
G_{34}	6	$6, 12, 18, 24, 30, 42$	39191040		[GM03]

TABLE 4.1. The exceptional complex reflection groups whose reflection representation has character field $\mathbb{Q}(\zeta_3) = K$, according to [GM03, Table 1]; the order is the product of the degrees (Proposition 4.6).

Table 4.1 lists these groups. The three groups $G_4 \subset G_{25} \subset G_{32}$ form a chain: G_{25} is generated by r_1, r_2, r_3 (acting on the span of a_1, a_2, a_3), G_4 by r_1, r_2, and in each case the generators are reflections of order 3 related as in Figure 4.1. One may think of the diagrams as Dynkin diagrams of type A_2, A_3, A_4 in which every node has become a reflection of order 3; for real reflections of order 2 the same diagrams would give the Weyl groups S_3, S_4, S_5. The group G_{26} contains G_{25} as a normal subgroup of index 2, obtained by adjoining a reflection of order 2.

EXAMPLE 4.3 (G_4 as a reflection group). Take the matrices $r_1 = \left(\begin{smallmatrix} 1 & 1 \\ 0 & \omega \end{smallmatrix}\right)$ and $r_2 = \left(\begin{smallmatrix} \omega & 0 \\ -\omega & 1 \end{smallmatrix}\right)$ of Example 3.14. They are reflections of order 3 with eigenvalue ω, satisfy $r_1 r_2 r_1 = r_2 r_1 r_2$, and generate the group $G_4 \cong \mathrm{SL}(2, 3)$ of order 24. From the class list of Example 3.14, the reflections are the four elements of class $3b$ (eigenvalue ω, trace $-\omega^2$) and their four squares

in class $3a$: 8 reflections. Their fixed lines are spanned by

$$\begin{pmatrix}1\\0\end{pmatrix},\quad \begin{pmatrix}0\\1\end{pmatrix},\quad \begin{pmatrix}1\\-1\end{pmatrix},\quad \begin{pmatrix}1\\\omega\end{pmatrix},$$

so there are 4 reflecting lines, each the fixed line of one reflection s and of s^2. Conjugation permutes the four lines; the centre $\{\pm1\}$ acts trivially, and $G_4/\{\pm1\}\cong A_4$ acts on them as the rotation group of a tetrahedron on its four vertices. The product $c=r_1r_2$ has order 6 and $c^3=-1$, so the centre is generated by a power of the product of the generators. The invariants and degrees are computed in Example 4.7.

4.3. Invariants, degrees and the centre

For a finite group $G\subset \mathrm{GL}(V)$ let $\mathbb{C}[V]=\mathrm{Sym}(V^*)$ be the ring of polynomial functions on V, graded by degree, with G acting by $(gf)(v)=f(g^{-1}v)$, and $\mathbb{C}[V]^G$ the ring of invariant polynomials. For the permutation representation of S_n on $\mathbb{C}^n$ the invariants are generated by the elementary symmetric polynomials of degrees $1,\dots,n$, with no relations; a finite group rarely does this well. The theorem that singles out reflection groups is the following.

Theorem 4.4 (Chevalley–Shephard–Todd). *Let $G\subset\mathrm{GL}(V)$ be a finite group, $n=\dim V$. Then $\mathbb{C}[V]^G$ is a polynomial ring, i.e. $\mathbb{C}[V]^G=\mathbb{C}[f_1,\dots,f_n]$ for algebraically independent homogeneous f_i, if and only if G is generated by reflections. In that case the degrees $d_1\le\cdots\le d_n$ of the f_i do not depend on the choice of generators; they are the* degrees *of G.*

Shephard and Todd proved "reflection group $\Rightarrow$ polynomial invariants" by going through their classification, Chevalley gave a classification-free proof the next year, and the converse follows from Molien's formula below [ST54; Che55], [GM03, Thm. 1.3]. We use the theorem only through the degrees, which can be read off from a generating function that is computable from the character of V alone.

Theorem 4.5 (Molien). *For a finite group $G\subset\mathrm{GL}(V)$,*

$$\sum_{k\ge0}\dim\mathbb{C}[V]^G_k\,t^k=\frac{1}{|G|}\sum_{g\in G}\frac{1}{\det(1-tg)}\ .$$

Proof. The operator $P=\frac{1}{|G|}\sum_g g$ on the finite-dimensional space $\mathbb{C}[V]_k=\mathrm{Sym}^k V^*$ is a projection onto the invariants (it fixes invariants, and $hP=P$ for $h\in G$), so $\dim\mathbb{C}[V]^G_k=\operatorname{Tr}P=\frac{1}{|G|}\sum_g\operatorname{Tr}(g|\,\mathrm{Sym}^k V^*)$. If g acts on V^* with eigenvalues $\mu_1,\dots,\mu_n$ (it is diagonalisable, having finite order), the monomials in an eigenbasis are eigenvectors on $\mathrm{Sym}^k V^*$, so $\sum_k t^k\operatorname{Tr}(g|\,\mathrm{Sym}^k V^*)=\prod_i(1-\mu_it)^{-1}=\det(1-tg|_{V^*})^{-1}$. Since g acts on V^* by the inverse transpose, $\det(1-tg|_{V^*})=\det(1-tg^{-1})$, and summing over g is the same as summing over g^{-1}. □

If $\mathbb{C}[V]^G=\mathbb{C}[f_1,\dots,f_n]$, the monomials $f_1^{k_1}\cdots f_n^{k_n}$ form a basis, and the left side is $\prod_i(1-t^{d_i})^{-1}$. Comparing the two sides near $t=1$ gives the basic numerical facts.

Proposition 4.6. *Let $G\subset\mathrm{GL}(V)$ be a reflection group with degrees $d_1,\dots,d_n$ and N reflections. Then*

$$|G|=d_1d_2\cdots d_n\ ,\qquad N=\sum_{i=1}^n(d_i-1)\ .$$

Proof. Put $u=1-t$ and expand both sides of Molien's formula in powers of u. On the left, $1-t^d=u(1+t+\cdots+t^{d-1})=u\big(d-\frac{d(d-1)}{2}u+O(u^2)\big)$, so

$$\prod_i\frac{1}{1-t^{d_i}}=\frac{1}{u^n\prod_i d_i}\Big(1+\frac{u}{2}\sum_i(d_i-1)+O(u^2)\Big)\ .$$

On the right, $\det(1-tg)^{-1}$ has a pole at $t=1$ of order $\dim\ker(g-1)$. Only $g=1$ gives order n, with contribution exactly $u^{-n}/|G|$; only reflections give order $n-1$, a reflection

with eigenvalue λ contributing $u^{-(n-1)}(1-\lambda t)^{-1}/|G| = u^{-(n-1)}(1-\lambda)^{-1}/|G| + O(u^{-(n-2)})$. Comparing the coefficients of u^{-n} gives $|G| = \prod_i d_i$. The reflections in a fixed hyperplane, together with 1, form a cyclic group of order e whose non-trivial elements have the eigenvalues $\lambda \neq 1$ with $\lambda^e = 1$; pairing λ with $\bar\lambda = \lambda^{-1}$ and using $\frac{1}{1-\lambda} + \frac{1}{1-\bar\lambda} = 1$ (Exercise 4.2; for $\lambda = -1$ the single term is $\frac12$), the reflections contribute $\frac{N}{2}$ in total. Comparing the coefficients of $u^{-(n-1)}$ gives $\frac{N}{2|G|} = \frac{\sum_i (d_i-1)}{2\prod_i d_i}$, hence $N = \sum_i (d_i - 1)$. □

Exercise 4.2. Show that $\frac{1}{1-\lambda} + \frac{1}{1-\bar\lambda} = 1$ for every complex number $\lambda \neq 1$ with $|\lambda| = 1$. (Write the sum over the common denominator $|1-\lambda|^2$.) Then verify the identity of the $u^{-(n-1)}$ coefficients for G_4 directly: $\frac{8}{2\cdot 24} = \frac{3+5}{2\cdot 4\cdot 6}$.

Example 4.7 (Molien series of G_4, and of $\mathrm{SL}(2,3)$ without reflections). For a 2×2 matrix, $\det(1-tg) = 1 - (\operatorname{Tr} g)t + (\det g)t^2$, so the class table of Example 3.14 (sizes $1,1,4,4,6,4,4$, traces $2,-2,\omega,-\omega,0,-\omega^2,\omega^2$, determinants $1,1,\omega^2,\omega^2,1,\omega,\omega$) gives

$$\frac{1}{24}\Big(\frac{1}{(1-t)^2} + \frac{1}{(1+t)^2} + \frac{4}{1-\omega t+\omega^2 t^2} + \frac{4}{1+\omega t+\omega^2 t^2} + \frac{6}{1+t^2} + \frac{4}{1+\omega^2 t+\omega t^2} + \frac{4}{1-\omega^2 t+\omega t^2}\Big) = \frac{1}{(1-t^4)(1-t^6)}\,,$$

an identity of rational functions, which can be checked by clearing denominators. So G_4 has degrees $4, 6$, and indeed $4\cdot 6 = 24$ and $(4-1)+(6-1) = 8$ reflections, as counted in Example 4.3.

The same abstract group has a second faithful two-dimensional representation over K, $\chi_4 = \chi_2\chi_5$, realized by $\omega r_1, \omega r_2$ (Example 3.25). These matrices have determinant $\omega^2\cdot\omega = 1$, so the group they generate lies in $\mathrm{SL}_2(K)$; it is the binary tetrahedral group, and it contains no reflection at all, because an element of SL_2 with eigenvalue 1 has both eigenvalues 1. Its Molien series is

$$\frac{1+t^{12}}{(1-t^6)(1-t^8)} = 1 + t^6 + t^8 + 2t^{12} + t^{14} + t^{16} + 2t^{18} + \cdots\,,$$

and a computation confirms what it suggests: there are invariants f_6, f_8 and a further invariant f_{12} that is not a polynomial in them, and f_{12}^2 is a polynomial in f_6, f_8. This is a hypersurface ring, not a polynomial ring, exactly as Theorem 4.4 predicts for a group without reflections. The two representations differ only by the one-dimensional character χ_2, which is invisible to the abstract group structure but decides whether the generators are reflections.

Exercise 4.3. (a) Use Proposition 4.6 to check $|G_4| = 24$ and the count of 8 reflections from the degrees $4, 6$. (b) For the binary tetrahedral group of Example 4.7, expand its Molien series at $t = 1$, $u = 1-t$, and check that the coefficient of u^{-2} is $1/24$ and that the coefficient of u^{-1} is 0, as the proof of Proposition 4.6 requires for a group without reflections. (Hint: $\frac{1+t^{12}}{(1-t^6)(1-t^8)} = \frac{2-12u+O(u^2)}{48u^2(1-\frac{12}{2}u+O(u^2))}$.)

The centre of an irreducible G consists of scalars, by Schur's lemma (Lemma 3.1). The degrees also control it.

Proposition 4.8. *Let $G \subset \mathrm{GL}(V)$ be an irreducible reflection group with degrees $d_1, \ldots, d_n$. Then $Z(G)$ is cyclic of order* $\gcd(d_1, \ldots, d_n)$.

Proof of one inequality, and a citation for the other. A central element is a scalar $\xi\cdot 1$, and $Z(G)$ is a finite subgroup of $\mathbb{C}^\times$, hence cyclic. A scalar $\xi\cdot 1$ acts on a homogeneous polynomial of degree d by ξ^{-d}, and it fixes the basic invariant f_i, so $\xi^{d_i} = 1$ for all i: every element of $Z(G)$ has order dividing $d = \gcd(d_i)$, so $|Z(G)|$ divides d. Conversely, let $\xi = e^{2\pi \mathrm{i}/d}$. Springer's theory of regular elements shows that the largest dimension of an eigenspace $\ker(g-\xi)$, $g \in G$, equals the number of degrees divisible by d [Spr74, Thm. 3.4], as stated in [GM03, Thm. 1.10(i)]; here that number is n, so some g is $\xi\cdot 1$, and $\xi\cdot 1 \in Z(G)$. □

For G_4, $\gcd(4,6)=2$ and $Z=\{\pm 1\}$; for G_{25}, $\gcd(6,9,12)=3$; for G_{26}, $\gcd(6,12,18)=6$; for G_{32}, $\gcd(12,18,24,30)=6$. For G_4 and G_{32} the centre is also found directly (Example 4.3, Proposition 4.10(d)).

EXERCISE 4.4. Using only Propositions 4.6 and 4.8 and the degrees in Table 4.1: (a) find the order, the number of reflections and the order of the centre of G_{25}, G_{26} and G_{32}; (b) given that all reflections of G_{25} and G_{32} have order 3, find their numbers of reflecting hyperplanes; (c) G_{26} has 24 reflections of order 3 (those of G_{25}); how many of order 2 does it have? Compare with a direct computation: G_{26} has 21 reflecting hyperplanes.

What just happened. A finite group is a reflection group exactly when its invariant polynomials form a polynomial ring (Theorem 4.4). Molien's formula computes the Hilbert series of the invariants from the eigenvalues of the group elements; near $t=1$ the identity contributes the leading pole and the reflections the next one, which gives $|G|=\prod d_i$ and $N=\sum(d_i-1)$. Invariance of the basic invariants under scalars bounds the centre by $\gcd(d_i)$, with equality by Springer's theorem. For G_4: $24=4\cdot 6$, $8=3+5$, $|Z|=2$; a group without reflections, such as $\mathrm{SL}(2,3)$ inside SL_2, has non-polynomial invariants.

4.4. The group G_{32}

Intuition 4.9 (a big brother of G_4). G_{32} does in four dimensions what G_4 does in two: it is generated by reflections of order 3 arranged in a chain, it has a centre generated by a power of the product of the generators, and its reflection representation is defined over K with non-real character. What is new is size. With 155520 elements, 102 conjugacy classes and irreducible representations of degree up to 81, the data are computed, and the theory of this chapter predicts some of them: the order must be the product of the degrees, the number of reflections their sum minus four, and the degrees of the irreducibles with a given central character must have squares summing to $|G|\,/\,|Z|$.

PROPOSITION 4.10 (the data of G_{32}). *Let $G=G_{32}=\langle r_1,r_2,r_3,r_4\rangle\subset \mathrm{GL}_4(K)$ with the reflections of Section 4.1, and let ρ be the inclusion.*

(a) $r_i^3=1$, $r_ir_{i+1}r_i=r_{i+1}r_ir_{i+1}$ for $i=1,2,3$, and $r_ir_j=r_jr_i$ for $|i-j|\geq 2$; these relations present a group of order 155520, so they are a presentation of G.

(b) $|G|=155520=2^7\cdot 3^5\cdot 5$, the degrees are $12,18,24,30$, and $|G|=12\cdot 18\cdot 24\cdot 30$.

(c) G contains exactly 80 reflections, all of order 3: one conjugacy class of 40 with eigenvalue ω, which contains $r_1,\dots,r_4$, and the class of their squares. They have 40 reflecting hyperplanes, and $80=11+17+23+29$.

(d) $Z(G)=\{\zeta^k\cdot 1:0\leq k\leq 5\}\cong\mu_6$, and $(r_1r_2r_3r_4)^5=\zeta\cdot 1$; the element $r_1r_2r_3r_4$ has order 30 and eigenvalues $e^{2\pi ik/30}$, $k=1,7,13,19$.

(e) The derived group $G'=[G,G]$ has order 51840, lies in $\mathrm{SL}_4(K)$, is isomorphic to $\mathrm{Sp}_4(\mathbb{F}_3)$, and $G=\langle\omega\cdot 1\rangle\times G'\cong C_3\times\mathrm{Sp}_4(\mathbb{F}_3)$. In particular $G/Z(G)\cong\mathrm{PSp}_4(\mathbb{F}_3)$, a simple group of order 25920.

(f) G has 102 conjugacy classes. All irreducible characters take values in K. The character of ρ is irreducible, non-real, with character field K and Frobenius–Schur indicator 0; hence $\mathrm{End}_{\mathbb{Q}[G]}(V_\rho)=K$ and $V_\rho\otimes\mathbb{C}=\rho\oplus\bar\rho$ by Proposition 3.27.

(g) Exactly 14 irreducible characters have central character κ_0, $\kappa_0(\zeta^k\cdot 1)=\zeta^k$. Their degrees are $4,4,20,20,20,20,20,36,36,60,60,60,64,80$, and $\sum\chi(1)^2=25920=|G|\,/\,|Z(G)|$. The two of degree 4 are ρ and $\bar\rho\otimes\det^2$; $\bar\rho$ itself is not among them.

PROOF (COMPUTATION). Every item is an exact computation in GAP: the group is built from the four matrices over $\mathbb{Q}(\omega)$; (a) by checking the relations on the matrices and by coset enumeration for the abstract presentation (a surjection from a group of order 155520 onto G is an isomorphism); (b) by GAP's order computation and by the Molien series of the natural character, which equals $\prod_i(1-t^{d_i})^{-1}$ for $d=(12,18,24,30)$ as rational functions; (c)

and (d) from the conjugacy classes and the centre; (e) by computing G', testing $\omega \cdot 1 \notin G'$ (its determinant is $\omega^4 = \omega$) and an explicit isomorphism with $\mathrm{Sp}_4(\mathbb{F}_3)$; (f) and (g) from the character table that GAP computes from the group. The identities $|G| = \prod d_i$, $80 = \sum(d_i - 1)$ and $|Z| = \gcd(d_i)$ are then instances of Propositions 4.6 and 4.8 and serve as checks; the eigenvalues in (d) are $e^{-2\pi \mathrm{i}(d_i-1)/30}$, consistent with the eigenvalues of Coxeter elements in Bessis's theorem as stated in [GM03, Thm. 3.2(vi)]. The character values in K and the order 25920 match the entries $\mathbb{Q}(\zeta_3)$ and $U_4(2)$ of [GM03, Table 1]; the computation also checks that $\mathrm{PSp}_4(\mathbb{F}_3)$ is simple and isomorphic to $U_4(2) = \mathrm{PSU}_4(\mathbb{F}_2)$. □

Two remarks on what (f) means. First, ρ is defined over K for the most concrete reason: the four generating matrices have entries in K (the vector a_3 contains $\sqrt{-3} = \omega - \omega^2 \in K$). Its character cannot be realized over $\mathbb{Q}$, because it is not real: the trace of r_1 is $3 + \omega$. Second, the same holds for every irreducible representation of G_{32} by a general theorem: for a complex reflection group, the character field of the reflection representation is a splitting field, i.e. every irreducible representation can be realized over it [Ben76], [GM03, Thm. 4.2]. The proof does not need this beyond ρ, but it explains why all 14 irreducibles of Proposition 4.10(g) have character field exactly K (Exercise 4.7).

Exercise 4.5. (a) Compute $\gcd(12, 18, 24, 30)$ and conclude from the easy half of Proposition 4.8 that $|Z(G_{32})|$ divides 6. (b) From the eigenvalues $e^{2\pi \mathrm{i}k/30}$, $k = 1, 7, 13, 19$, of $c = r_1 r_2 r_3 r_4$, show that c has order 30 and that c^5 is the scalar $\zeta \cdot 1$, so that $Z(G_{32}) = \mu_6$. (c) Show that $c^{10} = \omega \cdot 1$ and $c^{15} = -1$, and find the analogous statements for G_4 with $c = r_1 r_2$ of order 6.

Exercise 4.6. Let G be a finite group with centre Z and κ a character of Z. Show that $\sum \chi(1)^2 = |G| \,/\, |Z|$, the sum over the irreducible χ with central character κ. (Restrict the regular representation $\mathbb{C}[G]$ to Z: each character of Z occurs $|G| \,/\, |Z|$ times; on the other hand the χ-isotypic part of $\mathbb{C}[G]$ has dimension $\chi(1)^2$ and Z acts on it by κ_χ.) Check the identity for the degrees in Proposition 4.10(g), and for G_4 with κ the sign of $\{\pm 1\}$ (Example 3.14).

Exercise 4.7. (a) Show that $\bar{\rho} \otimes \det^2$ has central character κ_0, using $\det(\zeta \cdot 1) = \zeta^4$. (b) Explain why a representation with central character κ_0 can never be realized over $\mathbb{Q}$ or over $\mathbb{R}$, and why its character field contains K. (c) Suppose χ is irreducible with character values in K. Explain why one still needs the Schur index of χ over K to be 1 to realize χ by matrices over K, and how the splitting-field theorem quoted above [Ben76] supplies this for every irreducible representation of G_{32}. Compare with χ_4 of G_4, which has Schur index 2 over $\mathbb{Q}$ but 1 over K (Example 3.25).

4.5. The conjugacy classes of the branch data

The local monodromies of a G-cover of $\mathbb{P}^1$ are elements of G defined up to conjugacy (Chapter 5), so branch data are lists of conjugacy classes. The family uses three classes of G_{32}, whose data are in Table 4.2. A class is determined here by the order, the size and the eigenvalues of a representative; among the 102 classes, these are the only classes of reflections, and the only class of involutions other than the central -1.

The involution can be described as "an involution with $\mathrm{rk}(\gamma - 1) = 2$ from the conjugacy class of size 90"; the table shows that this specifies one class: G_{32} has exactly two classes of involutions, $\{-1\}$ and $\mathcal{I}_{90}$, and $\mathcal{I}_{90}$ is the only class of any order with $\mathrm{rk}(\gamma - 1) = 2$ and size 90 (the other classes with $\mathrm{rk}(\gamma - 1) = 2$ have sizes 240, 360, 480 and 540). Note that G_{32} has no reflections of order 2, so an element of $\mathcal{I}_{90}$ is not a product of commuting real reflections inside G_{32}, although on $\mathbb{C}^4$ it looks like one.

The tuple $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_{13})$ consists of explicit matrices in $\mathrm{GL}_4(K)$: six reflections from $\mathcal{R}_\omega$, six from $\mathcal{R}_{\omega^2}$ and one element of $\mathcal{I}_{90}$, with product 1, generating G_{32}. The matrices and these properties are listed in Chapter C. They are the hypotheses under which Chapter 5 builds the curve C. Two numbers from Table 4.2 feed that chapter directly. The Chevalley–Weil formula expresses the multiplicity of ρ in $H^1(C, \mathbb{C})$ through the fixed spaces of the

class	in $\boldsymbol{\gamma}$	order	size	$\lvert C_G(\gamma)\rvert$	$\det\gamma$	eigenvalues	$\dim\ker(\gamma-1)$	$\operatorname{Tr}\rho(\gamma)$
$\mathcal{R}_\omega$	6	3	40	3888	$\omega=\zeta^2$	$1,1,1,\omega$	3	$3+\omega$
$\mathcal{R}_{\omega^2}$	6	3	40	3888	$\omega^2=\zeta^4$	$1,1,1,\omega^2$	3	$3+\omega^2$
$\mathcal{I}_{90}$	1	2	90	1728	1	$1,1,-1,-1$	2	0

TABLE 4.2. The three conjugacy classes of G_{32} from which the branch data of the family are drawn, with the size of the centralizer $C_G(\gamma)$ ($=155520/\text{size}$), the determinant, the eigenvalues on $\mathbb{C}^4$, the dimension of the fixed space and the trace. The second column says how often each class occurs in the tuple $\boldsymbol{\gamma}$. Computed in GAP.

local monodromies, $\dim M_\rho=-2\dim\rho+\sum_i\operatorname{rk}(\rho(\gamma_i)-1)$ for ρ without invariant vectors (Chapter 5); with the column $\dim\ker(\gamma-1)$ this is $-8+12\cdot1+2=6$. And Riemann–Hurwitz uses only the orders 3 and 2. The eigenvalues themselves, not only their number, decide the Hodge numbers $(3,3)$ of M_ρ (Chapter 5).

EXERCISE 4.8. (a) Check that the determinants in a tuple of the type of $\boldsymbol{\gamma}$ multiply to 1, as they must since $\gamma_1\cdots\gamma_{13}=1$: six factors ω, six ω^2 and one 1. Would a tuple with seven reflections from $\mathcal{R}_\omega$ and five from $\mathcal{R}_{\omega^2}$, plus one element of $\mathcal{I}_{90}$, be possible? (b) Compute $\dim M_\rho=-8+\sum_i\operatorname{rk}(\gamma_i-1)$ for the type of $\boldsymbol{\gamma}$, and for a pattern of seven reflections of each kind. (c) Verify the sizes 40 and 90 from the centralizer orders 3888 and 1728. (d) Multiplication by the central element -1 permutes the conjugacy classes. Show that it maps $\mathcal{I}_{90}$ to itself and $\mathcal{R}_\omega$ to a class of elements of order 6 that are not reflections.

REMARK 4.11 (Where this is used). Proposition 4.10(f) and Proposition 3.27 give $\operatorname{End}_{\mathbb{Q}[G]}(V_\rho)=K$ and make $\mathcal{B}_C$ a K-vector space in Chapter 5 (step (i)); Table 4.2 gives the branch data of the family and the inputs of the Chevalley–Weil and Riemann–Hurwitz formulas there, and the tuple checks of Section 4.5 are the starting point of the Hurwitz component $\mathcal{H}$. The centre μ_6 and the 14 irreducibles with central character κ_0 of Proposition 4.10(d,g) define the family of pieces (H,ψ), $H\supseteq Z(G)$, $\psi|_Z=\kappa_0$, and the coordinates of the vectors $v_{H,\psi}$ in Chapter 7 (step (ii)); the fact that κ_0 is not real is the isotropy used in the contraction of Chapter 11 (step (vi)). The reflections of $\boldsymbol{\gamma}$, and the pairs among them with product 1, reappear in the boundary stratum used for the dominance in Chapter 6. The group G_4 remains the toy model: its covers give the small examples of pieces in Chapter 7 and the exact model of the contraction in Chapter 11.

CHAPTER 5

G-covers of the line: Hurwitz spaces, braid monodromy and Chevalley–Weil

Proof steps: (i) family, $\mathcal{B}_C$, Hodge numbers, invariant form; (ii) multiplicities $M_{\rho'}$

Where we are. This chapter builds the family of step (i). A finite group G acting on a curve C with quotient $\mathbb{P}^1$ is encoded by a tuple of group elements, one for each branch point; moving the branch points moves the curve, and the tuples are permuted by braids. From this one gets three things the proof needs: the genus and the representation of G on $H^1(C)$, including its Hodge numbers (Chevalley–Weil), which give the multiplicity spaces $M_{\rho'}$ of step (ii); the rational Hodge structure $\mathcal{B}_C = \mathrm{Hom}_{\mathbb{Q}[G]}(V_\rho, H^1(C,\mathbb{Q}))$, a Weil sixfold for the G_{32}-covers of the family, with its polarization E_u; and the action of braids on parabolic cohomology, from which the invariant Hermitian form of Example 2.13 is computed. Background: cyclic covers of the line [Moo10, §2], the Riemann–Hurwitz formula [Stacks, Tag 0C1F], local systems and their monodromy [Voi03, §3.1.1], Chapter 3 (characters, isotypic parts, V_ρ) and the group G_{32} of Chapter 4. Notation as in Chapters 2 and 3: $K = \mathbb{Q}(\sqrt{-3})$, $\theta = \sqrt{-3}$, $\zeta = e^{2\pi \mathrm{i}/6}$, $\omega = \zeta^2$, subgroups are called H, and κ_0 is the central character of ρ.

5.1. The family of step (i) in two pages

Cyclic covers $y^m = \prod (x - t_i)^{a_i}$ are described by one integer a_i per branch point, the power of the generator by which y is multiplied when one walks once around t_i [Moo10, pp. 2.1–2.2]. For a non-abelian group there is no equation to write down, but the description survives: a connected curve C with an action of a finite group G and quotient $C/G = \mathbb{P}^1$, branched over $t_1, \dots, t_r$, is the same thing as a tuple $(\gamma_1, \dots, \gamma_r)$ of elements of G that generate G and multiply to 1, up to simultaneous conjugation (Riemann's existence theorem, Theorem 5.2). The element γ_i is the *local monodromy* at t_i; its order e_i is the ramification index of every point of C over t_i, so the genus is $2g(C) - 2 = |G|\,(-2 + \sum_i (1 - 1/e_i))$ (Proposition 5.4).

When the branch points move, the curve moves with them, and nothing discrete can change except the tuple, which is transformed by braids: when t_j and t_{j+1} exchange places, (γ_j, γ_{j+1}) becomes $(\gamma_j \gamma_{j+1} \gamma_j^{-1}, \gamma_j)$ (a *Hurwitz move*). The covers with a given set of conjugacy classes therefore form a space, the *Hurwitz space*, a finite cover of the space of configurations of r points, and its connected components correspond to the orbits of the braid group on tuples (Definition 5.6). The family of the proof is one such component, which we call $\mathcal{H}$: $G = G_{32}$, $r = 13$, and the tuple $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_{13})$ of Example 5.19, with twelve reflections of order 3 and one involution. Its base is a finite cover of the 13-dimensional space of configurations of 13 points in $\mathbb{C}$; configurations related by a Möbius transformation give isomorphic curves, so the curves of $\mathcal{H}$ depend on $13 - 3 = 10$ moduli. We say that a G_{32}-cover has the *branching type of the family* if its local monodromies are six reflections of determinant ω, six of determinant ω^2 and one involution from the conjugacy class of size 90 (the three classes of Section 4.5).

The representation of G on $H^1(C,\mathbb{C})$ is determined by the tuple. The *Chevalley–Weil formula* (Theorem 5.10) gives, for each irreducible representation ρ', the dimension of the multiplicity space $M_{\rho'} = \mathrm{Hom}_G(\rho', H^1(C,\mathbb{C}))$ of Proposition 3.5 and its Hodge numbers, from the eigenvalues of the matrices $\rho'(\gamma_i)$ alone. For the reflection representation ρ of G_{32} and a

cover of the branching type of the family: a reflection moves one dimension, the involution two, so

$$\dim M_\rho = -2\cdot 4 + 12\cdot 1 + 2 = 6, \qquad g(C) = 1 + \tfrac{1}{2}\cdot 155520\cdot\left(-2 + 12\cdot\tfrac{2}{3} + \tfrac{1}{2}\right) = 505441,$$

with Hodge numbers $(3,3)$ on M_ρ (Example 5.13). By Proposition 3.27, the rational Hodge structure

$$\mathcal{B}_C = \mathrm{Hom}_{\mathbb{Q}[G]}(V_\rho, H^1(C,\mathbb{Q}))$$

is then a six-dimensional K-vector space of weight one with Hodge numbers $(3,3)$ on the two eigenspaces of K; with the polarization $E_u(f,f') = E_C(f(u), f'(u))$ it is H^1 of a Weil sixfold A_C (Definition 5.17 and Proposition 5.18).

Finally, the monodromy of the family. Over a Hurwitz component the spaces $\mathcal{B}_C$ form a local system, and $\mathcal{B}_C \otimes \mathbb{C}$ contains M_ρ, which is identified with the *parabolic cohomology* of the rank-4 local system on $\mathbb{P}^1 \smallsetminus \{t_i\}$ defined by ρ and the tuple. Parabolic cohomology is linear algebra with the matrices $\rho(\gamma_i)$, and braids that bring the tuple back to itself act on it by explicit matrices (Section 5.5). An exact computation with 67 such matrices shows that the invariant Hermitian forms on $\mathcal{B}_C$ form a line, spanned by a form of determinant $-128 \equiv -2$; by Lemma 2.10 this is the class of the polarization on every member of $\mathcal{H}$. The whole mechanism is also carried out on an S_3-cover of genus 4, where every number fits on a line (Examples 5.5 and 5.16).

Sources. Riemann's existence theorem and Hurwitz spaces: [Völ96, Ch. 5, Ch. 10], [FV91]. The Riemann–Hurwitz formula: [Stacks, Tag 0C1F]. The Chevalley–Weil formula goes back to [CW34]; its H^1 part is proved here from the Lefschetz fixed-point formula and its $H^{1,0}$ part derived from the Eichler trace formula [FK92, Ch. V]. Parabolic cohomology and its braid monodromy: [DW06]. The data of the family (the tuple, the group it generates, $\dim M_\rho = 6$, the form of determinant -128) are exact computations in GAP over K.

5.2. G-covers and generating tuples

Definition 5.1 (G-cover of the line). Let G be a finite group. A *G-cover of $\mathbb{P}^1$* is a compact connected Riemann surface C with a faithful action of G by biholomorphic maps and a holomorphic map $\pi\colon C \to \mathbb{P}^1$ that identifies C/G with $\mathbb{P}^1$. Its *branch points* are the points $t \in \mathbb{P}^1$ over which π is ramified. Two G-covers are isomorphic if there is a G-equivariant biholomorphic map between them over $\mathbb{P}^1$.

The stabilizer G_P of a point $P \in C$ acts faithfully on the tangent line T_PC, because a holomorphic map that fixes P and has derivative 1 there, and has finite order, is the identity near P (linearize it by averaging the local coordinate over the group it generates). So G_P is cyclic, of order e_P equal to the ramification index of π at P, and it has a distinguished generator, the element that acts on T_PC by $e^{2\pi\mathrm{i}/e_P}$. We call it the *canonical generator* at P. The points over one branch point t_i form one G-orbit, their stabilizers are conjugate, and so are their canonical generators.

Theorem 5.2 (Riemann's existence theorem for G-covers of the line). *Let $t_1,\dots,t_r \in \mathbb{P}^1$ be distinct, $U = \mathbb{P}^1 \smallsetminus \{t_1,\dots,t_r\}$, $t_0 \in U$, and let $\ell_1,\dots,\ell_r$ be standard generators of $\pi_1(U,t_0)$: ℓ_i runs from t_0 to near t_i, once around t_i counterclockwise and back, the paths do not cross, and $\ell_1\ell_2\cdots\ell_r = 1$ is the only relation. For every finite group G there are bijections between*

(a) isomorphism classes of G-covers of $\mathbb{P}^1$ unramified over U;
(b) surjective homomorphisms $\pi_1(U,t_0) \to G$, modulo conjugation by elements of G;
(c) tuples $(\gamma_1,\dots,\gamma_r) \in G^r$ with $\gamma_1\gamma_2\cdots\gamma_r = 1$ that generate G, modulo simultaneous conjugation.

The map from (b) to (c) is $\phi \mapsto (\phi(\ell_1),\dots,\phi(\ell_r))$. Under the bijection, the stabilizers of the points over t_i are the conjugates of $\langle\gamma_i\rangle$, and with the normalization of Remark 5.3, γ_i is the canonical generator at a point P_i over t_i. In particular t_i is a branch point if and only if $\gamma_i \neq 1$, and every point over t_i has ramification index $e_i = \operatorname{ord}\gamma_i$. Every G-cover is a smooth projective algebraic curve on which G acts by algebraic automorphisms.

The bijection from (a) to (b) is covering-space theory: $\pi^{-1}(U) \to U$ is a connected unramified covering on which G acts simply transitively on fibres, and a choice of point P_0 over t_0 turns path lifting into a surjection $\pi_1(U, t_0) \to G$; another choice of P_0 conjugates it. Conversely, a surjection gives a connected covering of U with G acting; over a small punctured disc around t_i it is a disjoint union of copies of $z \mapsto z^{e_i}$, each of which is completed by one point, and the complex structure extends over these points. That the result is algebraic is the Riemann existence theorem proper (every compact Riemann surface is a projective curve, with holomorphic maps algebraic). The bijection from (b) to (c) holds because $\pi_1(U, t_0)$ is free on $\ell_1, \dots, \ell_{r-1}$. For proofs see [Völ96, Ch. 5] (cited by chapter). Cyclic covers are the case $G = \mu_m = \langle\gamma\rangle$ with $\gamma(x, y) = (x, \zeta_m y)$: at a point P_i over t_i with $\gcd(a_i, m) = 1$ a local coordinate z has $x - t_i = z^m$ and $y = z^{a_i} \cdot (\text{unit})$, so the canonical generator, which multiplies z by ζ_m, multiplies y by $\zeta_m^{a_i}$ and is γ^{a_i} (a deck transformation is determined by its action on y). For general a_i the same computation with $x - t_i = z^{m/\gcd(a_i,m)}$ gives γ^{a_i} again. So the tuple is $(\gamma^{a_1}, \dots, \gamma^{a_r})$, and the condition $\sum a_i \equiv 0$ is $\gamma_1 \cdots \gamma_r = 1$.

Remark 5.3 (two readings of a tuple). Whether path lifting gives a homomorphism or an anti-homomorphism, and whether the lift ends at gP_0 or at $g^{-1}P_0$, are conventions, and different sources make different choices. Changing them replaces each γ_i by γ_i^{-1} (in some conventions together with a reversal of the order). In this chapter the tuple is always normalized so that γ_i is the canonical generator at some point over t_i; the formulas of Theorem 5.10 hold in this normalization, and the other reading exchanges $h^{1,0}$ and $h^{0,1}$ in part (b) (Exercise 5.6). For real branch points, complex conjugation of the cover reverses the orientation of the standard loops, so the other reading describes the complex-conjugate cover and, for a family, the complex-conjugate Hurwitz component. For the numbers of the family used in this chapter both readings give the same result (Example 5.13).

Proposition 5.4 (Riemann–Hurwitz for G-covers). *If $C \to \mathbb{P}^1$ is a G-cover with tuple $(\gamma_1, \dots, \gamma_r)$ and $e_i = \operatorname{ord}\gamma_i$, then*

$$2g(C) - 2 = |G| \left(-2 + \sum_{i=1}^{r} \Big(1 - \frac{1}{e_i}\Big)\right).$$

Proof. Apply the Riemann–Hurwitz formula [Stacks, Tag 0C1F] to π, of degree $|G|$ (in characteristic 0 every ramification is tame, so a point of index e contributes $e - 1$): over t_i lie $|G|/e_i$ points (the orbit of P_i, whose stabilizer has order e_i), each with ramification index e_i, contributing $(|G|/e_i)(e_i - 1)$. □

Example 5.5 (an S_3-cover of genus 4). Let $G = S_3$ with $a = (12)$, $b = (13)$, $c = (23)$, products composed from the right ($(xy)(i) = x(y(i))$). The tuple (a, a, b, b, c, c) has product 1 and generates S_3, so it defines an S_3-cover $C \to \mathbb{P}^1$ branched at six points, each with two-element stabilizers. By Proposition 5.4, $2g - 2 = 6\,(-2 + 6 \cdot \frac{1}{2}) = 6$, so $g(C) = 4$. Two quotients will be used. The cyclic group $A_3 = \langle(123)\rangle$ contains no stabilizer, so $C \to C/A_3$ is unramified of degree 3, and $C/A_3 \to \mathbb{P}^1$ is the double cover branched at the six points: a curve of genus 2, and indeed $6 = 3\,(2 \cdot 2 - 2)$. The subgroup $\langle a\rangle$ fixes one point over each t_i (six fixed points, Example 5.11), so $6 = 2\,(2g' - 2) + 6$ and $E = C/\langle a\rangle$ is an elliptic curve; $E \to \mathbb{P}^1$ is a three-sheeted cover with simple branching at six points.

Exercise 5.1. (a) Compute $g(C)$ for a G_{32}-cover of the branching type of the family (twelve local monodromies of order 3, one of order 2, $|G| = 155520$), and check that $g(C)$ is an integer although $-2 + \sum_i (1 - 1/e_i) = \frac{13}{2}$ is not. (b) The centre $Z \cong \mu_6$ of G_{32} consists of scalar matrices. Show that Z acts freely on C, deduce $g(C/Z) = 84241$, and show that the part of $H^1(C, \mathbb{C})$ on which Z acts through κ_0 has dimension 168480 (show that a nontrivial character of a group acting freely on a curve X with quotient Y occurs in $H^1(X, \mathbb{C})$ with multiplicity $2g(Y) - 2$ times its dimension, by the Lefschetz argument in the proof of Theorem 5.10(a) below, where now no element $g \neq 1$ has fixed points). (c) Check (b) against the table of Example 5.13: $\sum_{\rho'} \dim \rho' \cdot \dim M_{\rho'}$ over the 14 irreducibles with central character κ_0.

5.3. Hurwitz spaces and braids

Fix the group and let the branch points move. The configuration space

$$\mathcal{U}_r = \{\text{sets } \{t_1, \dots, t_r\} \subset \mathbb{C} \text{ of } r \text{ distinct points}\}$$

is a complex manifold of dimension r whose fundamental group is Artin's braid group B_r, generated by the half-twists $\sigma_1, \dots, \sigma_{r-1}$: with the points on the real line in increasing order, σ_j moves t_j and t_{j+1} around each other counterclockwise until they have exchanged places. A half-twist changes the standard loops: ℓ_j becomes a loop conjugate to ℓ_{j+1}, and ℓ_{j+1} becomes ℓ_j, while the other loops do not change. On a tuple this is the Hurwitz move.

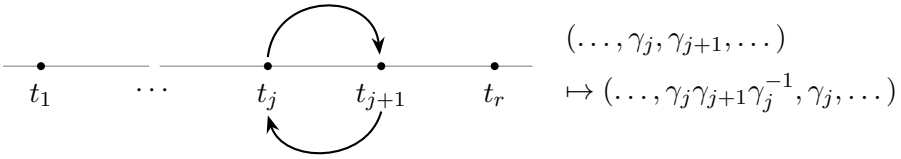


FIGURE 5.1. The half-twist σ_j: the branch points t_j and t_{j+1} exchange places counterclockwise, and the tuple undergoes a Hurwitz move. The product of the tuple and the multiset of conjugacy classes of its entries do not change (Exercise 5.2).

DEFINITION 5.6 (Hurwitz space). Let G be a finite group, $r \geq 3$, and let Ni (the *Nielsen class*) be a set of generating tuples $(\gamma_1, \dots, \gamma_r)$ with product 1, modulo simultaneous conjugation, that is stable under the Hurwitz moves

$$\sigma_j : (\dots, \gamma_j, \gamma_{j+1}, \dots) \longmapsto (\dots, \gamma_j\gamma_{j+1}\gamma_j^{-1}, \gamma_j, \dots)$$

(for instance, all tuples whose entries lie in prescribed conjugacy classes, counted with multiplicity). The *Hurwitz space* $\mathcal{H}(\mathrm{Ni})$ is the covering space of $\mathcal{U}_r$ with fibre Ni over the base configuration and monodromy given by the action of B_r through Hurwitz moves. Its points are the isomorphism classes of G-covers whose tuple lies in Ni (Theorem 5.2), and its connected components (*Hurwitz components*) correspond to the B_r-orbits in Ni.

Over every simply connected open subset of $\mathcal{H}(\mathrm{Ni})$ the covers form a family of smooth projective curves with a fibrewise action of G, unique up to the action of the centre $Z(G)$: a central element acts on every G-cover as a G-equivariant automorphism over $\mathbb{P}^1$, so the local families glue only up to $Z(G)$, and when $Z(G) \neq 1$ a global family need not exist. For G_{32}, whose centre is μ_6, monodromy is therefore taken modulo $Z(G)$ (Section 12.2); Chapter 6 uses a family over a simply connected base. The Hurwitz space is a quasi-projective variety, finite and étale over $\mathcal{U}_r$, by Fried and Völklein [FV91], [Völ96, Ch. 10] (we use only the analytic statement, which is covering-space theory, and cite the algebraic structure); in particular it has dimension r. The isomorphism class of the cover depends only on the configuration of the branch points on $\mathbb{P}^1$ up to the action of $\mathrm{PGL}_2(\mathbb{C})$, a group of dimension 3: the Möbius transformations close to the identity move a configuration of $r \geq 3$ points of $\mathbb{C}$ inside a three-dimensional submanifold of $\mathcal{U}_r$ along which the cover does not change, so the curves of a component depend on $r - 3$ moduli. (Modulo the affine maps $x \mapsto \alpha x + \beta$ alone one would count $r - 2$; the extra parameter is the position of the unbranched point ∞ relative to the others.) For the family, $r = 13$: $\mathcal{H}$ has dimension 13 and its curves depend on 10 moduli, one more than $\dim X^+ = 9$ (Proposition 2.15), so the period map can at best be dominant with fibres of dimension $13 - 9 = 4$, three of them along the Möbius directions; Chapter 6 shows that it is dominant.

EXAMPLE 5.7 (one Hurwitz move written out). In S_3, take the three-point tuple $\tau = ((12), (13), (123))$; its product is $(12)(13)(123) = 1$ and it generates S_3. The cover has

$2g-2 = 6\,(-2+\tfrac12+\tfrac12+\tfrac23) = -2$, so $C \cong \mathbb{P}^1$ with S_3 acting through Möbius transformations. The half-twist σ_1 gives

$$\sigma_1\tau = \big((12)(13)(12)^{-1}, (12), (123)\big) = \big((23), (12), (123)\big),$$

again with product 1 and the same classes; then $\sigma_1^2\tau = ((13), (23), (123))$ and $\sigma_1^3\tau = \tau$. So the full twist σ_1^2 does not fix this tuple, while σ_1^3 does: the loop σ_1^3 in configuration space lifts to a closed loop in the Hurwitz space through τ. Such stabilizing braids are the source of monodromy matrices in Section 5.5.

Exercise 5.2. (a) Show that a Hurwitz move preserves the product $\gamma_1 \cdots \gamma_r$, the subgroup generated by the tuple, and the multiset of conjugacy classes of the entries. (b) Write down the inverse move σ_j^{-1} and check $\sigma_j^{-1}\sigma_j = \mathrm{id}$ on tuples. (c) Show that if $\gamma_j = \gamma_{j+1}$, then σ_j fixes the tuple, and that if $\gamma_j\gamma_{j+1} = 1$, then σ_j^2 fixes the tuple.

Example 5.8 (the Hurwitz space of Example 5.5). Let Ni consist of the generating six-tuples of transpositions of S_3 with product 1, modulo conjugation. There are 240 such tuples; S_3 acts freely on them by conjugation (only the identity centralizes a generating set, as the centre is trivial), so $|\mathrm{Ni}| = 40$, and a direct computation shows that the braid group B_6 acts transitively on Ni. So the S_3-covers branched at six points with transpositions as local monodromies form one connected family, a covering of degree 40 of $\mathcal{U}_6$. That one orbit is no accident: for simply branched covers (all local monodromies transpositions in S_n) the Hurwitz space is connected, a classical theorem of Clebsch and Hurwitz which we do not use. For G_{32} the proof needs only the single component $\mathcal{H}$ of the tuple $\boldsymbol{\gamma}$, and the number of Hurwitz components of its branching type plays no role.

Exercise 5.3. Count the tuples of Example 5.8 by hand: show that $x_1, \dots, x_5$ can be any transpositions and x_6 is then determined and is a transposition, that exactly 3 of the resulting $3^5 = 243$ tuples fail to generate, and conclude 240 tuples and 40 conjugacy classes of tuples.

Intuition 5.9 (branch points as anyons). A tuple is a labelling of the punctures by group elements, read along a fixed system of loops. Moving punctures around each other drags the loops, and the labels change by conjugation. This is how the labels of non-abelian anyons (or of twist fields in an orbifold conformal field theory) change under braiding, and the Hurwitz space is the configuration space of these labelled punctures. Where the analogy stops: there is no Hilbert space of fusion channels here. The curve C and its cohomology are the only "state space", and the braid group acts on $H^1(C)$ by honest monodromy of a family of algebraic curves, which is what makes the action compatible with Hodge theory.

5.4. The Chevalley–Weil formula

The group G acts on $H^1(C, \mathbb{Q})$; we let $g \in G$ act by $g \cdot x = (g^{-1})^* x$, so that this is a representation. It preserves $H^{1,0}$ and $H^{0,1}$, because G acts holomorphically. Here $H^{1,0}(C) = H^0(C, \Omega_C)$ is the space of holomorphic 1-forms, Ω_C being the canonical sheaf of C; for a nodal curve, Ω_C denotes the dualizing sheaf (Section 6.4), and where linear systems are concerned (Chapters 8 and 10) the canonical line bundle of a curve Y is written K_Y, as usual. By Proposition 3.5, $H^1(C, \mathbb{C}) = \bigoplus_{\rho'} \rho' \otimes M_{\rho'}$, and each $M_{\rho'} = \mathrm{Hom}_G(\rho', H^1(C, \mathbb{C}))$ has a Hodge decomposition $M_{\rho'} = \mathrm{Hom}_G(\rho', H^{1,0}) \oplus \mathrm{Hom}_G(\rho', H^{0,1})$. The formula below computes the dimensions from the tuple.

Theorem 5.10 (Chevalley–Weil for G-covers of the line). *Let $C \to \mathbb{P}^1$ be a G-cover with tuple $(\gamma_1, \dots, \gamma_r)$ normalized as in Remark 5.3, $e_i = \operatorname{ord}\gamma_i$. Let ρ' be an irreducible representation of dimension d, $\delta = 1$ if ρ' is trivial and $\delta = 0$ otherwise, and let $N_{i,k}$ be the multiplicity of $e^{2\pi \mathrm{i} k/e_i}$ as an eigenvalue of $\rho'(\gamma_i)$, $0 \le k < e_i$. Then*

(a) $\dim M_{\rho'} = 2\delta - 2d + \sum_{i=1}^{r}(d - N_{i,0})$, *where* $d - N_{i,0} = \operatorname{rk}(\rho'(\gamma_i) - 1)$*;*

(b) $\dim \operatorname{Hom}_G(\rho', H^{1,0}(C)) \;=\; \delta \,-\, d + \sum_{i=1}^{r}\sum_{k=0}^{e_i-1} N_{i,k}\,\dfrac{k}{e_i}$, *and* $\dim \operatorname{Hom}_G(\rho', H^{0,1}(C))$ *is the difference of (a) and (b).*

Eigenspaces of cyclic covers. For a cyclic cover with data $(m; a_1, \dots, a_r)$ and generator γ we write, as in Chapter 2,

$$V_n = \{x \in H^1(C, \mathbb{C}) : \gamma^* x = \zeta_m^n x\}, \qquad d_n = \dim V_n^{1,0} \qquad (n \not\equiv 0 \bmod m),$$

the eigenspaces of the *pull-back*. For the action $g \cdot x = (g^{-1})^* x$ the χ^n-isotypic part, $\chi^n \colon \gamma \mapsto \zeta_m^n$, is $\{x : (\gamma^{-1})^* x = \zeta_m^n x\} = V_{-n}$. Applied to $G = \mu_m$ and $\rho' = \chi^n$, the theorem gives $\dim V_{-n} = -2 + r_n$ with $r_n = \#\{i : na_i \not\equiv 0\}$, and $\dim V_{-n}^{1,0} = -1 + \sum_i \langle na_i/m \rangle$, with $\langle\cdot\rangle$ the fractional part; replacing n by $-n$,

$$d_n = -1 + \sum_i \Big\langle \frac{-na_i}{m} \Big\rangle, \qquad \dim V_n = r_n - 2.$$

This agrees with [Moo10, Lemma 2.7, (3.2.2), 4.5], whose (n)-eigenspace is our V_n: his basis forms $y^n \prod_i (x - t_i)^{\lfloor -na_i/m \rfloor} x^\nu \, \mathrm{d}x$ are multiplied by ζ_m^n under γ^*, since $\gamma^* y = \zeta_m y$. Example 5.12 checks the bookkeeping on one cover.

Proof. (a) We compute the character χ_{H^1} of $H^1(C, \mathbb{C})$ and take inner products. Let $g \neq 1$. Triangulate $C/G = \mathbb{P}^1$ with the branch values among the vertices and lift the triangulation to C; then g is a simplicial homeomorphism whose fixed points are vertices, and its Lefschetz number $\sum_k (-1)^k \operatorname{tr}(g^* | H^k(C, \mathbb{Q}))$ equals the number of fixed points [Hat02, Thm. 2C.3 and Exercise 2C.4] (traces on H^k and on H_k agree by duality). Equivalently, at each fixed point g is $z \mapsto \epsilon z$ in a local coordinate, $\epsilon \neq 1$, and as a real linear map $\det(1 - \mathrm{d}g) = |1 - \epsilon|^2 > 0$, so each fixed point has index $+1$. This gives $1 - \operatorname{tr}(g^* | H^1) + 1 = \#\mathrm{Fix}(g)$. The character of $H^1(C, \mathbb{Q})$ is rational-valued, so $\operatorname{tr}((g^{-1})^*) = \operatorname{tr}(g^*)$ and

$$\chi_{H^1}(g) = 2 - \#\mathrm{Fix}(g) \quad (g \neq 1), \qquad \chi_{H^1}(1) = 2g(C).$$

The points over t_i are xP_i, $x \in G/\langle\gamma_i\rangle$, with stabilizers $x\langle\gamma_i\rangle x^{-1}$. Hence

$$\sum_{g \neq 1} \#\mathrm{Fix}(g)\,\overline{\chi'(g)} = \sum_i \frac{|G|}{e_i} \sum_{k=1}^{e_i - 1} \overline{\chi'(\gamma_i^k)} = \sum_i |G| \left(N_{i,0} - \frac{d}{e_i} \right),$$

where χ' is the character of ρ', and we used $\sum_{k=0}^{e-1} \overline{\chi'(\gamma^k)} = e\, N_0$ (the average of a representation of the cyclic group $\langle\gamma\rangle$ is the projection onto the fixed vectors). With $\sum_{g \neq 1} \overline{\chi'(g)} = |G|\,\delta - d$ and Proposition 5.4,

$$\dim M_{\rho'} = \frac{1}{|G|} \sum_g \chi_{H^1}(g) \overline{\chi'(g)} = \frac{(2g(C) - 2)d}{|G|} + 2\delta - \sum_i \Big(N_{i,0} - \frac{d}{e_i} \Big) = -2d + 2\delta + \sum_i (d - N_{i,0}).$$

(b) The same bookkeeping with the character of $H^{1,0}$. The Eichler trace formula [FK92, Ch. V] says that for an automorphism $f \neq 1$ of C, $\operatorname{tr}(f^* | H^{1,0}) = 1 + \sum_P \epsilon_P / (1 - \epsilon_P)$, the sum over the fixed points, where f acts on $T_P C$ by ϵ_P. (It follows from the holomorphic Lefschetz formula $\operatorname{tr}(f^* | H^{0,0}) - \operatorname{tr}(f^* | H^{0,1}) = \sum_P (1 - \epsilon_P)^{-1}$, $H^{0,1} = \overline{H^{1,0}}$; on $\mathbb{P}^1$ with $f(z) = \epsilon z$ both sides are 1.) Applied to $f = g^{-1}$, whose rotation numbers are $\bar\epsilon_P$, it gives $\chi_{1,0}(g) = 1 + \sum_{P \in \mathrm{Fix}(g)} (\epsilon_P - 1)^{-1}$, and $g = x\gamma_i^k x^{-1}$ acts at xP_i by $\epsilon = e^{2\pi \mathrm{i} k/e_i}$. Expanding $\overline{\chi'(\gamma_i^k)} = \sum_j N_{i,j} e^{-2\pi \mathrm{i} jk/e_i}$ and using

$$\frac{1}{e} \sum_{k=1}^{e-1} \frac{\varepsilon^{-jk}}{\varepsilon^k - 1} = \frac{j}{e} - \frac{e - 1}{2e} \qquad (\varepsilon = e^{2\pi \mathrm{i}/e},\ 0 \le j < e),$$

one finds $\dim \mathrm{Hom}_G(\rho', H^{1,0}) = (g(C)-1)d/|G| + \delta + \sum_i \sum_j N_{i,j}(\frac{j}{e_i} - \frac{e_i-1}{2e_i})$, which is (b) after inserting Proposition 5.4. The identity is Exercise 5.4. □

Chevalley and Weil [CW34] determined the representation on holomorphic differentials of any Galois cover of curves; the form (a), (b) for covers of $\mathbb{P}^1$ is the one used in the literature on families of G-covers. Three consequences are used repeatedly. First, the trivial representation has $\dim M_{\mathbf{1}} = 0$, since $H^1(\mathbb{P}^1) = 0$. Second, by (a) the multiplicity depends only on the ranks $\mathrm{rk}(\rho'(\gamma_i) - 1)$, the number of dimensions each local monodromy moves. Third, replacing every γ_i by its inverse (the other reading of Remark 5.3) replaces k by $e_i - k$ in (b) and exchanges $h^{1,0}$ and $h^{0,1}$.

Exercise 5.4. Prove the identity in the proof of (b). (Hint: for $j = 0$ pair k with $e-k$; then show that the left side increases by $1/e$ when j increases by 1, using $\varepsilon^{-(j+1)k} = \varepsilon^{-jk} - \varepsilon^{-jk}(1-\varepsilon^{-k})$ and $\sum_{k=0}^{e-1} \varepsilon^{-jk} = 0$ for $j \not\equiv 0$.)

Example 5.11 (Chevalley–Weil for the S_3-cover). For the cover of Example 5.5 (characters $\mathbf{1}, \mathrm{sgn}, \rho_2$ as in Example 3.4): the points over t_i fixed by a transposition τ are the xP_i with $x^{-1}\tau x \in \langle\gamma_i\rangle$, i.e. $x\gamma_i x^{-1} = \tau$; the two such x form one coset of $\langle\gamma_i\rangle$, so τ fixes exactly one point over each t_i, six in all, and acts there by -1. A 3-cycle lies in no stabilizer and has no fixed points. Hence

	1	τ	(123)
#Fix	–	6	0
χ_{H^1}	8	$2-6=-4$	2
$\chi_{1,0}$	4	$1+6\cdot(-\frac{1}{2})=-2$	1

(for τ, each fixed point contributes $(\epsilon-1)^{-1} = -\frac{1}{2}$). With class sizes $1, 3, 2$:

$$\langle\chi_{H^1}, \mathrm{sgn}\rangle = \tfrac{1}{6}(8+12+4) = 4, \quad \langle\chi_{H^1}, \rho_2\rangle = \tfrac{1}{6}(16+0-4) = 2, \quad \langle\chi_{H^1}, \mathbf{1}\rangle = \tfrac{1}{6}(8-12+4) = 0,$$

so $H^1(C,\mathbb{C}) \cong \mathrm{sgn}^{\oplus 4} \oplus \rho_2^{\oplus 2}$, and in the same way $H^{1,0} \cong \mathrm{sgn}^{\oplus 2} \oplus \rho_2$. The closed formulas agree: for ρ_2, each $\rho_2(\gamma_i)$ has eigenvalues $1, -1$, so (a) gives $-4+6\cdot 1 = 2$ and (b) gives $-2+6\cdot\frac{1}{2} = 1$; for sgn, $-2+6 = 4$ and $-1+6\cdot\frac{1}{2} = 2$. So M_{sgn} has Hodge numbers $(2,2)$ and M_{ρ_2} has $(1,1)$. Both are forced also by reality: sgn and ρ_2 are real representations, so complex conjugation maps $\mathrm{Hom}_G(\rho', H^{1,0})$ onto $\mathrm{Hom}_G(\rho', H^{0,1})$. The quotients of Example 5.5 see these pieces: $H^1(C)^{A_3} = H^1(C/A_3)$ is the sign part (dimension $4 = 2\cdot 2$), and $H^1(C)^{\langle a\rangle} = \rho_2^{\langle a\rangle} \otimes M_{\rho_2}$ is two-dimensional, $= H^1(E)$, since sgn has no vector fixed by a and ρ_2 has one (Frobenius reciprocity, Theorem 3.10). So $M_{\rho_2} \cong H^1(E)$ for the elliptic curve $E = C/\langle a\rangle$: a two-dimensional multiplicity space that is the cohomology of a curve. This is the toy version of step (ii), where every piece $W_{H,\psi}$ is an eigenspace of a cyclic quotient (Chapter 7).

Exercise 5.5. For the S_3-cover of Example 5.5: (a) verify $\chi_{1,0} = (4,-2,1)$ and decompose it; (b) compute $\dim M_{\rho'}$ and its Hodge numbers for an S_3-cover branched at $2k$ points with transpositions as local monodromies, and check $\sum_{\rho'} \dim \rho' \dim M_{\rho'} = 2g(C)$; (c) what changes if two of the six transpositions are replaced by the two 3-cycles $(123), (132)$ (check first that a generating tuple with product 1 exists)?

Example 5.12 (a cyclic cover as a G-cover). The μ_6-cover with exponents $(3,1,4,4)$ of Example 2.9 is the G-cover with $G = \mu_6 = \langle\gamma\rangle$ and tuple $(\gamma^3, \gamma, \gamma^4, \gamma^4)$, of orders $2, 6, 3, 3$; Proposition 5.4 gives $2g-2 = 6\,(-2+\frac{1}{2}+\frac{5}{6}+\frac{2}{3}+\frac{2}{3}) = 4$, $g = 3$. For $\chi^n(\gamma) = \zeta^n$, $\chi^n(\gamma^{a_i}) = \zeta^{na_i}$, so (a) gives $\dim M_{\chi^n} = -2 + \#\{i : na_i \not\equiv 0 \bmod 6\} = 2,1,0,1,2$ for $n = 1,\dots,5$, and (b) gives $h^{1,0}(M_{\chi^n}) = -1 + \sum_i \langle na_i/6\rangle = 1,0,0,1,1$. The pull-back dimensions $d_n = \dim V_n^{1,0}$ are $1,1,0,0,1$; the two lists match after $n \mapsto -n$, because M_{χ^n} is V_{-n}. In particular $d_1 = d_5 = 1$: the Φ_6-part is a Weil surface (Example 2.9).

Exercise 5.6. (a) Redo Example 5.12 with the inverse tuple $(\gamma^{-3}, \gamma^{-1}, \gamma^{-4}, \gamma^{-4})$, i.e. exponents $(3,5,2,2)$, and check that $h^{1,0}$ and $h^{0,1}$ of each M_{χ^n} are exchanged. (b) Show

in general, from Theorem 5.10(b), that replacing every γ_i by γ_i^{-1} replaces $h^{1,0}(M_{\rho'})$ by $\dim M_{\rho'} - h^{1,0}(M_{\rho'})$.

EXAMPLE 5.13 (the G_{32}-covers of the family). Let $G = G_{32} \subset \mathrm{GL}_4(K)$ and let the tuple have the branching type of the family: six reflections of determinant ω (eigenvalues $1,1,1,\omega$), six of determinant ω^2 (eigenvalues $1,1,1,\omega^2$) and one involution with eigenvalues $1,1,-1,-1$. Then $g(C) = 505441$ (Exercise 5.1), and for the reflection representation ρ, with $d = 4$ and $\delta = 0$, Theorem 5.10 gives

$$\dim M_\rho = -8 + 12\cdot 1 + 2 = 6, \qquad h^{1,0}(M_\rho) = -4 + 6\cdot\tfrac{1}{3} + 6\cdot\tfrac{2}{3} + 2\cdot\tfrac{1}{2} = 3,$$

so M_ρ has Hodge numbers $(3,3)$. The inverse reading of the tuple exchanges the roles of $\frac{1}{3}$ and $\frac{2}{3}$ and gives 3 again. For the other irreducibles one needs the character table of G_{32}. Evaluating (a) and (b) for all 102 irreducible characters (in GAP, from the eigenvalue multiplicities of $\rho'(\gamma_i)$) gives the totals $\sum_{\rho'} \dim\rho' \cdot \dim M_{\rho'} = 1010882 = 2g(C)$ and $\sum_{\rho'} \dim\rho' \cdot h^{1,0}(M_{\rho'}) = 505441 = g(C)$, as they must be. The 14 irreducibles with central character κ_0, the ones that enter step (ii) (Exercise 3.2), are:

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$\dim\rho_k$	4	4	20	20	20	20	20	36	36	60	60	60	64	80
$\dim M_{\rho_k}$	6	42	102	102	138	174	174	162	270	342	378	450	384	552
$h^{1,0} = h^{0,1}$	3	21	51	51	69	87	87	81	135	171	189	225	192	276

The first column is $\rho_1 = \rho$. We number these irreducibles $\rho_1,\dots,\rho_{14}$ in the order of the table, by degree and then by $\dim M_{\rho'}$; in the two pairs of degree 20 with equal $\dim M_{\rho'}$ (102 and 174) the order is fixed by the character value at γ_1, listed in Table 7.2. Where no confusion is possible we also name an irreducible by the pair $(\dim\rho', \dim M_{\rho'})$. All fourteen are balanced, $h^{1,0} = h^{0,1}$, and $\sum \dim\rho' \cdot \dim M_{\rho'} = 168480$, the dimension of the κ_0-part (Exercise 5.1).

5.5. Monodromy and parabolic cohomology

Over a simply connected open subset of a Hurwitz component the curves form a smooth projective family (Section 5.3), a family of polarized algebraic manifolds in the sense of [Sch73, §2], so $H^1(C_t,\mathbb{Q})$ is a local system underlying a polarized variation of Hodge structure [Sch73, §2]; over the whole component it is a local system up to the action of $Z(G)$, which on $M_{\rho'}$ is multiplication by the scalars of the central character of ρ'. Monodromy commutes with the fibrewise action of G (it is induced by continuation along paths of a G-equivariant family) and preserves the cup product. Hence each $M_{\rho'}$, and each $\mathcal{B}_C$, is a local system on the base in its own right, with monodromy acting by composition $f \mapsto T \circ f$. To compute this monodromy without the genus-505441 curve, one passes to the base.

Recall that a *local system* on a connected, locally simply connected space X is a locally constant sheaf of finite-dimensional vector spaces; taking the stalk at a base point x_0 with the action of loops by continuation is an equivalence with finite-dimensional representations of $\pi_1(X,x_0)$ [Voi03, §3.1.1]. Let $U = \mathbb{P}^1 \smallsetminus \{t_1,\dots,t_r\}$ and let ρ' be a representation on V. The tuple defines a homomorphism $\pi_1(U,t_0) \to G \to \mathrm{GL}(V)$, $\ell_i \mapsto g_i := \rho'(\gamma_i)$, hence a local system $\mathbb{L}$ of rank $\dim V$ on U. Its first cohomology is the group cohomology $H^1(\pi_1(U),V)$: cocycles $u\colon \pi_1 \to V$ with $u(xy) = u(x) + x\,u(y)$, modulo coboundaries $u(x) = (x-1)v$. A cocycle is determined by $u_i = u(\ell_i)$, subject to the single condition coming from $\ell_1\cdots\ell_r = 1$.

DEFINITION 5.14 (parabolic cohomology). The *parabolic cohomology* $H^1_{\mathrm{par}}(U,\mathbb{L})$ is the quotient Z^1_{par}/B^1, where

$$Z^1_{\mathrm{par}} = \Big\{(u_1,\dots,u_r) \in V^r : u_i \in \mathrm{im}(g_i - 1),\ \sum_{i=1}^{r} g_1\cdots g_{i-1}u_i = 0\Big\},$$
$$B^1 = \{((g_1-1)v,\dots,(g_r-1)v) : v \in V\}.$$

The condition $u_i \in \operatorname{im}(g_i - 1)$ says that the cocycle restricted to a small loop around t_i is a coboundary: the class extends over the puncture. Sheaf-theoretically, $H^1_{\mathrm{par}} = H^1(\mathbb{P}^1, j_*\mathbb{L})$ for $j\colon U \hookrightarrow \mathbb{P}^1$, the image of compactly supported cohomology in ordinary cohomology [DW06]. The link with the curve is the following.

PROPOSITION 5.15 (parabolic cohomology and multiplicity spaces). *Let ρ' be irreducible and nontrivial on V, with local monodromies $g_i = \rho'(\gamma_i)$ generating an irreducible group.*

(a) $\dim H^1_{\mathrm{par}}(U, \mathbb{L}) = \sum_i \operatorname{rk}(g_i - 1) - 2\dim V$, *which equals* $\dim M_{\rho'}$ *by Theorem 5.10(a).*

(b) $H^1_{\mathrm{par}}(U, \mathbb{L})$ *is identified, compatibly with the monodromy of the family, with* $M_{\rho'} = \operatorname{Hom}_G(\rho', H^1(C, \mathbb{C}))$ *or with* $M_{\rho'^\vee}$*; which of the two depends on the direction conventions of Remark 5.3, and the choice does not affect the determinant class of an invariant Hermitian form (Remark 2.11).*

Part (a) is linear algebra: the map $\prod_i \operatorname{im}(g_i - 1) \to V$, $(u_i) \mapsto \sum_i g_1 \cdots g_{i-1} u_i$, is onto when no nonzero linear form on V is fixed by all g_i (its image is $\sum_i g_1 \cdots g_{i-1} \operatorname{im}(g_i - 1)$, whose annihilator is the fixed forms), so $\dim Z^1_{\mathrm{par}} = \sum \operatorname{rk}(g_i - 1) - \dim V$; and $v \mapsto ((g_i - 1)v)$ is injective when no nonzero vector is fixed by all g_i. Part (b) is Shapiro's lemma for the cover $C^\circ = \pi^{-1}(U)$ together with the identification of the classes that extend over the punctures; we use it as stated in [DW06]. For the family: $\dim Z^1_{\mathrm{par}} = 12 + 2 - 4 = 10$, $\dim B^1 = 4$, and an exact computation over K confirms $\dim_K H^1_{\mathrm{par}} = 6 = \dim M_\rho$.

EXERCISE 5.7. Let $\mathbb{L}$ be a rank-one local system on $U = \mathbb{P}^1 \smallsetminus \{t_1, \dots, t_r\}$ with monodromies $c_i \in \mathbb{C}^\times$, $\prod_i c_i = 1$, not all equal to 1. (a) Show $\dim H^1_{\mathrm{par}}(U, \mathbb{L}) = \#\{i : c_i \neq 1\} - 2$. (b) Compare with $\dim V_n = r_n - 2$ for the eigenspaces of a cyclic cover (Section 5.4). (c) What is $\dim H^1(U, \mathbb{L})$, without the parabolic condition, and which classes does the parabolic condition remove?

Braids acting on cocycles. A half-twist σ_j acts on $\pi_1(U)$ by the automorphism β with $\ell_j \mapsto \ell_j \ell_{j+1} \ell_j^{-1}$, $\ell_{j+1} \mapsto \ell_j$, fixing the other ℓ_i and the product $\ell_1 \cdots \ell_r$. Composing with β turns the tuple into its Hurwitz move and a cocycle u into $u \circ \beta$, whose components are, by the cocycle rule,

$$u'_j = u(\ell_j \ell_{j+1} \ell_j^{-1}) = u_j + g_j u_{j+1} - g_j g_{j+1} g_j^{-1} u_j, \qquad u'_{j+1} = u_j, \qquad u'_i = u_i \ (i \neq j, j+1).$$

This preserves the parabolic condition and coboundaries. A braid b that returns the tuple to itself therefore acts on H^1_{par} for the *same* local system, and this action is the monodromy of the family of parabolic cohomologies along the loop b in the Hurwitz space, or its inverse, according to the direction convention [DW06]. Such braids are easy to produce: if $\gamma_j = \gamma_{j+1}$, σ_j itself; in general, take any braid word w, apply it, find the least k with σ_j^k restoring the pair at positions $j, j+1$, and undo w. The braid $w\sigma_j^k w^{-1}$ stabilizes the tuple. The 67 matrices used for the family in Example 5.19 are of this form.

EXAMPLE 5.16 (a braid matrix and an invariant form, for the S_3-cover). Take the tuple (a, a, b, b, c, c) of Example 5.5 and $\rho' = \rho_2$, realized over $\mathbb{Q}$ on $\{x \in \mathbb{Q}^3 : \sum x_i = 0\}$ with basis $e_1 - e_2$, $e_2 - e_3$. Each $g_i = \rho_2(\gamma_i)$ is a reflection, so Proposition 5.15(a) gives $\dim Z^1_{\mathrm{par}} = 6 - 2 = 4$, $\dim B^1 = 2$ and $\dim H^1_{\mathrm{par}} = 2 = \dim M_{\rho_2}$, as computed in Example 5.11. Since $\gamma_1 = \gamma_2$, the half-twist σ_1 stabilizes the tuple; in a suitable basis of H^1_{par} it acts by

$$\begin{pmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{pmatrix},$$

a transvection: unipotent with $(T - 1)^2 = 0$. This fits the geometry: as t_1 and t_2, which carry the same transposition, come together, the elliptic curve $E = C/\langle a \rangle$ with $H^1(E) \cong M_{\rho_2}$ acquires a node (two simple branch points of the three-sheeted cover $E \to \mathbb{P}^1$ collide), and the half-twist acts on $H^1(E)$ by the Dehn twist about the vanishing circle, by the Picard–Lefschetz formula [Voi03, §3.2.1], as in the Legendre family of Section 6.4. Eleven further matrices from braids $w\sigma_j^k w^{-1}$ all have determinant 1; the space of bilinear forms X with $T^{\mathsf{T}} X T = X$

for all twelve is one-dimensional, spanned by an alternating form, the cup product up to a scalar, and the Hermitian form $\mathrm{i}X$ has signature $(1,1)$, the signature of H^1 of an elliptic curve. This is the computation of Example 2.13 in miniature: a few monodromy matrices cut the invariant forms down to a line, and the line contains the polarization (Lemma 2.10(d)).

5.6. The Weil sixfold $\mathcal{B}_C$ and the component $\mathcal{H}$

Definition 5.17 (the Hodge structure $\mathcal{B}_C$). Let $G = G_{32}$, $\rho\colon G \to \mathrm{GL}_4(K)$ its reflection representation and $V_\rho = K^4$ viewed as a $\mathbb{Q}[G]$-module (Proposition 3.27). For a G-cover $C \to \mathbb{P}^1$ put

$$\mathcal{B}_C = \mathrm{Hom}_{\mathbb{Q}[G]}(V_\rho, H^1(C,\mathbb{Q})),$$

a K-vector space through $(af)(v) = f(av)$, $a \in K = \mathrm{End}_{\mathbb{Q}[G]}(V_\rho)$.

Proposition 5.18 ($\mathcal{B}_C$ is a Weil sixfold). *Let C be a G_{32}-cover of the branching type of the family. Fix a G-invariant positive definite Hermitian form k on V_ρ over K and put $s = \mathrm{Tr}_{K/\mathbb{Q}}\, k$.*

(a) $\mathcal{B}_C$ is a rational Hodge structure of weight one with $\mathcal{B}_C \otimes \mathbb{C} = M_\rho \oplus M_{\bar\rho}$, the two eigenspaces of K. By Example 5.13, $\dim_K \mathcal{B}_C = \dim M_\rho = 6$, and the Hodge numbers of M_ρ and of $M_{\bar\rho}$ are $(3,3)$.

(b) For $u \in V_\rho \smallsetminus \{0\}$, the evaluation $f \mapsto f(u)$ embeds $\mathcal{B}_C$ into $H^1(C,\mathbb{Q})$ as a sub-Hodge structure, and $E_u(f,f') = E_C(f(u), f'(u))$, with E_C the cup-product polarization, is a polarization of $\mathcal{B}_C$, invariant under monodromy, with $E_u(af,f') = E_u(f,\bar a f')$ for $a \in K$.

(c) For $f, f' \in \mathcal{B}_C$ there is a unique $\beta(f,f') \in K$ with $E_C(f(v), f'(v')) = s(\beta(f,f')v, v')$ for all $v, v' \in V_\rho$. The map β is K-linear in f, conjugate-linear in f', skew-Hermitian, and

$$E_u = k(u,u)\ \mathrm{Tr}_{K/\mathbb{Q}}\,\beta, \qquad h_u = k(u,u)\,\frac{\beta}{\theta}$$

is the Hermitian form of E_u in the sense of Definition 2.1. In particular the polarizations E_u for different u are positive rational multiples of one another.

Hence $\mathcal{B}_C = H^1(A_C,\mathbb{Q})$ for an abelian sixfold A_C, defined up to isogeny, of Weil type for K; this is the sixfold A_C of the whole text. Its Hermitian form h_u has signature $(3,3)$ by the argument of Proposition 2.15(a), which uses the signature only in its last sentence (Exercise 2.8), since K acts on $\mathrm{Lie}\, A_C$ with multiplicities $(3,3)$.

Proof. (a) G acts on $H^1(C,\mathbb{Q})$ by morphisms of Hodge structures, so the G-equivariant homomorphisms from V_ρ (pure of type $(0,0)$) form a sub-Hodge structure of $\mathrm{Hom}(V_\rho, H^1(C,\mathbb{Q}))$, of weight one. By Proposition 3.27(a), $\mathcal{B}_C \otimes \mathbb{C} = \mathrm{Hom}_G(\rho \oplus \bar\rho, H^1(C,\mathbb{C})) = M_\rho \oplus M_{\bar\rho}$, with $a \in K$ acting by its two complex embeddings, and the Hodge decomposition is that of the multiplicity spaces. By Proposition 3.27(d) and Example 5.13, $\dim_K \mathcal{B}_C = \dim M_\rho = 6$ with Hodge numbers $(3,3)$; complex conjugation maps $\mathrm{Hom}_G(\rho, H^{1,0})$ onto $\mathrm{Hom}_G(\bar\rho, H^{0,1})$, so $M_{\bar\rho}$ has the Hodge numbers of M_ρ in reverse order, again $(3,3)$.

(b) Evaluation at u is a morphism of Hodge structures and injective by Proposition 3.27(e). The restriction of a polarization to a sub-Hodge structure is a polarization, since positivity of $E_C(x, Cx)$ is inherited by subspaces stable under the Weil operator C. Monodromy preserves E_C and acts on f by composition, so it preserves E_u. For compatibility with K: for fixed f, f' the form $(v,v') \mapsto E_C(f(v), f'(v'))$ is G-invariant on V_ρ, hence equal to $s(bv, v')$ for some $b \in K$; the s-adjoint of $a \in K$ is $\bar a$, since $s(av,v') = \mathrm{Tr}_{K/\mathbb{Q}}(a\,k(v,v')) = s(v, \bar a v')$; so $E_u(af,f') = s(bau,u) = s(bu,\bar a u) = E_u(f,\bar a f')$. Part (c) refines this computation.

(c) The form $(v,v') \mapsto E_C(f(v), f'(v'))$ is $\mathbb{Q}$-bilinear and G-invariant, since f, f' are G-maps and E_C is G-invariant. A G-invariant bilinear form is a G-map $V_\rho \to V_\rho^\vee$; composing with the inverse of $v \mapsto s(v,\cdot)$, which is a G-isomorphism, gives an element of $\mathrm{End}_{\mathbb{Q}[G]}(V_\rho) = K$. This is $\beta(f,f')$. Since $s(av,v') = \mathrm{Tr}(a\,k(v,v')) = s(v,\bar a v')$, the s-adjoint of a is $\bar a$; together with $(af)(v) = f(av)$ this gives $\beta(af,f') = a\,\beta(f,f')$ and $\beta(f,af') = \bar a\,\beta(f,f')$. Skew-symmetry of E_C and symmetry of s give $s(\beta(f',f)v',v) = -s(\beta(f,f')v,v') = -s(v', \overline{\beta(f,f')}v)$,

so $\beta(f',f) = -\overline{\beta(f,f')}$. Now $E_u(f,f') = s(\beta u,u) = \operatorname{Tr}_{K/\mathbb{Q}}(\beta\, k(u,u)) = k(u,u)\operatorname{Tr}_{K/\mathbb{Q}}\beta$, because $k(u,u)$ is a positive rational number. The form $h_u = k(u,u)\beta/\theta$ is K-linear in the first variable, Hermitian (as $\bar\theta = -\theta$ and β is skew-Hermitian), and $\operatorname{Tr}_{K/\mathbb{Q}}(\theta h_u) = E_u$; by the uniqueness in Definition 2.1 [Del82, Lemma 4.6] it is the Hermitian form of E_u. The compatibility $E_u(af,f') = E_u(f,\bar a f')$ follows from the sesquilinearity of β. □

Exercise 5.8. Explain why $\dim_K \mathcal{B}_C = \dim_{\mathbb{C}} M_\rho$, in two ways: (a) from $\mathcal{B}_C \otimes_{\mathbb{Q}} \mathbb{C} = M_\rho \oplus M_{\bar\rho}$ and complex conjugation; (b) from $\mathcal{B}_C \cong \operatorname{ev}_u(\mathcal{B}_C) \subset H^1(C,\mathbb{Q})$ and the complexification $u'\otimes M_\rho \oplus u''\otimes M_{\bar\rho}$ of the image, where $u = u'+u''$ in $\rho\oplus\bar\rho$. Then check, with Theorem 5.10, that the branching type of the family gives $\dim_{\mathbb{Q}} \mathcal{B}_C = 12 = 2\dim A_C$.

Example 5.19 (the component $\mathcal{H}$). The family of the proof is the Hurwitz component $\mathcal{H}$ of the tuple $\boldsymbol{\gamma} = (\gamma_1,\dots,\gamma_{13})$ printed in Chapter C, matrices in $\mathrm{GL}_4(K)$ with entries in $\frac{1}{36}\mathbb{Z}[\omega]$. Exact computations over K (in GAP) establish the facts used in this chapter: $\gamma_1\cdots\gamma_{13} = 1$; the group the γ_i generate has order 155520, 102 conjugacy classes, a scalar centre of order 6 and 80 reflections, all of order 3 (the data of G_{32}, Chapter 4); $\gamma_1,\dots,\gamma_{12}$ are reflections of order 3 in classes of size 40, with $\det\gamma_i = \omega$ exactly for $i \in \{2,3,4,7,8,12\}$, and γ_{13} is an involution with $\operatorname{rk}(\gamma_{13}-1) = 2$ in the class of size 90, so the tuple has the branching type of the family; and $\dim_K H^1_{\mathrm{par}} = 6$. (The facts about the boundary point that Chapter 6 needs are stated there.) From 67 nontrivial braid-monodromy matrices on H^1_{par}, given by braids of the form $w\sigma_j^k w^{-1}$, one finds a one-dimensional space of invariant sesquilinear forms, spanned by a Hermitian form of signature $(3,3)$ and determinant $-128 = -2^7$. With Lemma 2.10 and Proposition 5.18 this is Example 2.13: every member of $\mathcal{H}$ is a Weil sixfold with $\det h \equiv -2$, $\operatorname{disc} h = 2$, not split. The component $\mathcal{H}$ has dimension 13 and its curves depend on 10 moduli (Section 5.3); that its period map is dominant is Chapter 6.

What just happened. A G-cover of the line is a tuple of group elements with product 1, one per branch point; its genus is Riemann–Hurwitz, and its representation on H^1, Hodge numbers included, is Chevalley–Weil, read off from the eigenvalues of the local monodromies. Moving the branch points acts on tuples by Hurwitz moves, and the components of the space of covers are the braid orbits. For G_{32} and the branching type of the family the multiplicity space of the reflection representation has dimension 6 and Hodge numbers $(3,3)$, so $\mathcal{B}_C = \operatorname{Hom}_{\mathbb{Q}[G]}(V_\rho, H^1(C,\mathbb{Q}))$ is a Weil sixfold, polarized by $E_u(f,f') = E_C(f(u),f'(u))$, whose Hermitian form is $k(u,u)\beta/\theta$ for one K-valued cup product β. Its monodromy over the component $\mathcal{H}$ is visible on parabolic cohomology, where braids that fix the tuple act by explicit matrices; invariance under 67 of them pins down the Hermitian form up to a rational scalar, with determinant $-128 \equiv -2$.

Remark 5.20 (Where this is used). Theorem 5.2 and Definition 5.6 and Example 5.19 define the family of step (i), whose dominance is Chapter 6 and whose determinant class is computed in Example 2.13. Theorem 5.10 and the table of Example 5.13 give the dimensions $\dim M_{\rho'}$ used in step (ii) to check $\dim W_j = \sum_{\rho'} v_j(\rho')\dim M_{\rho'}$ and the balance of the pieces (Chapter 7); the same formula, for cyclic quotients, gives the local exponents and genera of the pieces, and in Chapter 6 it gives $\dim F^1_\rho = 3$ at smooth fibres. Proposition 5.18 gives the Weil sixfold A_C whose Weil classes step (vi) makes algebraic (Chapter 11) and whose period point is moved around in step (vii) (Chapter 12). Parabolic cohomology and the Dehn-twist logarithms computed on it reappear in the first-order term of the dominance certificate (Chapter 6).

CHAPTER 6

The period map of the G_{32} family and the rank-nine certificate

Proof steps: (i)(b) dominance; used in (vii)

Where we are. This chapter explains the dominance half of step (i): the period map of the family $\mathcal{H}$ of G_{32}-covers (Example 5.19) to the nine-dimensional Shimura variety of class 2 has rank 9 at a generic point, so its image contains a dense open subset. This is what lets step (vii) (Chapter 12) spread the algebraicity of Weil classes from the members of one family to every point of $\Gamma'\backslash X^+$. The argument has three layers, and the chapter keeps them apart: standard facts from Hodge theory and the deformation theory of curves, listed as (S1)–(S7) in Section 6.1; two lemmas and an expansion, derived in this chapter; and one computation in ball arithmetic, which certifies that a certain 9×9 determinant, computed from those formulas, is not zero. What the certificate does and does not establish is spelled out in Section 6.6.

Background: period domains and period maps [Sch73, §3], Griffiths transversality [Sch73, (2.13)], the Picard–Lefschetz formula [Voi03, §3.2.1]; the domain X^+ (Definition 2.14 and Proposition 2.15); the family, $\mathcal{B}_C$, M_ρ and parabolic cohomology (Definition 5.17, Proposition 5.15, and Example 5.19); the conjugacy classes of the branching type (Section 4.5). Notation as in Chapter 5: $G = G_{32}$, ρ its reflection representation over $K = \mathbb{Q}(\sqrt{-3})$, $\bar{\rho} \cong \rho^\vee$, $M_\rho = \mathrm{Hom}_G(\rho, H^1(C, \mathbb{C}))$, ϕ denotes forms in F^1_ρ and η forms in $F^1_{\bar{\rho}}$.

6.1. Dominance in two pages

A point of $\mathcal{H}$ is a G-cover $C \to \mathbb{P}^1$ branched at 13 points, and C determines a Weil sixfold through $\mathcal{B}_C = \mathrm{Hom}_{\mathbb{Q}[G]}(V_\rho, H^1(C, \mathbb{Q}))$ (Definition 5.17). Its Hodge structure is the three-dimensional subspace $F^1_\rho = \mathrm{Hom}_G(\rho, H^0(C, \Omega_C))$ of the six-dimensional M_ρ, a point of the nine-dimensional domain X^+ of Proposition 2.15. The *period map* $\mathcal{P}$ sends C to this point, modulo monodromy. The base $\mathcal{H}$ has dimension 13, but Möbius transformations of the branch points do not change the curve, so the curves depend on $13 - 3 = 10$ moduli (Section 5.3); the target has dimension 9, and the claim is that the derivative of $\mathcal{P}$ has rank 9 somewhere, hence at a generic point. The generic fibre then has dimension $13 - 9 = 4$ in $\mathcal{H}$, three of its directions being Möbius ones, and dimension $10 - 9 = 1$ in moduli.

The derivative is computable in principle. By Griffiths' formula the codifferential of a period map of curves is a multiplication of holomorphic forms: two forms $\phi \in F^1_\rho$ and $\eta \in F^1_{\bar{\rho}}$ multiply to a G-invariant quadratic differential $\phi\eta$, and the codifferential sends $\phi \otimes \eta$ to the linear form "pair $\phi\eta$ with the deformation" (Proposition 6.1). So the rank of $\mathcal{P}$ at C is the rank of a 9×10 matrix: rows indexed by a basis $\phi_a \otimes \eta_b$ $(a, b = 1, 2, 3)$, columns by the ten deformation directions of the curve (the thirteen motions of the branch points modulo the three Möbius ones). In practice it is not computable at a smooth member, because C has genus 505441 and no equation; its holomorphic forms are known only through their local behaviour at the branch points.

The way out is a degeneration. Let five pairs of neighbouring branch points collide, $\{1, 2\}, \{3, 4\}, \{6, 7\}, \{8, 9\}, \{11, 12\}$. The curve breaks into pieces (Figure 6.1): a main piece over a $\mathbb{P}^1$ with eight special points, on which the monodromy is abelian, so that forms are explicit products of powers $(x - p_j)^{e_j}$; four small pieces over the clusters whose two monodromies are inverse to each other; and one small piece over $\{3, 4\}$ with non-abelian

monodromy. Near the broken curve one reopens each node by a parameter t_e (*plumbing*, $XY = t_e$) and keeps five parameters τ moving the special points of the main piece. This is a ten-parameter family $\mathcal{C} \to \Delta^{10}$ whose generic members lie in $\mathcal{H}$ (or in its complex conjugate, which has the same rank). The codifferential matrix $M(t, \tau)$ of this family extends holomorphically to the broken curve $t = 0$. Rank can only drop in a limit (lower semicontinuity), so a limit matrix of rank 9 at $t = 0$ proves rank 9 nearby, hence generically (Section 6.3).

At $t = 0$ the four node columns are not independent, because the two nodes $(6, 7)$ and $(8, 9)$ have the same monodromy logarithm on M_ρ ($N_{(6,7)} = N_{(8,9)}$, an exact computation), so the naive limit loses a direction. The missing direction comes from the next order in t: along the combination of directions that kills the order-zero term, the matrix is ϵ times a first-order column F built from the values of the forms at the nodes (Equation (6.4)). The limit matrix $M^* = [R \,|\, T \,|\, F]$ (R node columns, T point-moving columns, F the first-order column) is computed from exact frames and series coefficients, and one of its 9×9 minors is evaluated in ball arithmetic at 320 bits:

$$\det = [2.6207710\ldots \cdot 10^{-29} \pm 3.7 \cdot 10^{-106}] + [-2.9287922\ldots \cdot 10^{-29} \pm 4.6 \cdot 10^{-106}]\,\mathrm{i},$$

a ball that does not contain 0 (Theorem 6.6).

Standard inputs. The argument uses the following standard facts.

(S1) Griffiths' description of the differential of the period map as cup product with the Kodaira–Spencer class, so that the codifferential of a period map of curves is the multiplication map [CFG15, p. 4 of the arXiv version]; its isotypic form is Proposition 6.1.
(S2) Grauert's base change theorem [Gra60] for $\pi_*\Omega_{\mathcal{C}/\Delta}$.
(S3) The Chevalley–Weil formula (Theorem 5.10).
(S4) Rosenlicht's description of the dualizing sheaf of a nodal curve (simple poles with opposite residues) [Ser88].
(S5) The pairing of point-moving and node-opening directions with quadratic differentials by residues, consistent with [Wol13, Lemma 4].
(S6) The monodromy of the plumbing $XY = t$ around $t = 0$ is the Dehn twist about the vanishing circle, the case of complex dimension one of the Picard–Lefschetz formula [Lam81], [Voi03, §3.2.1], together with the formula for the action of a Dehn twist on H^1 [FM12].
(S7) Lower semicontinuity of the rank of a holomorphic matrix, and the chain rule (Section 6.3).

6.2. The period map and its codifferential

The period map. Over $\mathcal{H}$ the spaces $M_\rho = \mathrm{Hom}_G(\rho, H^1(C, \mathbb{C}))$ form a flat bundle of rank 6, with flat structure from the local system $\mathcal{B}_C$ (Section 5.6; flat up to the scalars by which $Z(G)$ acts, which do not move subspaces, Section 5.5), and the subspaces $F^1_\rho = \mathrm{Hom}_G(\rho, H^0(C, \Omega_C))$ form a holomorphic subbundle of rank 3: the Hodge bundle F^1 of the family of curves is a holomorphic subbundle by Griffiths' theorem [Sch73, (2.13)], and F^1_ρ is its image under the ρ-isotypic projection. In the language of Proposition 2.15, F^1_ρ is the subspace on which K acts through one embedding; it determines the complex structure J, so a point of X^+. The *period map* is $\mathcal{P}\colon \mathcal{H} \to \Gamma\backslash X^+$, $C \mapsto F^1_\rho(C)$, with Γ the monodromy group; it is holomorphic [Sch73, (3.27)], and after passing to a finite étale cover on which monodromy lies in a torsion-free group Γ' it is a map of algebraic varieties (Borel; Chapter 12). In weight one the Hodge filtration has only the step $F^1 \subset F^0 = H^1$, so Griffiths transversality $\nabla F^1 \subset F^0 \otimes \Omega^1$ imposes no condition: the whole tangent space of X^+ is available, and the question is only how much of it the family reaches.

The tangent space of the Grassmannian of 3-planes in M_ρ at F^1_ρ is $\mathrm{Hom}(F^1_\rho, M_\rho/F^1_\rho)$, of dimension $3 \cdot 3 = 9$, and this is the tangent space of the open subset X^+ (Proposition 2.15(c): graphs of 3×3 matrices Z). The polarization pairs M_ρ with $M_{\bar\rho}$ through $Q_\rho(f, f') = \sum_i Q(f(e_i), f'(e^i))$, where $e_1, \ldots, e_4$ is a basis of ρ and $e^1, \ldots, e^4$ the dual basis of $\rho^\vee \cong \bar\rho$,

and F^1_ρ is orthogonal to $F^1_{\bar\rho}$, so $M_\rho/F^1_\rho \cong (F^1_{\bar\rho})^\vee$ and

$$T_{F^1}X^+ \cong \operatorname{Hom}(F^1_\rho, (F^1_{\bar\rho})^\vee) \cong (F^1_\rho \otimes F^1_{\bar\rho})^\vee, \qquad \dim = 9. \tag{6.1}$$

So the cotangent space of X^+ is $F^1_\rho \otimes F^1_{\bar\rho}$, and the codifferential of $\mathcal{P}$ is a linear map from this nine-dimensional space to the cotangent space of the base, with values in the ten-dimensional space of covectors that vanish on the Möbius directions, along which $\mathcal{P}$ is constant.

The cotangent space of the base. The base of a family of G-covers of $\mathbb{P}^1$ is (étale over) the configuration space $\mathcal{U}_{13}$ of the branch points, of dimension 13. The covectors that vanish on the three Möbius directions form the cotangent space of the space of 13 points on $\mathbb{P}^1$ modulo $\mathrm{PGL}_2(\mathbb{C})$, that is, of $\mathcal{M}_{0,13}$: the quadratic differentials on $\mathbb{P}^1$ with at most simple poles at $t_1, \dots, t_{13}$, $H^0(\mathbb{P}^1, \Omega^{\otimes 2}_{\mathbb{P}^1}(t_1 + \cdots + t_{13}))$, of dimension $-4 + 13 + 1 = 10$ (a line bundle of degree 9 on $\mathbb{P}^1$). Pulled back to C these are exactly the G-invariant holomorphic quadratic differentials, $H^0(C, \Omega^{\otimes 2}_C)^G$, and the pairing with a tangent vector is Serre duality with the Kodaira–Spencer class, which for the direction that moves t_j alone is the residue at t_j (up to a universal constant). This identification is standard in the study of families of Galois covers; we follow [CFG15, p. 4 of the arXiv version], where it is used in the same form.

Proposition 6.1 (the codifferential is a multiplication map). *At a member C of $\mathcal{H}$, with the identification (6.1) and the identification of the covectors on $\mathcal{H}$ that vanish on the Möbius directions with $H^0(C, \Omega^{\otimes 2}_C)^G$, the codifferential of $\mathcal{P}$ is, up to a sign fixed by conventions, the multiplication map*

$$\mu_C \colon F^1_\rho \otimes F^1_{\bar\rho} \longrightarrow H^0(C, \Omega^{\otimes 2}_C)^G, \qquad \phi \otimes \eta \longmapsto \phi\eta = \sum_{i=1}^{4} \phi(e_i)\,\eta(e^i).$$

In particular $\operatorname{rk} d\mathcal{P}_C = \operatorname{rk} \mu_C$*, and $\mathcal{P}$ has generic rank 9 if and only if μ_C is injective at one member C.*

Proof. Let $\xi \in H^1(C, T_C)$ be the Kodaira–Spencer class of a tangent vector v. Griffiths' formula (S1) says that the derivative of $F^1H^1(C)$ along v is cup product with ξ, $F^1 \to H^1/F^1$, and that $Q(\xi \cup \alpha, \alpha') = \pm\langle \xi, \alpha\alpha' \rangle$ for holomorphic forms α, α' (Serre duality between $H^1(T_C)$ and $H^0(\Omega^{\otimes 2}_C)$). The family is G-equivariant, so ξ is G-invariant and cup product with ξ commutes with G; hence the derivative of $\mathcal{P}$ is composition with $\xi\cup$, from F^1_ρ to $\operatorname{Hom}_G(\rho, H^1/F^1) = M_\rho/F^1_\rho \cong (F^1_{\bar\rho})^\vee$. Evaluating on $\phi \otimes \eta$ gives $Q_\rho(\xi \cup \phi, \eta) = \pm \sum_i \langle \xi, \phi(e_i)\eta(e^i) \rangle = \pm\langle \xi, \phi\eta \rangle$. The sum $\sum_i e_i \otimes e^i$ is the invariant identity tensor of $\rho \otimes \rho^\vee$, so $\phi\eta$ does not depend on the basis and is G-invariant. The final sentence is linear algebra plus Section 6.3. □

Intuition 6.2 (Why multiplication). A first-order deformation of a curve is a way of tilting $\mathrm{d}z$ by a little $\mathrm{d}\bar z$ (a Beltrami differential). A holomorphic form ϕ then acquires a small anti-holomorphic part, and the size of that part, measured against another holomorphic form η, is $\int \phi\,\eta\,\bar\mu$, where $\bar\mu$ is the Beltrami differential. This integral only sees the product $\phi\eta$, a quadratic differential. So "how much does the Hodge structure move" is "how many independent quadratic differentials do the products $\phi\eta$ give", and dominance is the statement that the nine products $\phi_a\eta_b$ are linearly independent among the ten invariant quadratic differentials. This criterion is not evaluated at a smooth point, for the reason in the next paragraph.

Exercise 6.1 (dimension count). (a) Show that the configuration space of 13 distinct points of $\mathbb{P}^1$ modulo $\mathrm{PGL}_2(\mathbb{C})$ has dimension 10, and that $H^0(\mathbb{P}^1, \Omega^{\otimes 2}_{\mathbb{P}^1}(t_1 + \cdots + t_{13}))$ has dimension 10. (b) Using (6.1), show that a dominant $\mathcal{P}$ has generic fibres of dimension 4 in the 13-dimensional $\mathcal{H}$, and of dimension 1 modulo the Möbius directions.

Why not compute at a smooth member. To evaluate μ_C one needs bases of F^1_ρ and $F^1_{\bar\rho}$. On $\mathbb{P}^1$ minus the branch points, a form $\phi \in F^1_\rho$ is a vector-valued multivalued differential $f(x)\,\mathrm{d}x$, f with values in $\mathbb{C}^4$, which changes by $\rho(\gamma_j)$ around t_j and has prescribed local exponents. For abelian monodromy such f are products $\prod_j (x-t_j)^{e_j}$ times polynomials, as for cyclic covers, whose forms are $y^n \prod_i (x-t_i)^{\lfloor -na_i/m \rfloor} x^\nu \,\mathrm{d}x$ [Moo10, (2.6.1), Lemma 2.7]; for the non-abelian G_{32} they are solutions of a rank-four Fuchsian system with thirteen singular points, whose accessory parameters are not explicit. The degeneration replaces this by explicit algebraic data on the pieces.

6.3. One point suffices: lower semicontinuity and limits

The logic that turns a computation at one (even degenerate) point into a statement about the generic point uses only two facts about matrices.

Proposition 6.3 (rank is lower semicontinuous). *Let $M(s)$ be a $p \times q$ matrix depending continuously on s in a topological space S. For every k the set $\{s : \operatorname{rk} M(s) \geq k\}$ is open. If S is a connected complex manifold and M is holomorphic, then $\{s : \operatorname{rk} M(s) < r\}$, with $r = \max_s \operatorname{rk} M(s)$, is a proper closed analytic subset, so $\operatorname{rk} M(s) = r$ on a dense open set: the* generic rank.

Proof. $\operatorname{rk} M(s) \geq k$ if and only if some $k \times k$ minor is non-zero (Exercise 6.2), and each minor is a continuous function, so the set is a finite union of open sets. In the holomorphic case $\{\operatorname{rk} M < r\}$ is the common zero set of the $r \times r$ minors, holomorphic functions not all identically zero; on a connected manifold the zero set of a holomorphic function that is not identically zero is closed with dense complement. □

Exercise 6.2. Show that a matrix has rank $\geq k$ if and only if it has a non-zero $k \times k$ minor. Deduce that if one 9×9 minor of a 9×13 matrix is non-zero, the matrix has rank 9.

Exercise 6.3. (a) Give a holomorphic family $M(s)$ of 2×2 matrices, $s \in \mathbb{C}$, with $\operatorname{rk} M(0) = 1$ and $\operatorname{rk} M(s) = 2$ for $s \neq 0$, and another with $\operatorname{rk} M(s) = 1$ for all s. Explain why there is none with $\operatorname{rk} M(0) = 2$ and $\operatorname{rk} M(s) = 1$ for all small $s \neq 0$. (b) Show that the generic rank of a holomorphic map $f\colon S \to T$ between connected complex manifolds is at least $\operatorname{rk} df_s$ for every single s.

Two consequences are used. First, *the chain rule*: if a family over a base B maps to $\mathcal{H}$ by a holomorphic map c, then the period map of the family is $\mathcal{P} \circ c$, and its rank at b is at most the rank of $\mathcal{P}$ at $c(b)$, which by Proposition 6.3 is at most the generic rank of $\mathcal{P}$ on the irreducible $\mathcal{H}$. So rank 9 at one point of any family mapping to $\mathcal{H}$ suffices. Second, *limits*: if $M(\epsilon) \to M^*$ as $\epsilon \to 0$, then $\operatorname{rk} M^* \leq \operatorname{rk} M(\epsilon)$ for small $\epsilon \neq 0$. One may also first multiply $M(\epsilon)$ on the right by any matrix $P(\epsilon)$, which cannot raise its rank, $\operatorname{rk}(M(\epsilon)P(\epsilon)) \leq \operatorname{rk} M(\epsilon)$, and then take the limit: if $M(\epsilon)P(\epsilon) \to M^*$, then $\operatorname{rk} M^* \leq \operatorname{rk} M(\epsilon)$ for small $\epsilon \neq 0$. This is how a first-order term enters.

Example 6.4 (the limit trick in two dimensions). Let $M(\epsilon) = \left(\begin{smallmatrix} 1 & 1+\epsilon \\ 1 & 1+2\epsilon \end{smallmatrix}\right)$. Then $\det M(\epsilon) = \epsilon$, so $\operatorname{rk} M(\epsilon) = 2$ for $\epsilon \neq 0$, but $M(0)$ has rank 1: the naive limit loses a direction. The combination $v = (-1, 1)$ of the columns kills the order-zero term, $M(\epsilon)v = (\epsilon, 2\epsilon)$. Replace the second column by $M(\epsilon)v/\epsilon = (1, 2)$: this is right multiplication by $P(\epsilon) = \left(\begin{smallmatrix} 1 & -1/\epsilon \\ 0 & 1/\epsilon \end{smallmatrix}\right)$, invertible for $\epsilon \neq 0$, and the limit $M^* = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 2 \end{smallmatrix}\right)$ has determinant 1. So $\operatorname{rk} M(\epsilon) \geq \operatorname{rk} M^* = 2$. In the G_{32} computation the first column plays the role of the order-zero columns R, T, and $(1, 2)$ the role of the first-order column F.

6.4. Degenerations and plumbing in plain terms

Colliding branch points. When two branch points t_i, t_{i+1} of a cover $C \to \mathbb{P}^1$ approach each other, the base $\mathbb{P}^1$ "bubbles": in the stable-curve picture of $\overline{\mathcal{M}}_{0,13}$ the limit is a $\mathbb{P}^1$ on which the two points have merged into a node point c, glued at c to a small $\mathbb{P}^1$ carrying t_i, t_{i+1}

and the node. Over this nodal base the cover becomes an *admissible cover* [HM82]: a G-cover of each component, branched only at the marked points and the nodes, with compatible ramification on the two sides of each node. The monodromy around the node is $\gamma_i\gamma_{i+1}$. If $\gamma_{i+1} = \gamma_i^{-1}$, the node is *trivial*: the cover is unramified over it, and the curve C_0 has $|G|$ ordinary nodes there, permuted simply transitively by G.

For the tuple $\boldsymbol{\gamma}$ we use the real configuration of the branch points with the five consecutive clusters $\{1,2\}$, $\{3,4\}$, $\{6,7\}$, $\{8,9\}$, $\{11,12\}$. The pairs $(1,2),(6,7),(8,9),(11,12)$ multiply to 1 (Chapter C), and $\gamma_3\gamma_4$ has order 6; both are exact computations with the matrices of the tuple. The main component v_0 then carries the three unclustered points $5, 10, 13$ and the five node points, eight special points in all, and the eight monodromies there commute. On v_0 the ρ-forms are therefore explicit: in a joint eigenbasis of the commuting matrices, each component is $\prod_j (x-p_j)^{e_j}\,\mathrm{d}x$ times a polynomial, with exponents read off from the eigenvalues. The four leaves of the trivial nodes are explicit too. The leaf of $\{3,4\}$ is not, and the argument is arranged so that it is never needed (the column-deletion step below).

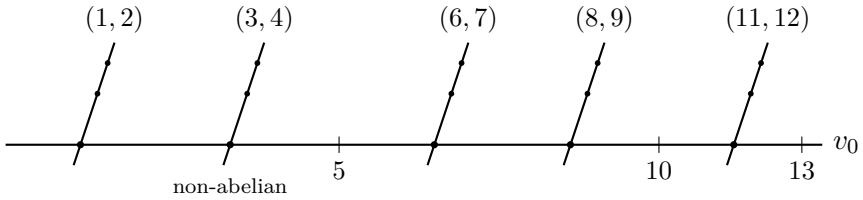


FIGURE 6.1. The stable base at the boundary point: the main component v_0 with eight special points (three branch points and five nodes) and five leaves, each with two branch points. Four leaves have trivial nodes; the leaf $(3,4)$ has node monodromy of order 6.

Plumbing. To move away from the broken curve one reopens each node. Near a node the curve looks like $XY = 0$ in $\mathbb{C}^2$ (two branches, X on one, Y on the other); *plumbing* replaces a neighbourhood of it by $XY = t$, an annulus for $t \neq 0$, whose core circle shrinks to the node as $t \to 0$. For a trivial node this is done at all $|G|$ nodes of the orbit at once, G-equivariantly; since XY is invariant under the stabilizer of a node, equivariant plumbing is possible at every node. On the base, plumbing a trivial node is exactly the substitution $x - c_e = t_e\, y$: the plumbed $\mathbb{P}^1$ is the original line with the cluster placed at $x = c_e + t_e y_i$, where y_i are the positions of the two branch points on the leaf. With five plumbing parameters t_e and five parameters τ that move the eight special points of v_0 modulo Möbius maps, one gets a flat proper family $\mathcal{C} \to \Delta^{10}$ with smooth total space; its fibres over the complement of the coordinate hyperplanes $\{t_e = 0\}$ are smooth G-covers in one Hurwitz component, $\mathcal{H}$ or its complex conjugate (the two readings of Remark 5.3; either way the rank is the same).

Physics aside 6.5 (Plumbing is the sewing of worldsheets). In string perturbation theory two surfaces are sewn by identifying discs through $XY = q$; the long thin tube that results has length $\sim -\log|q|$ and twist $\arg q$, and degenerations of the worldsheet are the limits $q \to 0$ where a propagator goes on shell. The plumbing parameter t_e here is the same q, and the logarithmic behaviour of periods below is the familiar $\tau \sim \log q/2\pi\mathrm{i}$ of a pinching torus. The analogy stops at the questions asked: here nothing is integrated over moduli; one asks how a filtration in a fixed six-dimensional space moves, and the group G acts on everything.

What a period does near a node: the Legendre family. The Legendre family $E_\lambda\colon y^2 = x(x-1)(x-\lambda)$ near $\lambda = 0$ shows every feature of the computation in one variable. Near $(x,y,\lambda) = (0,0,0)$ the factor $1-x$ is a unit; with $w = y/\sqrt{1-x}$ the equation becomes $w^2 + (x - \frac{\lambda}{2})^2 = \frac{\lambda^2}{4}$, i.e. $uv = \lambda^2/4$ with $u, v = (x - \frac{\lambda}{2}) \pm \mathrm{i}w$. So near the node the total

space is the pull-back of the plumbing family $uv = s$ by $s = \lambda^2/4$: an honest, if doubled, plumbing. For $0 < \lambda < 1$ let A, B be lifts to E_λ of loops around $[0, \lambda]$ and $[\lambda, 1]$, oriented so that $\operatorname{Im}(\pi_B/\pi_A) > 0$. Shrinking the loops onto the segments (each period is twice a segment integral) and substituting $x = \lambda s$, resp. $x = 1 - (1-\lambda)s$, Euler's integral [DLMF, (15.6.1)] gives, with $F(\lambda) = {}_2F_1(\frac12, \frac12; 1; \lambda)$,

$$\pi_A = \int_A \frac{\mathrm{d}x}{y} = 2\pi F(\lambda), \qquad \pi_B = 2\pi\mathrm{i} F(1-\lambda).$$

Since $F(1-\lambda) = \frac{2}{\pi} K(\sqrt{1-\lambda})$ [DLMF, (19.5.1)], the expansion [DLMF, (19.12.1)] of the complete elliptic integral K with complementary modulus $k' = \sqrt{\lambda}$, together with $d(m) = \psi(1+m) - \psi(\frac12 + m) = 2\log 2 - 2(H_{2m} - H_m)$ [DLMF, (19.12.3)], gives the classical expansion

$$F(1-\lambda) = \frac{1}{\pi}\Big(F(\lambda)\log\frac{16}{\lambda} - 4G(\lambda)\Big), \qquad G(\lambda) = \sum_{n\geq 1} \binom{2n}{n}^2 \Big(\frac{\lambda}{16}\Big)^n (H_{2n} - H_n), \tag{6.2}$$

with H_n the harmonic numbers. Since $F(\lambda) = 1 + \lambda/4 + \dots$, the period π_B contains $\log\lambda$ and $\lambda\log\lambda$ terms: periods themselves are not holomorphic at the node. Replacing $\log\lambda$ by $\log\lambda + 2\pi\mathrm{i}$ in (6.2) shows that continuation once around $\lambda = 0$ fixes π_A and sends $\pi_B \mapsto \pi_B + 2\pi_A$: on the column $(\pi_A, \pi_B)^{\mathsf{T}}$ the monodromy is $M_0 = \left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right)$, with logarithm $N = \log M_0 = \left(\begin{smallmatrix} 0 & 0 \\ 2 & 0 \end{smallmatrix}\right)$, and $\tau = \pi_B/\pi_A \mapsto \tau + 2$. In the period map τ the logarithm appears only with a constant coefficient,

$$\tau(\lambda) = -\frac{\mathrm{i}}{\pi}\log\lambda + \frac{4\mathrm{i}\log 2}{\pi} - \frac{4\mathrm{i}}{\pi}\frac{G(\lambda)}{F(\lambda)},$$

and $-\frac{\mathrm{i}}{\pi}\log\lambda = 2z$ with $z = \log\lambda/2\pi\mathrm{i}$ is the nilpotent-orbit term: it is the term that makes $\tau(\lambda) - 2z$ single-valued near $\lambda = 0$, as Schmid's nilpotent orbit theorem [Sch73, (4.9)] predicts in general. Hence the logarithmic derivative is holomorphic at $\lambda = 0$, with a constant term given by the monodromy logarithm and a first-order term given by local data:

$$\lambda\frac{\mathrm{d}\tau}{\mathrm{d}\lambda} = -\frac{\mathrm{i}}{\pi} - \frac{\mathrm{i}}{2\pi}\lambda + O(\lambda^2). \tag{6.3}$$

Numerically $\lambda\tau'(\lambda) = -0.319914482406\,\mathrm{i}$, $-0.318469170555\,\mathrm{i}$, $-0.318325802971\,\mathrm{i}$ at $\lambda = 10^{-2}, 10^{-3}, 10^{-4}$, against $-\mathrm{i}/\pi = -0.3183098862\,\mathrm{i}$; the remainder divided by λ^2 is $-0.1305\,\mathrm{i}$, $-0.1294\,\mathrm{i}$, $-0.1293\,\mathrm{i}$, as an $O(\lambda^2)$ remainder should be. This is the whole G_{32} mechanism in one variable: the derivative along the plumbing direction $t\partial_t$ is holomorphic up to $t = 0$; its value at $t = 0$ is topological (the monodromy logarithm N, through Picard–Lefschetz); its first-order coefficient depends on the values of the forms at the node.

Exercise 6.4 (first-order expansion). Assuming (6.2), derive (6.3): compute $G(\lambda) = \lambda/8 + O(\lambda^2)$ and $F(\lambda) = 1 + \lambda/4 + O(\lambda^2)$, write τ as above, and differentiate. Which term of (6.3) would survive if the family were replaced by its base change $\lambda = \mu^2$, and how does it change?

Holomorphic frames through the nodal fibre. Two facts make the codifferential matrix of $\mathcal{C} \to \Delta^{10}$ holomorphic on all of Δ^{10}, including the broken curve.

(i) The forms do not jump. On a nodal curve the role of holomorphic 1-forms is played by the *Rosenlicht forms*, sections of the dualizing sheaf Ω_{C_0}: meromorphic differentials on the normalization with at most simple poles at the two preimages of each node and opposite residues there (S4). Because the fibres are connected and nodal, $h^0(\Omega_{C_t})$ is constant in the family, so $\pi_*\Omega_{\mathcal{C}/\Delta^{10}}$ is locally free and commutes with base change (S2). Its ρ-isotypic part is the image of an idempotent, hence locally free of constant rank, and the rank is 3 by Chevalley–Weil at a smooth fibre (S3). So $\dim H^0(C_0, \Omega_{C_0})_\rho = 3$, and there are holomorphic frames $\phi_a(t,\tau)$, $\eta_b(t,\tau)$, $a, b = 1, 2, 3$, of F^1_ρ and $F^{\bar 1}_{\bar\rho}$ over all of Δ^{10} whose values at 0 are Rosenlicht forms.

(ii) The matrix is holomorphic. Let $M(t,\tau)$ have rows $\phi_a\eta_b$ and columns the pairings with the ten vector fields $t_e\partial_{t_e}$ and ∂_τ. A point-moving column pairs a quadratic differential q with

the motion of a point p_j by the residue $\operatorname{Res}_{p_j} q$ (S5). For a trivial node, $t_e\partial_{t_e}$ is the cluster scaling $\sum_i (x_i - c_e)\partial_{x_i}$, so by the residue theorem its pairing is $\sum_i (x_i - c_e)\operatorname{Res}_{x_i} q = [X^{-2}]q$, the coefficient of $X^{-2}\,\mathrm{d}X^2$ of q on the collar, with the same normalization as the point-moving columns. Wolpert's formula for the pairing of a node-opening direction with a quadratic differential is $\langle \partial/\partial t, \varphi\rangle = (-\pi/t)\,\mathrm{Laurent}(\varphi)$, where Laurent is a constant coefficient on the collar [Wol13, Lemma 4, p. 15 of the arXiv version]; it is consistent with the residue computation, with a constant $-\pi$ per node that is the same for every node, and a constant common to all node columns does not change any rank. What matters is that the normalization is the same for all node columns and for the point-moving columns, so that the relation among node columns used below and the comparison inside the first-order column are not distorted. Writing the $\rho^\vee$-valued form $\sum_i \phi(e_i)\otimes e^i$ on the collar as $\sum_n A_n X^n\,\mathrm{d}X/X$ with $A_n \in \mathbb{C}^4$, and η likewise with coefficients B_n, the node entry is $\sum_n A_n(t)\cdot B_{-n}(t)$. The coefficients are integrals over a fixed circle of holomorphic sections of $\pi_*\Omega_{\mathcal{C}/\Delta^{10}}$, so M is holomorphic on Δ^{10}.

Exercise 6.5 (Möbius relations). Let q be a meromorphic quadratic differential on $\mathbb{P}^1$, holomorphic at ∞, with poles of order at most 2 at points p_j, and let $v(x) = a + bx + cx^2$, a holomorphic vector field on $\mathbb{P}^1$. Show that $\sum_j \operatorname{Res}_{p_j}(v\,q) = 0$ and deduce that, if $T_j = \operatorname{Res}_{p_j} q$ and $R_j = [X^{-2}]q$ at p_j (with $X = x - p_j$), then $\sum_j \big((a + bp_j + cp_j^2)T_j + (b + 2cp_j)R_j\big) = 0$. Conclude that the eight point-moving columns and the node columns (Section 6.5) together satisfy three linear relations (apply the identity to each row $q = \phi_a\eta_b$), so that the thirteen columns of M^* span at most ten dimensions.

6.5. The limit matrix and the rank-nine theorem

Order zero: the node columns are monodromy logarithms. Fix a trivial node e. On the collar the node entry is $\sum_n A_n B_{-n}$. On the other branch $Y = t_e/X$ and $\mathrm{d}X/X = -\mathrm{d}Y/Y$, so the negative coefficients are $A_{-k}(t) = t_e^k \cdot$ (holomorphic) for $k \geq 1$ [Wol13, eqs. (5)–(6)]. Hence

$$\sum_n A_n \cdot B_{-n} = A_0 \cdot B_0 - t_e(a_1 \cdot d_1 + c_1 \cdot b_1)\big|_{t=0} + O(|t|^2), \tag{6.4}$$

where a_1, b_1 are the values of ϕ, η at the node on the v_0 side (coefficients of $\mathrm{d}X$), and c_1, d_1 those on the leaf side (coefficients of $\mathrm{d}Y$). The constant term is topological. For $t_e \neq 0$, $2\pi\mathrm{i}A_0$ is the vector of periods of ϕ over the core circle δ of the collar, a vanishing cycle. Going once around $t_e = 0$ is a Dehn twist about each of the $|G|$ disjoint vanishing circles $g\delta$ ($g \in G$); by Picard–Lefschetz in dimension one (S6) the monodromy is $1 + N_e$ with $Q(N_e x, y) = \epsilon_0 \sum_g \int_{g\delta} x \int_{g\delta} y$. Summing over g and using invariance of $\sum_i e_i \otimes e^i$, all $|G|$ terms are equal, and

$$A_0^e(\phi)\cdot B_0^e(\eta) = c_{\mathrm{PL}}\, Q_\rho(N_e\phi, \eta), \qquad c_{\mathrm{PL}} = \frac{\epsilon_0}{(2\pi\mathrm{i})^2\,|G|}, \tag{6.5}$$

with a sign ϵ_0 fixed by orientation conventions and the same for all trivial nodes. Sign conventions for the Picard–Lefschetz formula differ between sources (orientation of δ, sign of the intersection form, transport versus its inverse); the argument needs only that one sign ϵ_0 serves all trivial nodes, with each vanishing circle δ oriented by the collar coordinate X in the same way. So the order-zero node column of e is the 3×3 matrix of the bilinear form $(\phi, \eta) \mapsto Q_\rho(N_e\phi, \eta)$ on $F^1_\rho \times F^1_{\bar\rho}$, up to one common constant. Here N_e acts on M_ρ by composition, since it commutes with G, and only this restriction $N_e|_{M_\rho}$ enters: the computation below is on M_ρ, and on all of $H^1(C)$ the two operators compared there have different images.

The relation. An exact computation in GAP of the Dehn-twist logarithms on the six-dimensional parabolic cohomology, which is M_ρ (Proposition 5.15), gives ranks $\operatorname{rk} N_e = 1, 0, 1, 1, 1$ for the five nodes, $\dim \operatorname{span}\{N_e\} = 3$, and the single relation $N_{(6,7)} = N_{(8,9)}$. By (6.5) the order-zero columns satisfy $\sum_e c_e A_0^e \cdot B_0^e \equiv 0$ for $c = (0, -1, 1, 0)$ on the trivial nodes $(1,2), (6,7), (8,9), (11,12)$. Since $N_e|_{M_\rho}$ is a flat endomorphism, the relation holds at every

smooth fibre, hence (by holomorphy) on all of Δ^{10} and in every holomorphic frame. The loops of the computation (full twists of the pairs at the real configuration of $\boldsymbol{\gamma}$) are the loops around $t_e = 0$: the points of the plumbing family on the real path $\epsilon > 0$ are real configurations with the same ordering of the branch points, joined to that configuration through such configurations.

Why order zero is not enough. Without the first-order term, a floating-point evaluation of the limit matrix $[R\,|\,T]$ at the positions of Theorem 6.6 gives rank 8; this is not used in the proof, but it explains why the first-order column is there. There is a structural reason why columns built from monodromy logarithms alone cannot give dominance for a non-split family. The image $W_0 = \operatorname{im} N$ of a sum of commuting monodromy logarithms is defined over $\mathbb{Q}$, stable under K and totally isotropic, $Q(Nx, Ny) = -Q(x, N^2 y) = 0$. A three-dimensional isotropic subspace of the six-dimensional Hermitian space would make h hyperbolic, $\det h \equiv (-1)^3$, that is, split (Exercise 2.3). So at every boundary point of a non-split family $\dim_K W_0 \le 2$ (Exercise 6.6). The certificate escapes this constraint because its point-moving columns T live on the non-degenerate main component and its column F is not a monodromy logarithm.

Exercise 6.6 (non-split means few vanishing cycles). Let (V, h) be a non-degenerate Hermitian space over K of dimension 6 and N a K-linear endomorphism with $N^2 = 0$ and $h(Nx, y) + h(x, Ny) = 0$, defined over $\mathbb{Q}$. (a) Show that $\operatorname{im} N$ is totally isotropic and has dimension at most 3. (b) Show that if $\dim_K \operatorname{im} N = 3$ then (V, h) is split, that is, an orthogonal sum of three hyperbolic planes, so $\det h \equiv -1 \bmod \operatorname{Nm} K^\times$ (use Exercise 2.3(c)). (c) Conclude that for the G_{32} family, $\det h \equiv -2$, every boundary point has $\dim_K \operatorname{im} \sum_e N_e \le 2$.

The first-order column. Move along the curve $t_e = \beta_e \epsilon$ in Δ^{10}, with fixed complex β_e and fixed τ. Instead of the five τ-columns of M it is convenient to use the eight *point-moving columns* T_j, the pairings with the motion of the j-th special point of v_0 alone (for a node point, together with its cluster), and next to them the four trivial-node columns R_e, the pairings with $t_e \partial_{t_e}$. Let $\tilde{M}(\epsilon)$ be the 9×12 matrix with these columns. Each of them pairs the codifferential with a motion of the branch points, a tangent vector of the base, so $\operatorname{rk} \tilde{M}(\epsilon) \le \operatorname{rk} d\mathcal{P}$ at the fibre over the point of parameter ϵ (Proposition 6.1). Let $v \in \mathbb{C}^{12}$ have the entries c_e at the node columns and 0 elsewhere, so that $\tilde{M}(\epsilon) v = \sum_e c_e R_e(\epsilon)$ is the combination that kills the order-zero term. By (6.4),

$$\tilde{M}(\epsilon) v = -\epsilon \sum_e c_e \beta_e \, \Xi_e + o(\epsilon), \qquad \Xi_e = a_1 \cdot d_1 + c_1 \cdot b_1.$$

Appending the column $-\tilde{M}(\epsilon) v / \epsilon$ to $\tilde{M}(\epsilon)$ is right multiplication by a 12×13 matrix $P(\epsilon)$; as in Example 6.4, except that the old columns are kept; this is harmless because $\operatorname{rk}(\tilde{M}(\epsilon) P(\epsilon)) \le \operatorname{rk} \tilde{M}(\epsilon)$ for any $P(\epsilon)$ (Section 6.3). The limit is

$$M^* = [\, R \,|\, T \,|\, F \,], \qquad F = \sum_e c_e \beta_e \Xi_e = \beta_{(8,9)} \Xi_{(8,9)} - \beta_{(6,7)} \Xi_{(6,7)},$$

with R the order-zero node columns and T the eight point-moving columns at $t = 0$ (they satisfy three Möbius relations, Exercise 6.5). Since $\tilde{M}(0) v = 0$, a change of holomorphic frame multiplies F by the invertible matrix $A(0) \otimes B(0)$, so M^* may be computed in any basis of $F^1(C_0)$; and since β_e is arbitrary, phase conventions inside Ξ_e are absorbed. The expansion (6.4) is the G_{32} analogue of the constant and linear terms of (6.3).

The non-abelian leaf. The leaf of $\{3, 4\}$ carries monodromies g_3, g_4 that do not commute; its forms would be solutions of a hypergeometric-type system. An exact computation gives $\dim \operatorname{Fix}(g_3) = \dim \operatorname{Fix}(g_4) = 3$ and $\dim \operatorname{Fix}(g_{34}) = \dim \operatorname{Fix}\langle g_3, g_4 \rangle = 2$ on $\mathbb{C}^4$, $g_{34} = g_3 g_4$, so its parabolic cohomology has dimension $(3 - 2) \cdot 4 - (3 + 3 + 2) + 2 \cdot 2 = 0$, and $\operatorname{Fix}(g_{34}) \cap (\operatorname{im}(g_3 - 1) + \operatorname{im}(g_4 - 1)) = 0$. So no ρ-form lives on that leaf, and every ρ-form has residue 0 at its node on the v_0 side. The candidate forms on v_0 and the explicit leaves therefore contain $F^1(C_0)$, and dropping the column of the $\{3, 4\}$ node passes to a submatrix,

which can only lower the rank. This is the column deletion; it is why $\tilde{M}$ has no column for the node $\{3,4\}$.

THEOREM 6.6 (rank nine). *At the positions* $p = (-13,-9,-5,-2,1,4,8,12)/7$ *of the eight special points of* v_0 *and* $\beta = (1,\ \frac{2}{3}+\frac{\mathrm{i}}{3},\ -\frac{1}{5}+\frac{2\mathrm{i}}{5},\ \frac{3}{4}-\frac{\mathrm{i}}{4})$ *for the trivial nodes* $(1,2),(6,7),(8,9),(11,12)$*, the* 9×9 *minor of* M^* *in the columns* $T_3,T_1,R_6,T_2,T_4,T_6,F,T_5,R_3$ *(indices are* 0*-based positions in the list* p*:* 0:$(1,2)$*,* 1:$(3,4)$*,* 2:5*,* 3:$(6,7)$*,* 4:$(8,9)$*,* 5:10*,* 6:$(11,12)$*,* 7:13*; so* R_6 *and* R_3 *are the columns of the trivial nodes* $(11,12)$ *and* $(6,7)$*) lies in the ball* (6.6) *below, which excludes* 0*. Hence* $\operatorname{rk} M^* = 9$*, the period map* $\mathcal{H} \to \Gamma\backslash X^+$ *has generic rank* 9*, it is dominant onto the connected component containing its image, and its generic fibres have dimension* $13-9=4$ *in* $\mathcal{H}$*, that is, dimension* 1 *modulo the Möbius directions.*

THE ARGUMENT. The certificate (Section 6.6) gives $\operatorname{rk} M^* \geq 9$, and 9 is the number of rows. M^* is the limit of $\tilde{M}(\epsilon)P(\epsilon)$, so for small $\epsilon \neq 0$ (Proposition 6.3) $9 \leq \operatorname{rk}(\tilde{M}(\epsilon)P(\epsilon)) \leq \operatorname{rk}\tilde{M}(\epsilon)$. For such ϵ the fibre is a smooth member of $\mathcal{H}$ or of its complex conjugate, and the columns of $\tilde{M}(\epsilon)$ are values of the codifferential of its period map (Proposition 6.1); so the period map of $\mathcal{H}$ has rank 9 there, and by Proposition 6.3 its generic rank is $9 = \dim X^+$. The image is constructible (Borel's algebraicity of the period map and Chevalley's theorem, Chapter 12) of dimension 9 in an irreducible variety of dimension 9, so it contains a dense Zariski-open subset of the component; the generic fibre dimension is $13-9=4$. □

What just happened. The codifferential of the period map is the multiplication $\phi\otimes\eta \mapsto \phi\eta$, a 9×10 matrix. It cannot be written down at a smooth member, so one degenerates to a broken curve where the forms are explicit, and uses that rank can only drop in a limit. At the broken curve the four trivial-node columns are the monodromy logarithms N_e restricted to F^1, and one exact relation $N_{(6,7)} = N_{(8,9)}$ makes them dependent. The combination that vanishes at order zero has a first-order term, built from the values of the forms at the two nodes, and the limit matrix $[R\,|\,T\,|\,F]$ has a non-zero 9×9 minor. The non-explicit non-abelian leaf supports no ρ-forms, and its column is simply dropped.

Remark on the proof 6.1: $\dim F^1_\rho(C_0) = 3$ at the nodal fibre. The certificate itself only shows that the space of candidate forms has dimension at most 3: the constraint system has a certified non-zero 18×18 pivot minor (Section 6.6). Equality, and hence that the computed basis is a basis of $F^1(C_0)$, comes from the base-change argument (i) of Section 6.4, that is, from Grauert's theorem and Rosenlicht's description of Ω_{C_0} applied at a G-equivariant nodal fibre. A ball computation cannot prove that the remaining constraint rows vanish on the basis, since it cannot prove that a matrix is singular; the dimension count is what makes them redundant.

6.6. Ball arithmetic and what the certificate certifies

Midpoint-radius arithmetic. A *ball* is a pair $[m \pm r]$, a midpoint m (a binary floating-point number, real or complex) and a radius $r \geq 0$, standing for the set $\{x : |x-m| \leq r\}$. Each operation returns a ball that is guaranteed to contain the exact result of the operation applied to *any* points of the input balls; the radius absorbs both the propagated input radii and the rounding error of computing the new midpoint, which is bounded by rounding in the safe direction. For instance $[m_1\pm r_1]\cdot[m_2\pm r_2] \subseteq [m_1m_2 \pm (|m_1|\,r_2 + |m_2|\,r_1 + r_1r_2 + \text{rounding})]$, and transcendental functions (exp, log, real powers x^a, π) are evaluated with rigorous error bounds. The Arb library [Joh17] implements this with arbitrary precision. The consequence is the only thing a certificate needs: if a computation, carried out entirely in balls, returns a ball that does not contain 0, then the exact value of the computed expression is not 0. If the ball contains 0, nothing follows.

EXERCISE 6.7 (a product of two balls by hand). Enclose $[3 \pm 0.1] \cdot [2 \pm 0.05]$ by the rule above (ignoring rounding), compare with the exact range of products, and explain where the overestimate comes from. Why can ball arithmetic never return a ball that is too small, and why can it return one that is much too large?

Physics aside 6.7 (Error propagation, worst case). Ball arithmetic is the error propagation of a laboratory course with two changes: errors add linearly in the worst case instead of in quadrature, and the rounding of every intermediate step is itself counted as an error. That makes the bars pessimistic but honest. The analogy stops at interpretation: a laboratory error bar is a statement about likelihood, a ball is a containment statement with probability one, provided the inputs are exact (rational numbers, π, $\sqrt{3}$) and the computation evaluates the intended expression.

EXAMPLE 6.8 (a toy certificate). Let $x = e^{\pi\sqrt{163}}$ and $n_0 = 640320^3 + 744$, an integer, and

$$A = \begin{pmatrix} x & \pi & 1 \\ n_0 & \pi & 1 \\ e & \sqrt{2} & 1 \end{pmatrix}, \qquad \det A = (x - n_0)(\pi - \sqrt{2})$$

(subtract the second row from the first). Ramanujan's near-integer gives $x - n_0 = -7.49927\ldots \cdot 10^{-13}$, so $\det A = -1.2954089155\ldots \cdot 10^{-12}$ (a 60-digit evaluation). In double precision, $x \approx 2.6 \cdot 10^{17}$ is rounded to a multiple of 32: the computed $x - n_0$ is -480, and LU decomposition returns $\det A \approx -856$, wrong by fourteen orders of magnitude, with no warning; nothing in the double-precision output tells whether $\det A$ is zero. In ball arithmetic:

precision (bits)	$\det A$
53	$[\pm 1.18 \cdot 10^{4}]$ (contains 0)
64	$[\pm 4.79]$ (contains 0)
100	$[\pm 8.13 \cdot 10^{-11}]$ (contains 0)
128	$[-1.295409 \cdot 10^{-12} \pm 2.18 \cdot 10^{-19}]$
200	$[-1.2954089155 \cdot 10^{-12} \pm 8.36 \cdot 10^{-41}]$

At 53, 64 and 100 bits the ball contains 0 and decides nothing; from 128 bits on it excludes 0, which proves $\det A < 0$, hence $\operatorname{rk} A = 3$. Note what was certified: the determinant of this matrix of these numbers. Had we typed $\sqrt{3}$ instead of $\sqrt{2}$ in the last row, the computation would have certified, just as rigorously, a statement about a different matrix.

The certificate. The matrix M^* is assembled as follows. The frames of the commuting monodromies on v_0, the residue allowances at the nodes and all series coefficients are exact, in $\mathbb{Q}(\zeta_3) = K$; the only non-exact inputs are the fractional powers $(p_j - p_i)^e$ of differences of the rational positions, with rational exponents e, evaluated on the principal branch as balls at 320 bits. The candidate forms on v_0 and the explicit leaves are the solutions of a linear system (residue matching at the nodes, behaviour at ∞), one for ρ and one for its dual; for each of the two, an 18×18 pivot minor of the constraint matrix is a ball determinant that excludes 0, so the solution space has dimension at most 3, hence exactly 3 by Remark on the proof 6.1, and the basis is solved in balls. The node relation is solved in ball arithmetic as well, and its balls contain the exact relation $c = (0, -1, 1, 0)$. A numerical computation picks which 9 of the 13 columns of M^* to use; that choice need not be rigorous, since only the evaluation of the chosen minor must be. The chosen 9×9 minor is a ball determinant,

(6.6)

$$\det = [2.62077101761\ldots \cdot 10^{-29} \pm 3.7 \cdot 10^{-106}] + [-2.92879226889\ldots \cdot 10^{-29} \pm 4.6 \cdot 10^{-106}]\,\mathrm{i},$$

which excludes 0. The ratio of the modulus to the radius is about 10^{77}, so the conclusion is far from the edge of the computation.

What is certified and what is not. The split is as follows.

Certified (ball arithmetic)	The exact value of the 9×9 determinant defined by the formulas of this chapter, at the given rational positions, is non-zero; the two 18×18 pivot minors are non-zero.
Exact (GAP)	$N_{(6,7)} = N_{(8,9)}$ on M_ρ, $\dim \operatorname{span}\{N_e\} = 3$; the non-abelian leaf supports no ρ-forms; the tuple, its classes and the clusters.
Derived in this chapter	That these formulas compute the limit codifferential: Proposition 6.1, the constant (6.5), the expansion (6.4), the normalization of node versus point columns, the frame conventions and the identification of the Hurwitz component.
Standard	(S1)–(S7) of Section 6.1; Borel and Chevalley for the passage to dominance.
Not part of the proof	An exact or modular version of the minor (fractional powers of position differences would need extra bookkeeping); a computation at a smooth member.

So the certificate checks arithmetic, not formulas: it shows that a specific matrix, assembled from the formulas derived above, has rank 9, and it cannot detect an error in a formula.

REMARK 6.9 (Where this is used). Theorem 6.6 is the input of step (vii) (Chapter 12): there the period map of a finite étale cover of $\mathcal{H}$ to $\Gamma' \backslash X^+$ is algebraic, its image contains a dense Zariski-open U', and the locus where a flat Weil class is algebraic, a countable union of closed algebraic subsets containing U', is everything by Baire's theorem. Nothing in step (ii)–step (vi) uses dominance, so a failure of dominance would be local to step (vii): the algebraicity of the Weil classes on every member of $\mathcal{H}$ holds without it, and only the passage to every sixfold of class 2 would lose its input.

Part 3

Algebraic cycles from cyclic pieces

CHAPTER 7

Cyclic pieces and the eleven-piece combination

Proof steps: (ii), (v)

Where we are. This chapter explains step (ii) and the short step (v). The target of the proof is the six-dimensional space M_ρ inside $H^1(C,\mathbb{C})$ of a curve C of genus 505441 in the family (Example 5.13): its top exterior power is the Weil line that has to be shown algebraic. The proof reaches M_ρ through cyclic covers. What is available, from Schoen's 1988 theorem and from Chapters 8 and 9, is algebraicity of Weil lines of *cyclic* covers $X \to Y$, and C has many cyclic subcovers $C/\ker\psi \to C/H$, one for each subgroup H and character ψ of H. Each of them cuts out a piece $W_{H,\psi}$ of $H^1(C)$, a sum of multiplicity spaces $M_{\rho'}$ with multiplicities given by an integer vector $v_{H,\psi}$. Step (ii) finds eleven pieces whose vectors combine with coefficients ±1 to the indicator vector of ρ; step (vi) (Chapter 11) turns that relation into geometry. Background: Chapter 3 (Frobenius reciprocity Theorem 3.10, central characters, $k_{\min}$ Definition 3.17), Chapter 5 (Riemann–Hurwitz Proposition 5.4, Chevalley–Weil Theorem 5.10 and its form for cyclic covers in Section 5.4, the data of the family Example 5.13), and Jacobians [Mil08, I, §18]. Notation as in Chapters 3 and 5: $G = G_{32}$, $Z = Z(G) \cong \mu_6$, κ_0 the central character of the reflection representation ρ, $\zeta_m = e^{2\pi\mathrm{i}/m}$, subgroups are called H; the fourteen irreducibles with central character κ_0 are $\rho_1 = \rho, \rho_2, \dots, \rho_{14}$, numbered as in Example 5.13 and Table 7.2.

7.1. Step (ii) in two pages

By Proposition 3.5, the first cohomology of a curve with an action of G splits as

$$H^1(C,\mathbb{C}) = \bigoplus_{\rho'} \rho' \otimes M_{\rho'}, \qquad M_{\rho'} = \mathrm{Hom}_G(\rho', H^1(C,\mathbb{C})),$$

with G acting on the first factor and the Hodge structure living on the second. For the covers of the family, 14 of the 102 irreducibles have central character κ_0, all fourteen multiplicity spaces are balanced ($h^{1,0} = h^{0,1}$), and $\dim M_\rho = 6$ with Hodge numbers $(3,3)$ (Example 5.13). The Weil line of the proof is $\bigwedge^6 M_\rho$, or rather $\Lambda_u = \bigwedge^6(u \otimes M_\rho)$ for a vector $u \in \rho$ (Chapter 11).

The one geometric source of algebraic classes in this business is the cyclic cover. Take a subgroup $H \supset Z$ and a one-dimensional character $\psi\colon H \to \mathbb{C}^\times$ with $\psi|_Z = \kappa_0$. The curve $X = C/\ker\psi$ carries an action of $H/\ker\psi \cong \mu_m$, $m = \mathrm{ord}\,\psi$, with quotient $Y = C/H$, and the ψ-eigenspace of $H^1(X)$, pulled back to C, is the subspace

$$W_{H,\psi} = \bigoplus_{\rho'} \rho'^{(H,\psi)} \otimes M_{\rho'}, \qquad v_{H,\psi}(\rho') = \dim \rho'^{(H,\psi)} = \langle \mathrm{Res}_H\, \rho', \psi \rangle,$$

where $\rho'^{(H,\psi)}$ is the subspace of ρ' on which H acts through ψ (Definition 7.1 and Proposition 7.4). The pair (H,ψ), or the cover $X \to Y$, is a *piece*. Its Weil line $\bigwedge^{\mathrm{top}} H^1(X)_\psi$ is algebraic on $J(X)$ when the cover satisfies the hypotheses of Schoen's theorem or of the cyclic-cover theorem (Theorem 8.1; exponents fully paired, Definition 7.6) or is étale over a base of genus at least 2; pulled back to $J(C)$ it gives the algebraic line $\bigwedge^{\mathrm{top}} W_{H,\psi}$ (Proposition 7.10). So each piece contributes the integer vector $v_{H,\psi} \in \mathbb{Z}^{14}$, and one wants to reach the indicator vector e_ρ by combining pieces. Van Geemen and Verra applied Schoen's cyclic theorem to a cyclic subcover of a quaternionic étale cover [GV03].

No single piece does it (ρ is not induced from a character of any subgroup containing Z), but an integral combination does. Up to conjugacy there are 334 pieces on a curve of the family, 69 of them étale. The 265 ramified ones span a lattice of rank only 13 in $\mathbb{Z}^{14}$, not containing e_ρ, so étale pieces are indispensable. A relation with eleven pieces and coefficients ±1 exists (Proposition 7.8, Tables 7.1 and 7.2):

$$e_\rho = \sum_{j=1}^{11} c_j v_{H_j,\psi_j}, \qquad c_j \in \{\pm1\}, \qquad \sum_j c_j \dim W_j = 28374 - 28368 = 6 = \dim M_\rho.$$

Eight of the pieces are ramified with fully paired exponents, three are étale. The statement of Schoen's theorem [Sch88, Thm. 2.0, Cor. 3.1] contains all eleven pieces. The proof uses it for the three étale pieces and obtains the Weil lines of the eight ramified pieces from the cyclic-cover theorem, proved in Chapters 8 and 9.

Sources. Isotypic decomposition, Frobenius reciprocity, Riemann–Hurwitz and the Chevalley–Weil count are standard. The enumeration of the pieces, the relation and the table are exact computations in GAP from the character table of G_{32}, its classes of subgroups and the coset actions of the tuple; the relation is an identity of integer vectors, printed in Table 7.2.

7.2. Pieces of a G-cover

Let $C \to \mathbb{P}^1$ be a G-cover with tuple $(\gamma_1, \dots, \gamma_r)$ (Definition 5.1 and Theorem 5.2); here G is any finite group and $Z \le Z(G)$ a central subgroup with a character κ, later $G = G_{32}$, $Z = Z(G)$, $\kappa = \kappa_0$. We let G act on $H^1(C,\mathbb{C})$ by $g \cdot x = (g^{-1})^* x$, as in Chapter 5.

Definition 7.1 (piece). A *piece* is a pair (H,ψ) of a subgroup $Z \le H \le G$ and a one-dimensional character $\psi\colon H \to \mathbb{C}^\times$ with $\psi|_Z = \kappa$, taken up to conjugation in G. Its *multiplicity vector* is $v_{H,\psi}(\rho') = \langle \mathrm{Res}_H \rho', \psi\rangle_H$ for the irreducible ρ' with central character κ on Z, its *space* is $W_{H,\psi} = H^1(C,\mathbb{C})^{(H,\psi)}$, the subspace on which H acts through ψ, and its *cover* is $X = C/\ker\psi \to Y = C/H$, Galois with group $H/\ker\psi \cong \mu_m$, $m = \operatorname{ord}\psi$. We write r_ψ for the number of branch points of $X \to Y$ and call the piece *étale* if $r_\psi = 0$ and *ramified* otherwise.

Conjugating (H,ψ) by $g \in G$ replaces $W_{H,\psi}$ by $g \cdot W_{H,\psi}$ and leaves $v_{H,\psi}$, the genera and the exponents unchanged, so nothing below depends on the representative. Only irreducibles with central character κ can contribute to $W_{H,\psi}$: on $\rho'^{(H,\psi)}$ the centre acts by $\psi|_Z = \kappa$ and by the central character of ρ'. Since X is a quotient of the connected curve C, it is connected, and ψ induces a faithful character of $H/\ker\psi$; we identify $H/\ker\psi$ with μ_m through it, so that ψ becomes the tautological character $\chi(\sigma) = \zeta_m$ of a generator σ.

Intuition 7.2 (what a piece sees). $W_{H,\psi}$ is what an observer who can only measure the symmetry H, and only in the channel ψ, sees of $H^1(C)$. Inside each isotypic block $\rho' \otimes M_{\rho'}$ that observer sees $v_{H,\psi}(\rho')$ copies of the multiplicity space $M_{\rho'}$, the number of times ψ occurs when ρ' is restricted to H (the branching rule of physics, Theorem 3.10). Different pieces see the same $M_{\rho'}$ with different multiplicities, and that is the only handle on M_ρ that the proof has.

Lemma 7.3 (transfer). *Let a finite group N act on a compact Riemann surface C, with quotient map $\pi\colon C \to X = C/N$. Then $\pi^*\colon H^1(X,\mathbb{Q}) \to H^1(C,\mathbb{Q})^N$ is an isomorphism of Hodge structures.*

Proof. The pushforward (transfer) $\pi_*\colon H^1(C,\mathbb{Q}) \to H^1(X,\mathbb{Q})$, defined on forms by summing over the fibres, satisfies $\pi_*\pi^* = |N|$ and $\pi^*\pi_* = \sum_{n\in N} n^*$. The first identity makes π^* injective; the second shows that every N-invariant class x equals $\pi^*(\pi_* x/|N|)$. Both maps are induced by holomorphic maps (or by the trace on holomorphic forms), so they respect the Hodge decomposition. □

PROPOSITION 7.4 (pieces are eigenspaces of cyclic covers). *Let (H,ψ) be a piece with cover $X\to Y$, and let $g(Y)$ be the genus of Y. Then π^* maps the ψ-eigenspace $H^1(X,\mathbb{C})_\psi$ isomorphically onto $W_{H,\psi}$, and*

$$\dim W_{H,\psi}=\sum_{\rho'} v_{H,\psi}(\rho')\dim M_{\rho'}=2g(Y)-2+r_\psi.$$

The Hodge numbers of $W_{H,\psi}$ are $\sum_{\rho'} v_{H,\psi}(\rho')\,h^{1,0}(M_{\rho'})$ and $\sum_{\rho'} v_{H,\psi}(\rho')\,h^{0,1}(M_{\rho'})$.

PROOF. By Lemma 7.3 with $N=\ker\psi$, π^* identifies $H^1(X,\mathbb{C})$ with $H^1(C,\mathbb{C})^{\ker\psi}$, compatibly with the action of H; the ψ-part of the left side therefore goes to the ψ-part of the right side, which is $W_{H,\psi}$ (an H-eigenvector with eigencharacter ψ is automatically $\ker\psi$-invariant). In $\rho'\otimes M_{\rho'}$ the group acts on the first factor only, so the (H,ψ)-part is $\rho'^{(H,\psi)}\otimes M_{\rho'}$, of dimension $v_{H,\psi}(\rho')\dim M_{\rho'}$, and its Hodge numbers are $v_{H,\psi}(\rho')$ times those of $M_{\rho'}$. This gives the first equality and the Hodge numbers.

For the second equality we compute the multiplicity of χ in $H^1(X,\mathbb{C})$ for the μ_m-cover $X\to Y$ by the Lefschetz argument of Theorem 5.10(a). Let $b_1,\dots,b_s$ be the branch points of $X\to Y$ and $S_i\le\mu_m$ the (cyclic) stabilizer of the points over b_i, of order e_i; there are m/e_i such points. For $1\ne\tau\in\mu_m$ the Lefschetz formula gives $\operatorname{tr}(\tau^*|H^1)=2-\#\mathrm{Fix}(\tau)$, and $\operatorname{tr}(\tau^*)=\operatorname{tr}((\tau^{-1})^*)$ because the character of $H^1(X,\mathbb{Q})$ is rational. Hence, with $\sum_{\tau\ne1}\overline{\chi(\tau)}=-1$,

$$\langle H^1(X,\mathbb{C}),\chi\rangle=\frac{1}{m}\Big(2g(X)-2-\sum_{i=1}^{s}\frac{m}{e_i}\sum_{1\ne\tau\in S_i}\overline{\chi(\tau)}\Big)=\frac{1}{m}\Big(2g(X)-2-\sum_{i=1}^{s}\Big(m\,[\chi|_{S_i}=1]-\frac{m}{e_i}\Big)\Big).$$

Riemann–Hurwitz [Stacks, Tag 0C1F] gives $2g(X)-2=m(2g(Y)-2)+\sum_i(m-m/e_i)$. Substituting, the multiplicity is $2g(Y)-2+\#\{i:\chi|_{S_i}\ne1\}$. Since χ is faithful on μ_m and $S_i\ne1$, every branch point counts, and the multiplicity is $2g(Y)-2+r_\psi$. □

The two expressions for $\dim W$ are computed from different data: the first from the character table and the Chevalley–Weil dimensions $\dim M_{\rho'}$ over $\mathbb{P}^1$, the second from the permutation action of the tuple on the cosets of H (Section 7.3). They agree for all 334 pieces.

EXERCISE 7.1. (a) Let $H=\langle g\rangle$ be cyclic. Show that $v_{H,\psi}(\rho')$ is the multiplicity of $\psi(g)$ as an eigenvalue of $\rho'(g)$. (b) In $G_4\cong\mathrm{SL}(2,3)$ with the reflection representation ρ of Example 4.3, let $g=-r_1$, of order 6, and $H=\langle g\rangle\supseteq Z=\{\pm1\}$. List the three characters ψ of H with $\psi(-1)=-1$ and compute $v_{H,\psi}(\rho)$ for each. Which of the three pieces has $m=2$?

7.3. Local exponents, full pairing and balance

The hypotheses of Schoen's theorem and of the cyclic-cover theorem are conditions on the *local exponents*: at a branch point b of the μ_m-cover $X\to Y$ the canonical generator of the stabilizer (Definition 5.1 and the remark after it) is σ^a for an $a\in\mathbb{Z}/m$, $a\ne0$, the exponent at b. For cyclic covers of the line $y^m=\prod(x-t_i)^{a_i}$ these are the a_i (Section 5.2). For a piece of a G-cover they are read off from the tuple.

PROPOSITION 7.5 (exponents from double cosets). *Let (H,ψ) be a piece of the G-cover with tuple (γ_i), normalized as in Remark 5.3. The points of $Y=C/H$ over t_i correspond to the orbits of $\langle\gamma_i\rangle$ on the right cosets $H\backslash G$, i.e. to the double cosets $H\backslash G/\langle\gamma_i\rangle$. If Hx lies in an orbit of length ℓ, then $Y\to\mathbb{P}^1$ has ramification index ℓ at the corresponding point, the element $h=x\gamma_i^{\ell}x^{-1}$ lies in H, and the exponent of $X\to Y$ there is the $a\in\mathbb{Z}/m$ with $\psi(h)=\zeta_m^a$; the point is a branch point of $X\to Y$ if and only if $\psi(h)\ne1$. In particular*

$$2g(Y)-2=-2[G:H]+\sum_{i=1}^{r}\big([G:H]-\#\{\textit{orbits of }\gamma_i\textit{ on }H\backslash G\}\big).$$

PROOF. Over t_i the points of C are yP_i, $y\in G/\langle\gamma_i\rangle$, with stabilizer $y\langle\gamma_i\rangle y^{-1}$ and canonical generator $y\gamma_i y^{-1}$; their images in Y are the H-orbits, i.e. the double cosets $Hy\langle\gamma_i\rangle$, and

$Hy\langle\gamma_i\rangle \leftrightarrow$ the $\langle\gamma_i\rangle$-orbit of Hy under right multiplication (put $x = y$). The stabilizer of yP_i in H is $H \cap y\langle\gamma_i\rangle y^{-1} = \langle y\gamma_i^{\ell}y^{-1}\rangle$, where ℓ is the least exponent with $y\gamma_i^{\ell}y^{-1} \in H$, which is the orbit length of Hy. The canonical generator of a subgroup of index ℓ in a cyclic stabilizer is the ℓ-th power of the canonical generator (it acts on the tangent line by $e^{2\pi i\ell/e}$), so the local monodromy of $C \to Y$ at yP_i is $h = y\gamma_i^{\ell}y^{-1}$, and that of $X \to Y$ is its image in $H/\ker\psi \cong \mu_m$, which is σ^a with $\zeta_m^a = \psi(h)$. Ramification indices multiply along $C \to Y \to \mathbb{P}^1$: $e_i = \left|H \cap y\langle\gamma_i\rangle y^{-1}\right| \cdot e$ with $\left|H \cap y\langle\gamma_i\rangle y^{-1}\right| = e_i/\ell$, so $Y \to \mathbb{P}^1$ has index $e = \ell$ there, and Riemann–Hurwitz for $Y \to \mathbb{P}^1$ of degree $[G : H]$ is the displayed formula. □

The opposite reading convention for the tuple (Remark 5.3) replaces every exponent a by $-a$; none of the conditions below (full pairing, balance) sees the difference, since both are invariant under $a \mapsto -a$. The enumeration of Section 7.5 follows exactly this recipe: coset action, $h = x\gamma_i^{\ell}x^{-1}$, a from $\psi(h)$, $g(Y)$ from the cycle counts.

Definition 7.6 (fully paired, balanced). Let $X \to Y$ be a connected μ_m-cover with exponents $a_1, \dots, a_r \in \mathbb{Z}/m \smallsetminus \{0\}$ (a multiset). It is *fully paired* if for every a the values a and $-a$ occur equally often and, when m is even, $m/2$ occurs an even number of times; equivalently, the multiset can be arranged as $(b_1, -b_1, \dots, b_k, -b_k)$, $r = 2k$. It is *balanced* if $\sum_i \langle ta_i/m\rangle = r/2$ for every $t \in (\mathbb{Z}/m)^{\times}$, where $\langle x\rangle$ is the fractional part. A piece is fully paired (balanced) if its cover is.

Fully paired is Schoen's "simple" [Sch88, p. 11] (after a permutation, $a_{2i} = -a_{2i-1}$); it is the hypothesis of the cyclic-cover theorem, Theorem 8.1(b), and in Chapter 9 it is what makes the relevant character of a Fermat variety a "join of points", whose eigenline is spanned by linear subspaces. Balance has a Hodge-theoretic meaning: by the Chevalley–Weil formula [CW34] (for $g(Y) = 0$ this is Section 5.4), the subspace of $H^1(X, \mathbb{C})$ on which σ^* acts by ζ_m^t has Hodge numbers $\big(g(Y) - 1 + \sum_i\langle -ta_i/m\rangle,\ g(Y) - 1 + \sum_i\langle ta_i/m\rangle\big)$. Since $\langle x\rangle + \langle -x\rangle = 1$ for $x \notin \mathbb{Z}$, the two sums add up to r, and balance says exactly that every Galois conjugate eigenspace has $h^{1,0} = h^{0,1}$, which is necessary for its top exterior power to be a Hodge class at all.

Exercise 7.2. Decide which multisets are fully paired: (a) $(2, 2, 3, 3, 4, 4)$ in $\mathbb{Z}/6$; (b) $(1, 3, 4, 4)$ in $\mathbb{Z}/6$; (c) $(4, 6, 8)$ in $\mathbb{Z}/12$; (d) the exponents 4:324, 6:6, 8:324 of piece 4 in $\mathbb{Z}/12$ (value:multiplicity). Show that multiplying all exponents by a unit $t \in (\mathbb{Z}/m)^{\times}$ preserves full pairing, so a fully paired piece is fully paired for every Galois conjugate ψ^t.

Exercise 7.3. (a) Show that a fully paired multiset is balanced. (b) Show that $(1, 3, 4, 4)$ in $\mathbb{Z}/6$ is balanced, has sum $\equiv 0$, and is not fully paired; so balance is strictly weaker. (It is the datum of the Weil surface of Example 2.9.) (c) For $m = 6$ show that balance reduces to the single condition $\sum_i\langle a_i/6\rangle = r/2$.

Exercise 7.4. Let $X \to Y$ be an étale μ_m-cover, $g(Y) \geq 1$. Show that $g(X) = m(g(Y) - 1) + 1$, that every non-trivial character occurs in $H^1(X, \mathbb{C})$ with multiplicity $2g(Y) - 2$ and Hodge numbers $(g(Y) - 1, g(Y) - 1)$, so étale covers are balanced, and check the genera $g(X)$ of the étale pieces $5, 7, 8$ in Table 7.1.

For the pieces of the family, balance is in fact automatic: by Proposition 7.4 the Hodge numbers of $W_{H,\psi}$ are sums of those of the $M_{\rho'}$ with central character κ_0, which are all balanced (Example 5.13), and the conjugate pieces with central character $\bar\kappa_0$ involve the complex conjugate spaces $\overline{M_{\rho'}}$, balanced as well. Full pairing is not automatic in this way. In fact every one of the 334 pieces is fully paired for all Galois conjugates (Section 7.5); this is plausible because the branching type is closed under inversion (six reflections of determinant ω against six of determinant ω^2, and an involution), but we do not know a proof that it is forced. Only the eleven pieces of the relation need it, and for those it is checked directly (Table 7.1).

Example 7.7 (the pieces of an SL(2, 3)-cover). Take $G = G_4 \cong \mathrm{SL}(2, 3)$ with the reflections r_1, r_2 of Example 4.3, $Z = \{\pm 1\}$, and the tuple $(r_1, r_1^2, r_2, r_2^2, r_1, r_1^2)$: product

1, generating G, six branch points with stabilizers of order 3. Riemann–Hurwitz gives $2g(C)-2 = 24(-2+6\cdot\frac{2}{3}) = 48$, $g(C) = 25$; the centre acts freely. The irreducibles with $\kappa(-1) = -1$ are the three of degree 2. Call them ρ_a, ρ_b, ρ, with ρ the reflection representation; Chevalley–Weil (Theorem 5.10) gives $\dim M_{\rho_a} = 8$ (the reflections have no eigenvalue 1 on ρ_a) with Hodge numbers $(4,4)$, and $\dim M_{\rho_b} = \dim M_\rho = 2$ with $(1,1)$. For ρ: $-2\cdot 2 + 6\cdot 1 = 2$ and $h^{1,0} = -2 + 3\cdot\frac{1}{3} + 3\cdot\frac{2}{3} = 1$. There are five pieces up to conjugacy, with vectors $(v(\rho_a), v(\rho_b), v(\rho))$:

(H,ψ)	m	r_ψ	$g(Y)$	exponents	v	$\dim W$
(Z,sgn)	2	0	13	étale	$(2,2,2)$	24
(C_4,ψ_4) $(\sim (C_4,\bar\psi_4))$	4	0	7	étale	$(1,1,1)$	12
(C_6,ψ_2)	2	0	3	étale	$(0,1,1)$	4
(C_6,ψ_6)	6	6	3	2:3, 4:3	$(1,0,1)$	10
(C_6,ψ_6')	6	6	3	2:3, 4:3	$(1,1,0)$	10

In each row $\dim W = \sum v \dim M = 2g(Y) - 2 + r_\psi$; for (C_6,ψ_6) it reads $1\cdot 8 + 0\cdot 2 + 1\cdot 2 = 10 = 2\cdot 3 - 2 + 6$. Take the cyclic group $C_6 = \langle -r_1\rangle$, of index 4. The element r_1 acts on the four cosets with one fixed point and one 3-cycle, and so does every element of the tuple. By Proposition 7.5, the fixed coset gives a point of Y unramified over $\mathbb{P}^1$ whose stabilizer in C_6 is a conjugate of $r_1^{\pm 1}$, of order 3, on which the faithful ψ_6 is non-trivial: a branch point of $X \to Y$ with exponent 2 or 4. The 3-cycle gives a point of ramification index 3 with trivial stabilizer in C_6. So $r_\psi = 6$, and $2g(Y) - 2 = 4\cdot(-2) + 6\cdot 2 = 4$, $g(Y) = 3$. Here $X = C$ (the character ψ_6 is faithful), and Riemann–Hurwitz for $X \to Y$ is $48 = 6\cdot 4 + 6\cdot(6-2)$. The ramified pieces are fully paired; all pieces are balanced.

The two ramified vectors $(1,0,1)$ and $(1,1,0)$ span a plane that does not contain $e_\rho = (0,0,1)$: ramified pieces alone fail, exactly as for G_{32} below. With one étale piece,

$$e_\rho = v_{C_4,\psi_4} - v_{C_6,\psi_6'} = (1,1,1) - (1,1,0), \qquad 12 - 10 = 2 = \dim M_\rho,$$

two pieces with coefficients ± 1, one étale over a base of genus 7 and one ramified and fully paired. This is step (ii) in miniature. As an identity of vectors it is a Brauer relation of the kind found in Example 3.19; what the cover adds is the geometry, which decides which theorem applies to each piece (here: (C_4,ψ_4) étale over a base of genus $7 \geq 2$, (C_6,ψ_6') ramified and fully paired). The toy only illustrates the bookkeeping: its central character is real, whereas the contraction of Chapter 11 uses that κ_0 is not.

Exercise 7.5. In Example 7.7, check $\dim W = 2g(Y) - 2 + r_\psi$ in every row from the $\dim M$ values, and verify $24 = 2\cdot 13 - 2$ for (Z,sgn) by Riemann–Hurwitz for $C \to C/Z$. Then find a relation $e_\rho = \sum c_j v_j$ with $c_j = \pm 1$ that uses (C_6,ψ_6) instead of (C_6,ψ_6'), and check its dimension count.

7.4. One piece in full: piece 9

Piece 9 of Table 7.1 has the largest subgroup and the smallest genera, so every number can be followed by hand. Its subgroup H_9, of order $3888 = 155520/40$, is the stabilizer of one of the 40 reflecting hyperplanes of G_{32} (Proposition 4.10): the stabilizer of the line attached to a reflection (an orbit of length 40) is a subgroup of order 3888 containing Z, and it reproduces all the data of piece 9 below (a computation in GAP). The cosets $H_9\backslash G$ are then the 40 hyperplanes, and G acts on them by permutation.

The quotient Y. Each of the twelve reflections of the tuple fixes 13 of the 40 hyperplanes and permutes the other 27 in nine 3-cycles; the involution fixes 8 and exchanges the remaining 32 in 16 pairs. By Proposition 7.5,

$$2g(Y) - 2 = 40\cdot(-2) + 12\cdot 9\cdot 2 + 16 = -80 + 232 = 152, \qquad g(Y) = 77.$$

The character and the exponents. H_9 has three one-dimensional characters with $\psi|_Z = \kappa_0$, all of order $m = 6$. At a hyperplane fixed by a reflection γ_i the stabilizer in H_9 is a conjugate

of γ_i, of order 3; at a 3-cycle it is trivial. For the character ψ_9 of the table, ψ_9 is non-trivial on all $12 \cdot 13 = 156$ order-3 stabilizers, with values ω and ω^2 ($a = 2$ and $a = 4$) 78 times each, and non-trivial on 4 of the 8 order-2 stabilizers coming from the involution ($a = 3$). So $X_9 \to Y$ has $r = 160$ branch points with exponents 2:78, 3:4, 4:78. The multiset is fully paired (78 pairs $(2, 4)$ and two pairs $(3, 3)$) and balanced: $\sum_i \langle a_i/6 \rangle = 78 \cdot \frac{1}{3} + 4 \cdot \frac{1}{2} + 78 \cdot \frac{2}{3} = 80 = r/2$, and $t = 5$ exchanges 2 and 4.

The dimension. By Proposition 7.4, $\dim W_9 = 2 \cdot 77 - 2 + 160 = 312$. On the representation side, v_9 has two non-zero entries, both 1: at $\rho_2 = (4, 42)$, with Hodge numbers $(21, 21)$, and at $\rho_9 = (36, 270)$, with $(135, 135)$ (Example 5.13). Indeed $42 + 270 = 312$, with Hodge numbers $(156, 156)$.

The cover X_9. By Riemann–Hurwitz for the μ_6-cover $X_9 \to Y$, with $6 - \gcd(a, 6)$ equal to $4, 3, 4$ for $a = 2, 3, 4$,

$$2g(X_9) - 2 = 6 \cdot 152 + 78 \cdot 4 + 4 \cdot 3 + 78 \cdot 4 = 1548, \qquad g(X_9) = 775,$$

so $d = \dim W_9 = 312 \leq g(X_9)$, an inequality that holds for every cyclic cover with $m \geq 3$ (Remark 8.2). The exponents have $\gcd(a_i, 6) = 1$ overall, so $d' = 1$ and $m' = 6$: the Fermat source of Chapter 9 is the sextic X_6^{158}.

Two remarks put this in perspective. Of the two other characters of H_9 with $\psi|_Z = \kappa_0$, one gives a piece with $r = 16$, exponents 2:6, 3:4, 4:6, $\dim W = 168$, whose vector has entries 1 at $\rho_1 = \rho = (4, 6)$ itself and at $\rho_8 = (36, 162)$. It contains M_ρ, but together with a 162-dimensional companion; removing the companion is what the other ten pieces are for. The other gives the same r and exponents as ψ_9 but a different vector. Second, piece 9, like the other ramified pieces, is fully paired with $m > 2$, so it lies within the statement of Schoen's theorem; the proof takes its Weil lines from the cyclic-cover theorem (Table 9.1).

Exercise 7.6. (a) Recompute $\dim W_9 = 312$ in both ways. (b) Recompute $g(Y) = 77$ and $g(X_9) = 775$. (c) How many points does X_9 have over a general point of $\mathbb{P}^1$, and what is $2g(X_9) - 2$ by Riemann–Hurwitz for $X_9 \to \mathbb{P}^1$ directly? (Hint: $[G : \ker \psi_9] = 240$; find the ramification of $X_9 \to \mathbb{P}^1$ over the reflections and over the involution from the description above.)

7.5. All pieces, the lattice, and the eleven-piece relation

Enumeration. The subgroups $H \supseteq Z$ up to conjugacy are the preimages of the 116 classes of subgroups of G/Z (of order 25920). For each H one lists the one-dimensional characters with $\psi|_Z = \kappa_0$ and computes $v_{H,\psi}$ from the character table of H, the exponents and $g(Y)$ from the coset action (Proposition 7.5), and $\dim W$ and its Hodge numbers from the 14 Chevalley–Weil pairs. The result: 334 pieces, 69 of them étale; for every piece $\dim W$ computed in the two ways of Proposition 7.4 agrees; every piece is balanced and fully paired for all Galois conjugates; and every exponent of a ramified piece has order 2 or 3 in $\mathbb{Z}/m$, as it must, since the local monodromies of the branching type have orders 3 and 2.

Why étale pieces are needed. The 265 ramified vectors span a sublattice of $\mathbb{Z}^{14}$ of rank 13, and e_ρ is not in its $\mathbb{Q}$-span; together with the étale vectors the rank is 14. In the language of Definition 3.17, $k_{\min} = \infty$ for the ramified pieces and $k_{\min} = 1$ for all pieces. We know no conceptual reason for the defect of the ramified lattice; it is a computed fact for this tuple, and the toy Example 7.7 shows that the phenomenon is not peculiar to G_{32}.

Finding the relation. Integral solutions c of $\sum_j c_j v_j = e_\rho$, indexed by the distinct vectors v_j among the pieces, form an affine lattice $c_0 + L$, with L the integral kernel. A particular solution c_0 and a basis of L come from the Hermite normal form, and LLL [LLL82] applied to the basis $\{(\ell, 0) : \ell \in L\} \cup \{(c_0, B)\}$ with a large weight B produces short solutions: a reduced vector with last entry $\pm B$ is one. The shortest one found has eleven non-zero coefficients, all ± 1. Nothing in the proof depends on how the relation was found: it is an identity of integer vectors, which the reader can check in Table 7.2. We do not know whether eleven is the minimum.

j	c_j	$\lvert H_j\rvert$	m	r	$g(Y_j)$	$d=\dim W_j$	$g(X_j)$	m'	exponents
1	-1	432	6	544	989	2520	7015	6	2:270, 3:4, 4:270
2	$+1$	486	6	384	913	2208	6241	3	2:192, 4:192
3	-1	324	6	448	1361	3168	9049	6	2:216, 3:16, 4:216
4	$+1$	288	12	654	1537	3726	21043	6	4:324, 6:6, 8:324
5	$+1$	216	24	0	2341	4680	56161	–	étale
6	$+1$	144	6	336	3397	7128	21043	6	2:162, 3:12, 4:162
7	-1	120	24	0	4213	8424	101089	–	étale
8	$+1$	108	12	0	4681	9360	56161	–	étale
9	$+1$	3888	6	160	77	312	775	6	2:78, 3:4, 4:78
10	-1	72	12	672	6793	14256	84169	6	4:324, 6:24, 8:324
11	$+1$	972	6	64	449	960	2809	6	2:24, 3:16, 4:24

TABLE 7.1. The eleven pieces of the relation. Exponents are value:multiplicity in $\mathbb{Z}/m$; $m'=m/\gcd(m,a_i)$. The Weil lines of the eight ramified pieces are algebraic by the cyclic-cover theorem (Theorem 8.1(b)), and those of the étale pieces $5,7,8$ by Schoen's theorem [Sch88, Thm. 2.0, Cor. 3.1], with Patel–Zhang [PZ27] as an alternative; the statement of Schoen's theorem contains the ramified pieces as well (Table 9.1, Chapter 10).

k	$\dim\rho_k$	$\dim M_{\rho_k}$	$\chi_{\rho_k}(\gamma_1)$	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}
			c_j	-1	$+1$	-1	$+1$	$+1$	$+1$	-1	$+1$	$+1$	-1	$+1$
1	4	6	$2-\omega$	0	0	0	0	0	0	0	0	0	0	1
2	4	42	$-1+2\omega$	0	0	0	0	0	0	0	0	1	1	0
3	20	102	$7+7\omega$	0	0	0	0	1	1	1	1	0	2	0
4	20	102	$1-5\omega$	0	0	0	1	0	0	1	1	0	1	0
5	20	138	$-5-8\omega$	2	0	0	2	1	0	1	1	0	1	0
6	20	174	$-5+\omega$	0	1	1	1	0	1	1	1	0	2	0
7	20	174	$-8-5\omega$	2	0	1	2	1	1	1	1	0	1	0
8	36	162	9	0	0	1	0	1	1	2	2	0	2	1
9	36	270	9ω	0	1	0	0	1	2	2	2	1	5	0
10	60	342	$3-6\omega$	1	0	1	1	2	2	3	3	0	4	1
11	60	378	$3+3\omega$	0	1	1	1	1	3	3	3	0	5	0
12	60	450	$-6+3\omega$	1	1	1	1	2	3	3	3	0	6	1
13	64	384	$8+8\omega$	0	1	1	0	2	3	3	4	0	6	0
14	80	552	$-8-8\omega$	2	1	2	3	2	3	4	5	0	6	0

TABLE 7.2. The relation of Proposition 7.8. Rows: the fourteen irreducibles ρ_k with central character κ_0, with their degree, $\dim M_{\rho_k}$ (Example 5.13) and character value at the reflection γ_1 (of determinant ω^2), which tells apart the two pairs of degree 20 with equal $\dim M_{\rho_k}$. Columns: the multiplicity vectors $v_j=v_{H_j,\psi_j}$ of the pieces of Table 7.1, with the coefficients c_j; the entry in row k and column v_j is $v_j(\rho_k)$.

PROPOSITION 7.8 (the eleven-piece relation). *For the eleven pieces (H_j,ψ_j) of Table 7.1, with the vectors v_{H_j,ψ_j} of Table 7.2,*

$$e_\rho=\sum_{j=1}^{11}c_j v_{H_j,\psi_j}\quad\text{in }\mathbb{Z}^{14},\qquad \sum_{j=1}^{11}c_j\dim W_j=28374-28368=6=\dim M_\rho.$$

Every piece is fully paired and balanced for all Galois conjugates of ψ_j and satisfies $\dim W_j\le g(X_j)$; the étale pieces have $g(Y_j)\ge 2341$.

As an identity between characters of G the relation does not depend on how the fourteen irreducibles are named; in the numbering of Table 7.2 it says that the rows of the table, weighted by the c_j, sum to 1 in the row of $\rho_1 = \rho$ and to 0 in every other row (Exercise 7.7). The dimension identity is a consequence of the vector identity (apply the linear map $v \mapsto \sum_{\rho'} v(\rho') \dim M_{\rho'}$), but it is checked against the column $d = 2g(Y) - 2 + r$, which comes from the coset actions, so it tests the vectors against the geometry (Exercise 7.8). The plus side is $2208+3726+4680+7128+9360+312+960 = 28374$, the minus side $2520+3168+8424+14256 = 28368$. In Chapter 11 these two numbers reappear as the dimensions $D_+ = \dim U_+$ and $D_- = \dim U_-$ of the two sides of the correspondence.

Exercise 7.7. From Table 7.2: (a) check $\sum_j c_j v_j = e_\rho$ row by row; (b) compute $d_j = \sum_k v_j(\rho_k) \dim M_{\rho_k}$ for every j and compare with the column d of Table 7.1; (c) compute $\sum_k v_j(\rho_k) \dim \rho_k$ and compare with the index $[G : H_j] = 155520/|H_j|$. (By Frobenius reciprocity, $\sum_{\rho'} \langle \mathrm{Res}_H \rho', \psi \rangle \dim \rho'$, summed over all irreducibles, is $\dim \mathrm{Ind}_H^G \psi = [G : H]$; why do only the fourteen ρ_k contribute?)

Exercise 7.8. (a) Deduce $\sum_j c_j \dim W_j = \dim M_\rho$ from $\sum_j c_j v_j = e_\rho$ and Proposition 7.4. (b) Add up the column d of Table 7.1 with the signs c_j. (c) Which error in the enumeration would the check (b) detect, and which would it miss?

The prime condition. A ramified piece satisfies the *prime condition* if for each of its exponents a there is a prime $p \mid m$ with $p \nmid a$. For $m = 12$ the exponent 6 of the pairs $(6, 6)$ of pieces 4 and 10 is divisible by both primes of 12, so these two pieces fail the condition, and the other six ramified pieces of the relation satisfy it (Example 7.9). The computation below asks whether a relation for e_ρ can be formed from the étale pieces and the ramified pieces that satisfy the condition.

Example 7.9 (the $(6, 6)$ pairs at $m = 12$). The subcover of prime degree $p \mid m$, the quotient of X by the subgroup of index p, has local monodromy a mod p at a branch point with exponent a, so it is ramified there if and only if $p \nmid a$. Here $12 = 2^2 \cdot 3$, so the primes of m are 2 and 3. For a pair $(4, 8)$: $3 \nmid 4$, so $p = 3$ works (the μ_3-quotient is ramified at a point with exponent 4). For a pair $(6, 6)$: $2 \mid 6$ and $3 \mid 6$. The $\mathbb{Z}/2$-subcover (quotient by $\langle \sigma^2 \rangle$) has local monodromy $6 \equiv 0 \bmod 2$, and the $\mathbb{Z}/3$-subcover (quotient by $\langle \sigma^3 \rangle$) has $6 \equiv 0 \bmod 3$: both are unramified over the branch point, so the pair fails the prime condition. For $m = 6$ every exponent value occurring in the pieces satisfies it: $p = 3$ for $2, 4$ and $p = 2$ for 3. Among all exponent values of the eight pieces, $(m, a) = (12, 6)$ is the only one that fails the prime condition. Piece 4 contains three $(6, 6)$ pairs and piece 10 twelve (Table 7.1).

Of the 265 ramified pieces, 222 satisfy the prime condition (their m are $6, 12, 24$); the 43 others, pieces 4 and 10 among them, have $m = 12$ or 18, each with an exponent $m/2$ or $m/3$ divisible by every prime of m. The vectors of the 222 pieces together with the 69 étale pieces span $\mathbb{Q}^{14}$, and for this family of 291 pieces $k_{\min} = 2$; the $m = 6$ ramified pieces together with the étale ones also give $k_{\min} = 2$, while all 334 pieces give $k_{\min} = 1$. Since $\{k : ke_\rho \in \text{span}\}$ is an ideal of $\mathbb{Z}$ that can only shrink when the family shrinks, every family of pieces inside these 291 has $k_{\min}$ even or infinite. So pieces 4 and 10 cannot be replaced by pieces that satisfy the prime condition: those give only $2e_\rho$. Whether $k_{\min} = 1$ holds for the 332 pieces other than 4 and 10, which include 41 further pieces that fail the prime condition, was not computed.

A relation with $k = 2$ would make step (vi) produce $\Lambda_u \boxtimes \Lambda_{u'}$ instead of Λ_u; we do not know whether Λ_u can be extracted from that.

Exercise 7.9. (a) Show that there is no prime $p \mid m$ with $p \nmid a$ for an exponent $a \in \mathbb{Z}/m \setminus \{0\}$ if and only if a is a multiple of the radical $\mathrm{rad}(m)$, the product of the distinct primes dividing m. (b) List the failing a for $m = 12, 18, 24, 30$. (c) Conclude that for squarefree m every exponent of a simple tuple admits such a prime, and that for $m = 12$ the only pair without one is $(6, 6)$.

What just happened. Each pair (H,ψ) with $\psi|_Z = \kappa_0$ gives a cyclic cover $X \to Y$ whose ψ-eigenspace, pulled back to C, is the piece $W_{H,\psi} = \bigoplus \rho'^{(H,\psi)} \otimes M_{\rho'}$, of dimension $2g(Y)-2+r_\psi$; its exponents come from double cosets, and the hypotheses of the algebraicity theorems are full pairing (ramified) or $g(Y) \geq 2$ (étale). For the family there are 334 pieces; the ramified ones have rank $13 < 14$, and with the étale ones there is a relation $e_\rho = \sum c_j v_j$ with eleven pieces and $c_j = \pm 1$ (Table 7.2), with $28374 - 28368 = 6$ as a check. Pieces 4 and 10 cannot be replaced by pieces that satisfy the prime condition.

7.6. Step (v): from the pieces to the Jacobian of C

Steps (iii) and (iv) (Chapters 8 to 10) give, for each piece j, that the line $\Lambda_{\psi_j} = \bigwedge^{d_j} H^1(X_j,\mathbb{C})_{\psi_j} \subset H^{d_j}(J(X_j),\mathbb{C})$ lies in $\mathrm{Alg}(J(X_j)) \otimes \mathbb{C}$, the complex span of classes of algebraic cycles. Step (vi) works on products of copies of $P = J(C)$, so the lines have to be moved there.

Proposition 7.10 (pull-back to $J(C)$). *Let (H,ψ) be a piece with cover $X = C/\ker\psi$, $\pi\colon C \to X$, and let $\mathrm{Nm}\colon J(C) \to J(X)$ be the norm map (push-forward of divisors). Under $H^1(J(\cdot),\mathbb{Q}) = H^1(\cdot,\mathbb{Q})$, $\mathrm{Nm}^* = \pi^*$, and Nm^* maps $H^1(X,\mathbb{C})_\psi$ isomorphically onto $W_{H,\psi}$. Consequently $\mathrm{Nm}^*\Lambda_\psi = \bigwedge^{\mathrm{top}} W_{H,\psi} \subset H^d(J(C),\mathbb{C})$ for $\Lambda_\psi = \bigwedge^d H^1(X,\mathbb{C})_\psi$, and if $\Lambda_\psi \subset \mathrm{Alg}(J(X)) \otimes \mathbb{C}$ then $\Lambda_{H,\psi} := \bigwedge^{\mathrm{top}} W_{H,\psi} \subset \mathrm{Alg}(J(C)) \otimes \mathbb{C}$.*

Proof. Nm is the homomorphism of Jacobians induced by π_* on divisors of degree 0; on $H_1(J(C),\mathbb{Z}) = H_1(C,\mathbb{Z})$ [Mil08, I, §18, p. 73, and I, Thm. 2.3] it is π_*, so on $H^1 = \mathrm{Hom}(H_1,\mathbb{Z})$ it is the dual map, which under Poincaré duality on the curves is π^*. By Proposition 7.4, π^* maps $H^1(X,\mathbb{C})_\psi$ isomorphically onto $W_{H,\psi}$. Since $H^\bullet$ of an abelian variety is the exterior algebra on H^1, by cup product [Mil08, I, Thm. 2.3], and pull-back is a ring homomorphism, Nm^* maps $\bigwedge^d H^1(X)_\psi$ onto $\bigwedge^d W_{H,\psi}$. Pull-back along a morphism of smooth projective varieties sends cycle classes to cycle classes, hence preserves $\mathrm{Alg}\otimes\mathbb{C}$. □

So, granted steps (iii) and (iv), all eleven lines $\Lambda_j = \Lambda_{H_j,\psi_j} = \bigwedge^{\mathrm{top}} W_j$ are spanned by complex combinations of algebraic classes on P. This is the input of step (vi): there, the Λ_j with $c_j = +1$ live on one product of copies of P, those with $c_j = -1$ on another, and a correspondence with coefficients in $\mathbb{C}[G]$, built from Wedderburn's matrix units (Theorem 3.20), matches the two sides block by block. That only works because the multiplicity vectors match, which is Proposition 7.8.

Exercise 7.10. (a) For an étale double cover $\pi\colon X \to Y$ of a genus-2 curve, compute $g(X)$ and the dimensions of the ±1-eigenspaces of $H^1(X)$, and describe $\pi^*H^1(Y)$ inside $H^1(X)$. (b) Show that $\mathrm{Nm} \circ \pi^* = [\deg\pi]$ on $J(Y)$, where $\pi^*\colon J(Y) \to J(X)$ is pull-back of divisors, and deduce that Nm^* is injective on H^1.

Remark 7.11 (Where this is used). The relation of Proposition 7.8 and the lines Λ_j of Proposition 7.10 are the input of step (vi), Chapter 11, where $D_+ = 28374$ and $D_- = \dim U_- = 28368$ reappear. The hypotheses checked here (full pairing, $r \geq 64$, the exponents of order 2 and 3 and the Fermat orders $m' \in \{3,6\}$) are those of the cyclic-cover theorem in Chapters 8 and 9; the étale data ($g(Y) \geq 2341$, $d = 2g(Y)-2$) are used in Chapter 10. The review chapter Chapter 13 summarizes which pieces rest on which source.

CHAPTER 8

Symmetric products, the Abel–Jacobi map and the special fibre

Proof steps: (iii), steps 1–4

Where we are. This chapter and the next explain step (iii), the cyclic-cover theorem (Theorem 8.1): for a cyclic cover of curves $X \to Y$ with fully paired branching data, the top exterior power of a faithful eigenspace of $H^1(X)$ is spanned by algebraic classes on $J(X)$. In the proof of the Main Theorem it gives the Weil lines of the eight ramified pieces of Table 7.1. Here we do steps 1–4 of its proof: the eigenspace is carried by a quotient Z of the self-product X^d, which fibres over the symmetric product $\mathrm{Sym}^d Y$; the relevant cohomology of Z is concentrated on a single fibre of the Abel–Jacobi map, and that fibre is a cone over a projective space carrying the covering data of a Fermat variety. Chapter 9 (steps 5–9) turns this into algebraic cycles. The construction is Schoen's [Sch88, §2]. Background: Chapter 7 (pieces, $d = 2g - 2 + r$, full pairing); Jacobians and Abel's theorem [Mil08, I, §18]; Riemann–Roch on curves [Dem12, Ch. VI, (10.8)]; local systems (recalled in Section 5.5); direct images and the Leray spectral sequence [Stacks, Tags 01E4, 01F2]; Thom forms [BT82, §6]; Fermat varieties [Shi79, Thm. I], recalled in Section 8.1. *Notation warning:* in this chapter and the next, Z is the quotient $X^d/\mathcal{N}$, not the centre of G_{32} as in Chapter 7, and $\mathcal{H}_i$ are hypersurfaces in $\mathrm{Sym}^d Y$, unrelated to the Hurwitz component $\mathcal{H}$ of Chapter 5.

8.1. Steps 1–4 in two pages

Let Y be a smooth projective curve of genus g and $\pi\colon X \to Y$ a connected Galois cover with group $\mu_m = \langle\sigma\rangle$, $m \geq 3$, branched over $r \geq 3$ distinct points $b_1, \dots, b_r$ with local monodromy σ^{a_i} at b_i, $a_i \in \mathbb{Z}/m \setminus \{0\}$, $\sum_i a_i \equiv 0$. Let $\psi(\sigma) = \zeta_m$, let $W_\psi \subset H^1(X, \mathbb{C})$ be the ζ_m-eigenspace of σ^*, of dimension $d = 2g - 2 + r$ (Chevalley–Weil, Proposition 7.4), and put $d' = \gcd(m, a_1, \dots, a_r)$, $m' = m/d'$, $a_i' = a_i/d' \bmod m'$, $\alpha' = (a_1', \dots, a_r')$. Let $F = X_{m'}^{r-2} \subset \mathbb{P}^{r-1}$ be the Fermat hypersurface of degree m' and dimension $r-2$, $G_F = \mu_{m'}^r/\mu_{m'}$ its diagonal group, and $V(\alpha) \subset H^{r-2}(F, \mathbb{C})$ the eigenline of a character α; $\mathfrak{C}_{m'}^{r-2}$ is the set of characters whose line is spanned by algebraic classes (see below).

Fermat varieties. We fix the conventions for the Fermat hypersurface $F = X_m^n\colon \sum_{i=1}^{n+2} u_i^m = 0$ in $\mathbb{P}^{n+1}$ once [Shi79, (1.2)–(1.6), (1.9), Thm. I, pp. 176–177]. The group $G_F = \mu_m^{n+2}/\mu_m$ (modulo the diagonal) acts coordinatewise. A character of G_F is $\alpha = (a_1, \dots, a_{n+2}) \in (\mathbb{Z}/m)^{n+2}$ with $\sum_i a_i = 0$, $\alpha(g) = \prod_i \zeta_i^{a_i}$ for $g = (\zeta_i)$, and

$$V(\alpha) = \{\xi \in H^n_{\mathrm{prim}}(F, \mathbb{C}) : g^*\xi = \alpha(g)\xi \text{ for all } g \in G_F\},$$

defined by *pull-back*. Put $\mathfrak{A}_m^n = \{\alpha : a_i \neq 0 \text{ for all } i\}$ and $|\alpha| = \sum_i \langle a_i/m\rangle$, with $\langle\cdot\rangle$ the fractional part. Then $H^n_{\mathrm{prim}}(F, \mathbb{C}) = \bigoplus_{\alpha \in \mathfrak{A}_m^n} V(\alpha)$ with every $V(\alpha)$ a line, and $V(\alpha) \subset H^{n-q,q}$ with $q = |\alpha| - 1$ [Shi79, Thm. I(i), (ii), p. 176]; this is the assignment of Hodge types we use (some sources, for instance [SSL10, §3], use the opposite one; nothing below depends on the choice, since every set of characters we use is stable under $\alpha \mapsto -\alpha$). For even $n = 2p$, $\mathfrak{B}_m^n = \{\alpha \in \mathfrak{A}_m^n : |t\alpha| = p + 1 \text{ for all } t \in (\mathbb{Z}/m)^\times\}$ is the set of characters whose line consists of Hodge classes, and $\mathfrak{C}_m^n = \{\alpha : V(\alpha) \subset \mathrm{Alg}(F) \otimes \mathbb{C}\}$ the set of those whose line is spanned by algebraic classes.

THEOREM 8.1 (Weil lines of cyclic covers). *In this setting let $\Lambda_\psi = \bigwedge^d W_\psi$, a line in $H^d(J(X),\mathbb{C}) = \bigwedge^d H^1(X,\mathbb{C})$.*

(a) If r is even and $\pm\alpha' \in \mathfrak{C}^{r-2}_{m'}$, then $\Lambda_\psi \subset \mathrm{Alg}^{d/2}(J(X))_{\mathbb{Q}} \otimes \mathbb{C}$.

(b) If $\{a_1,\dots,a_r\}$ is fully paired (Definition 7.6), then (a) applies to ψ^t for every $t \in (\mathbb{Z}/m)^\times$, every W_{ψ^t} has Hodge numbers $(d/2,d/2)$, and the rational space $\bigwedge^d_{\mathbb{Q}(\zeta_m)} H^1(X,\mathbb{Q})_{[\psi]}$, whose complexification is $\bigoplus_t \Lambda_{\psi^t}$, is spanned by algebraic classes.

REMARK 8.2 (parity, and $d \le g(X)$). In (a) the parity of r is needed for the statement to make sense: $d = 2g-2+r$ is even exactly when r is, and $\mathfrak{C}^{r-2}_{m'}$ is defined for even $r-2$. Fully paired data have r even, so (b) is not restricted. The proof uses the inequality $d \le g(X)$ (Poincaré's formula, Proposition 9.7), which holds automatically: for every $t \in (\mathbb{Z}/m)^\times$ the character ψ^t is faithful, so $\dim W_{\psi^t} = 2g-2+r = d$ (Proposition 7.4), and these $\varphi(m)$ eigenspaces are independent in $H^1(X,\mathbb{C})$; hence $\varphi(m)\,d \le 2g(X)$, and $\varphi(m) \ge 2$ for $m \ge 3$ gives $d \le g(X)$.

The one idea of steps 1–4 is that the line Λ_ψ can be moved, without losing or gaining anything, through a chain of spaces until it becomes the Fermat line $V(\alpha)$:

$$\begin{aligned}\bigwedge^d W_\psi \overset{1}{=} H^d(Z,\mathbb{C})_\psi \overset{2}{=} H^d(S, j_!\mathbb{L}_\psi) \overset{3}{\cong} H^d_c(U_0,\mathbb{L}_\psi)\\ \overset{4}{\cong} H^{r-2}_c(U_R,\mathbb{L}') \overset{4}{=} V(\alpha), \qquad \alpha \in \{\alpha',-\alpha'\}.\end{aligned} \tag{8.1}$$

Step 1: $Z = X^d/\mathcal{N}$ is a quotient of X^d by a group that permutes the factors and twists them by σ with total twist 1; by Künneth and transfer its ψ-part is exactly the antisymmetric power of W_ψ, one line (Lemma 8.6). Step 2: Z is a μ_m-cover of $S = \mathrm{Sym}^d Y$, so its ψ-part is the cohomology of a rank-one local system $\mathbb{L}_\psi$ on the open set $U \subset S$ of divisors avoiding the b_i, extended by zero (Proposition 8.9). Step 3: for $d \ge 2g-1$ the Abel–Jacobi map $S \to \mathrm{Pic}^d(Y)$ is a projective bundle with fibres $|\mathcal{L}|$, and on every fibre except $|\mathcal{L}_0|$, $\mathcal{L}_0 = K_Y(b_1+\cdots+b_r)$, the open part is a torus $(\mathbb{C}^*)^{r-1}\times\mathbb{C}^{g-1}$ around which $\mathbb{L}_\psi$ winds non-trivially, so it contributes nothing (Lemma 8.11 and Proposition 8.12). Step 4: the special fibre $|\mathcal{L}_0|$ is the cone over $\mathbb{P}(R) = \mathbb{P}^{r-2}$, $R = \{w \in \mathbb{C}^r : \sum w_i = 0\}$, with vertex $\mathbb{P}^{g-1}$; the residue theorem $\sum w_i = 0$ becomes the Fermat equation after $w_i = u_i^{m'}$, and $\mathbb{L}_\psi$ is the local system of the Fermat covering $F \to \mathbb{P}(R)$ (Proposition 8.15). The degrees fit because $d = 2g + (r-2)$: the vertex direction contributes the Thom shift $2g$.

object	general	μ_6, $(2,4,2,4)$ over E	piece 9
$d = \dim W_\psi$	$2g-2+r$	4	312
fibre $\lvert\mathcal{L}\rvert$ of $\mathrm{Sym}^d Y \to \mathrm{Pic}^d(Y)$	$\mathbb{P}^{d-g}$	$\mathbb{P}^3$	$\mathbb{P}^{235}$
$\lvert\mathcal{L}\rvert \cap U$ for $\mathcal{L} \ne \mathcal{L}_0$	$(\mathbb{C}^*)^{r-1}\times\mathbb{C}^{g-1}$	$(\mathbb{C}^*)^3$	$(\mathbb{C}^*)^{159}\times\mathbb{C}^{76}$
vertex of $\lvert\mathcal{L}_0\rvert$	$\mathbb{P}^{g-1}$	a point	$\mathbb{P}^{76}$
base $\mathbb{P}(R)$ of the cone	$\mathbb{P}^{r-2}$	$\mathbb{P}^2$	$\mathbb{P}^{158}$
Fermat source F	$X^{r-2}_{m'}$	X^2_3 (cubic surface)	X^{158}_6

TABLE 8.1. The objects of steps 1–4, for the elliptic example of Example 8.17 and for piece 9 of Table 7.1 ($g = 77$, $r = 160$, $m = m' = 6$).

Credit. The quotient Z, the distinguished Abel–Jacobi fibre $|K_Y(D)|$ and its linear subspace $\{D + D'' : D'' \in |K_Y|\}$ (our vertex) are Schoen's: in his notation the fibre P_∞ of $S^h X \to \mathrm{Pic}^0 X$ over 0 is a projective space of dimension $h-q$ containing the $(q-1)$-dimensional $P_{-\infty}$ [Sch88, §2, (2.1) and p. 12]. The special fibre is a cone over a quotient of a Fermat variety (step 4), which brings in the cyclic covers of projective space branched along hyperplanes studied in [Sch89]. For fully paired data Schoen's Theorem 2.0 and Corollary 3.1 state the conclusion of Theorem 8.1(b) for every $m > 2$. For the related Lemma 3.4 of [Ope26a] see Remark 9.13.

8.2. Symmetric products and the Abel–Jacobi map

A divisor $D = p_1 + \cdots + p_d$ of degree d on Y is an unordered d-tuple of points with repetitions, so the effective divisors of degree d form the quotient

$$\operatorname{Sym}^d Y = Y^d/S_d.$$

Although S_d has fixed points on Y^d (the diagonals), the quotient of a curve is smooth: near a point where k of the p_j collide at a point with local coordinate z, the unordered k-tuple $(z_1, \dots, z_k)$ is the same as the monic polynomial $\prod_j (T - z_j)$, whose k coefficients (the elementary symmetric functions) are local coordinates. So $\operatorname{Sym}^d Y$ is a smooth projective variety of dimension d [ACGH85, Ch. IV], [Mac62]. Its cohomology is the S_d-invariant part of $H^*(Y)^{\otimes d}$ (transfer, as in Lemma 8.6 below), which Macdonald computed: the Poincaré polynomial of $\operatorname{Sym}^n Y$ is the coefficient of x^n in $(1+xt)^{2g}/((1-x)(1-xt^2))$ [Mac62].

Sending a divisor to its line bundle defines the *Abel–Jacobi map*

$$\mathrm{AJ}_d \colon \operatorname{Sym}^d Y \to \operatorname{Pic}^d(Y), \qquad D \mapsto \mathcal{O}_Y(D),$$

which after a base point is the Abel–Jacobi map of [Mil08, I, §18, p. 73], $\operatorname{Pic}^d(Y) \cong \operatorname{Pic}^0(Y) \cong J(Y)$ by the theorems of Abel and Jacobi [Mil08, I, Thms. 18.3–18.4 and p. 73]; we write AJ_d rather than a Greek letter, since α denotes characters below. Its fibre over $\mathcal{L}$ is the set of effective divisors D with $\mathcal{O}(D) \cong \mathcal{L}$, which are the zero divisors of the nonzero sections of $\mathcal{L}$, up to scalars:

$$\mathrm{AJ}_d^{-1}(\mathcal{L}) = |\mathcal{L}| = \mathbb{P}(H^0(Y, \mathcal{L})) \cong \mathbb{P}^{h^0(\mathcal{L})-1}$$

[Mil08, III, §5, p. 101].

Intuition 8.3 (a fibration by projective spaces). Two divisors lie in the same fibre when they differ by the divisor of a meromorphic function, so a fibre is a "linear system": the zero sets of one family of sections. For small d the number of sections jumps from line bundle to line bundle, but once $d \geq 2g-1$ Riemann–Roch leaves no room for jumps, and $\operatorname{Sym}^d Y$ becomes a bundle of projective spaces of one fixed dimension $d-g$ over the g-dimensional torus $J(Y)$. Dimensions check: $(d-g)+g = d$.

Proposition 8.4 (the Abel–Jacobi bundle). *Let $g \geq 1$. For every line bundle of degree $d \geq 2g-1$, $h^0(\mathcal{L}) = d-g+1$, and AJ_d is a $\mathbb{P}^{d-g}$-bundle (the projectivization of a vector bundle of rank $d-g+1$ on $\operatorname{Pic}^d(Y)$). For $d = 2g-2$, $h^0(\mathcal{L}) = g-1$ for $\mathcal{L} \ncong K_Y$ and $h^0(K_Y) = g$.*

Proof. Riemann–Roch with Serre duality [Dem12, Ch. VI, (10.8), p. 318] says $h^0(\mathcal{L}) - h^0(K_Y \otimes \mathcal{L}^{-1}) = d+1-g$. For $d \geq 2g-1$ the bundle $K_Y \otimes \mathcal{L}^{-1}$ has negative degree $2g-2-d$, hence no sections, so $h^0(\mathcal{L}) = d-g+1$ for every $\mathcal{L}$. For $d = 2g-2$ it has degree 0, and a degree-0 line bundle has a nonzero section only if it is trivial; so $h^0(K_Y \otimes \mathcal{L}^{-1})$ is 1 for $\mathcal{L} \cong K_Y$ and 0 otherwise. That constant fibre dimension makes AJ_d a projective bundle (the sections of a Poincaré line bundle, pushed forward to Pic^d, form a vector bundle of rank $d-g+1$ by Grauert's theorem) is standard [ACGH85, Chs. IV and VII] (cited by chapter). □

Example 8.5 (genus 2: Sym^2 and Sym^3). Let $g = 2$. For $d = 2 = 2g-2$ the fibre of AJ_2 over $\mathcal{L} \ncong K_Y$ is $\mathbb{P}^0$, a point, and over K_Y it is $|K_Y| = \mathbb{P}^1$, the canonical pencil (the fibres of the hyperelliptic double cover $Y \to \mathbb{P}^1$). So $\mathrm{AJ}_2 \colon \operatorname{Sym}^2 Y \to \operatorname{Pic}^2(Y) \cong J(Y)$ is birational and contracts one $\mathbb{P}^1$ to a point; classically it is the blow-up of $J(Y)$ at that point [ACGH85, Ch. IV] (cited by chapter). The cohomology agrees: by the blow-up formula (for instance [SK79, Lemma 2.1], stated there for ℓ-adic cohomology) a blown-up surface gains one class in degree 2, so the Betti numbers should be those of the torus, $1, 4, 6, 4, 1$, plus t^2, and Macdonald's formula gives $1, 4, 7, 4, 1$ for $\operatorname{Sym}^2 Y$; the Euler characteristics are $\chi(\operatorname{Sym}^2 Y) = 1 = 0 - 1 + 2$. For $d = 3 = 2g-1$, Proposition 8.4 makes $\operatorname{Sym}^3 Y$ a $\mathbb{P}^1$-bundle

over $J(Y)$, with Poincaré polynomial $(1+t)^4(1+t^2)$ and $\chi=0$; Macdonald's formula gives the same polynomial.

EXERCISE 8.1. (a) Show that $\dim|\mathcal{L}| = d-g$ for every line bundle of degree $d \geq 2g-1$ on a curve of genus $g \geq 1$, and that the bound is sharp: on a curve of genus $g \geq 1$ the fibre of AJ_{2g-2} over K_Y has dimension $g-1$, one more than the other fibres. (b) For $g=1$ show that $\mathrm{AJ}_d\colon \mathrm{Sym}^d E \to \mathrm{Pic}^d(E) \cong E$ is a $\mathbb{P}^{d-1}$-bundle for every $d \geq 1$, and identify the fibre over $\mathcal{O}(d\cdot O)$ for $d=2$ with the pencil of fibres of the x-coordinate of a Weierstrass model.

8.3. The variety Z and its ψ-part (step 1)

From now on we are in the setting of Section 8.1. The group

$$\mathcal{G} = \mu_m^d \rtimes S_d, \qquad \mathcal{N} = \ker\Big(\mathcal{G} \to \mu_m,\ (c_1,\dots,c_d;\tau) \mapsto c_1c_2\cdots c_d\Big),$$

acts on X^d: c_j acts by σ^{c_j} on the j-th factor (we write μ_m additively as $\mathbb{Z}/m$ when convenient) and $\tau \in S_d$ permutes the factors. The quotient of X^d by all of $\mathcal{G}$ is $Y^d/S_d = \mathrm{Sym}^d Y =: S$. The variety of the proof is the intermediate quotient

$$Z = X^d/\mathcal{N}, \qquad q\colon X^d \to Z, \qquad \pi_Z\colon Z \to S,$$

where π_Z is the quotient by $\mathcal{G}/\mathcal{N} \cong \mu_m$; a point of Z over a divisor $D' = p_1+\cdots+p_d$ is a choice of lifts $x_j \in \pi^{-1}(p_j)$, up to permutations and up to changing the lifts by σ^{c_j} with $\sum_j c_j \equiv 0$. Only the total twist survives, so π_Z has degree m. A subscript ψ denotes the subspace on which σ, or $\mathcal{G}/\mathcal{N}$, acts through ψ (by pull-back).

LEMMA 8.6 (step 1: Künneth and transfer). *$H^*(Z,\mathbb{C})_\psi = \bigwedge^d W_\psi$, a line in degree d, and q^* maps it isomorphically onto the line $\ell_\psi \subset H^d(X^d,\mathbb{C})$ of alternating tensors in $W_\psi^{\otimes d}$. More generally, for $k \not\equiv 0$ the ψ^k-part of $H^*(Z,\mathbb{C})$ is $\bigwedge^d W_{\psi^k}$, placed in degree d, and the invariant part is $H^*(\mathrm{Sym}^d Y,\mathbb{C})$.*

PROOF. For a finite group acting on a compact space, pull-back along the quotient map identifies the rational (or complex) cohomology of the quotient with the invariants ([Hat02, §3.G] treats covering spaces; the argument is the one of Lemma 7.3: $q_*q^* = |\mathcal{N}|$ and $q^*q_* = \sum_{n\in\mathcal{N}} n^*$, with q_* defined by summing over fibres). So $H^*(Z) = H^*(X^d)^{\mathcal{N}}$. By Künneth $H^*(X^d) = H^*(X)^{\otimes d}$, where a permutation acts with the Koszul sign $(-1)^{\deg x \deg y}$ for every transposition of two classes x,y. Decompose $H^*(X) = \bigoplus_\kappa H^*(X)_\kappa$ by characters κ of μ_m; $H^0(X)$ and $H^2(X)$ are in the trivial summand, and $H^1(X)_{\psi^k} = W_{\psi^k}$. The element of $\mathcal{N}$ with σ in slot i, σ^{-1} in slot j and trivial permutation acts on $H^*(X)_{\kappa_1}\otimes\cdots\otimes H^*(X)_{\kappa_d}$ by $\kappa_i(\sigma)\kappa_j(\sigma)^{-1}$, so invariance under the sum-zero subgroup forces $\kappa_1 = \cdots = \kappa_d = \kappa$. On such a summand σ in one slot acts by $\kappa(\sigma)$, so the summand belongs to the character κ of $\mathcal{G}/\mathcal{N}$; the ψ^k-part of $H^*(Z)$ is therefore the S_d-invariant part of $\big(H^*(X)_{\psi^k}\big)^{\otimes d} = W_{\psi^k}^{\otimes d}$ for $k \neq 0$. Since W_{ψ^k} sits in odd degree, the Koszul sign makes the S_d-invariants the antisymmetric tensors, $\bigwedge^d W_{\psi^k}$, in degree d; for $k=1$ it is a line because $\dim W_\psi = d$. For $\kappa = 1$ the $\mathcal{N}$-invariants are the $\mathcal{G}$-invariants, i.e. $H^*(Y^d/S_d)$. The image of $\bigwedge^d W_\psi$ under q^* is ℓ_ψ by construction. □

Physics aside 8.7 (fermions in d slots). The Koszul sign is the rule for fermions: classes of odd degree anticommute. Putting d identical fermions into a d-dimensional one-particle space W_ψ gives exactly one state, the Slater determinant $v_1\wedge\cdots\wedge v_d$; this is ℓ_ψ. The group $\mathcal{N}$ plays the role of a superselection rule that only lets through states in which every particle carries the same charge κ. The analogy is exact as linear algebra (both are the exterior power of a super vector space) and stops there: no dynamics or statistics theorem is involved.

EXAMPLE 8.8 (the μ_6-cover $(3,1,4,4)$ of $\mathbb{P}^1$). Take $Y=\mathbb{P}^1$, $m=6$, $(a_i)=(3,1,4,4)$, the genus-3 curve of Example 2.9 (Riemann–Hurwitz: $2g(X)-2 = -12+(3+5+4+4) = 4$).

Chevalley–Weil gives $\dim W_{\psi^k} = -2 + \#\{i : ka_i \not\equiv 0 \bmod 6\} = 2, 1, 0, 1, 2$ for $k = 1, \dots, 5$, so $d = 2$, and W_ψ has Hodge numbers $(1, 1)$ ($g - 1 + \sum\langle a_i/6\rangle = -1 + 12/6 = 1$). By Lemma 8.6, $H^*(Z) = H^*(\mathrm{Sym}^2 \mathbb{P}^1) \oplus \bigwedge^2 W_\psi \oplus \bigwedge^2 W_{\psi^5}$, since $\bigwedge^2$ of a space of dimension ≤ 1 vanishes. With $\mathrm{Sym}^2 \mathbb{P}^1 = \mathbb{P}^2$ the Betti numbers of Z are $1, 0, 3, 0, 1$ and $\chi(Z) = 5$. The ψ-line of $H^2(Z)$ has type $(1, 1)$; Example 8.16 identifies it with a line on the Fermat sextic surface.

Exercise 8.2. (a) For $d = 2$ prove $H^*(X^2/\mathcal{N}) = H^*(\mathrm{Sym}^2 Y) \oplus \bigoplus_{k\neq 0} \bigwedge^2 W_{\psi^k}$ by hand: list the $\mathcal{N}$-invariant Künneth components of $H^*(X) \otimes H^*(X)$, and show that the swap acts on $W \otimes W \subset H^1 \otimes H^1$ by $v \otimes w \mapsto -w \otimes v$. (b) Compute the Betti numbers of Z for the μ_4-cover of $\mathbb{P}^1$ with exponents $(1, 3, 1, 3)$. (c) Why does the transfer argument need coefficients in a field of characteristic 0 (or prime to $|\mathcal{N}|$)?

Exercise 8.3. Let $d_k = \dim W_{\psi^k}$, with $d_0 = 2g$. Using $d_k = 2g - 2 + \#\{i : ka_i \not\equiv 0 \bmod m\}$ for $k \neq 0$, show $\sum_{k=0}^{m-1} d_k = 2g(X)$ by comparing with Riemann–Hurwitz, $2g(X) - 2 = m(2g - 2) + \sum_i (m - \gcd(a_i, m))$. Check it for $(3, 1, 4, 4)$ ($g(X) = 3$) and for the μ_6-cover of an elliptic curve with exponents $(2, 4, 2, 4)$ ($g(X) = 9$).

8.4. The ψ-part as a sheaf on the symmetric product (step 2)

Since π_Z is finite, it is a closed map with finite fibres between Hausdorff spaces, so the stalk of $R^q\pi_{Z*}\mathbb{C}_Z$ at s is $H^q(\pi_Z^{-1}(s), \mathbb{C})$ [Stacks, Lemma 20.18.1, Tag 09V5], which vanishes for $q > 0$ because the fibre is finite and discrete. The Leray spectral sequence [Stacks, Lemma 20.13.4, Tag 01F2] therefore gives $H^*(Z, \mathbb{C}) = H^*(S, \pi_{Z*}\mathbb{C}_Z)$. The sheaf $\pi_{Z*}\mathbb{C}_Z$ has stalk $\mathbb{C}[\pi_Z^{-1}(s)]$ at $s \in S$, the functions on the fibre, on which $\mathcal{G}/\mathcal{N} \cong \mu_m$ acts; it splits into isotypic summands, and $H^*(Z)_\psi = H^*(S, (\pi_{Z*}\mathbb{C})_\psi)$. Let

$$\mathcal{H}_i = \{D' \in S : D' \ge b_i\} \cong \mathrm{Sym}^{d-1} Y, \qquad U = S \setminus \textstyle\bigcup_i \mathcal{H}_i, \qquad j\colon U \hookrightarrow S.$$

For a local system $\mathbb{L}$ on U, the *extension by zero* $j_!\mathbb{L}$ is the sheaf on S with stalks $\mathbb{L}_s$ for $s \in U$ and 0 for $s \notin U$; since S is compact, $H^*(S, j_!\mathbb{L}) = H^*_c(U, \mathbb{L})$, the cohomology with compact supports.

Proposition 8.9 (step 2). *$(\pi_{Z*}\mathbb{C}_Z)_\psi = j_!\mathbb{L}_\psi$ for a rank-one local system $\mathbb{L}_\psi$ on U whose monodromy around $\mathcal{H}_i$ (the meridian) is $\zeta_m^{a_i}$, or its inverse if loops are made to act by the inverse of continuation; in either convention it is $\neq 1$, which is all that is used below.*

Proof. *Over U the action of $\mathcal{G}/\mathcal{N}$ is free.* Let $(c; \tau) \in \mathcal{G}$ fix a point $P = (x_1, \dots, x_d) \in X^d$ none of whose coordinates lies over a branch point. Follow a cycle $j_1 \to j_2 \to \cdots \to j_\ell \to j_1$ of τ: each step says $x_{j_{s+1}} = \sigma^c x_{j_s}$ for the relevant $c = c_{j_{s+1}}$, and going once around gives $x_{j_1} = \sigma^{c_{j_1} + \cdots + c_{j_\ell}} x_{j_1}$. Since μ_m acts freely on the fibre of π through x_{j_1}, the twists along the cycle sum to 0; summing over all cycles, $\sum_j c_j \equiv 0$, i.e. $(c; \tau) \in \mathcal{N}$. If $\bar\gamma \in \mathcal{G}/\mathcal{N}$ fixes $q(P)$, then $\gamma P = nP$ for some $n \in \mathcal{N}$, so $n^{-1}\gamma \in \mathcal{N}$ and $\bar\gamma = 1$. Hence π_Z is an unramified μ_m-cover over U, $\pi_{Z*}\mathbb{C}$ is a local system of rank m there, and its ψ-part $\mathbb{L}_\psi$ has rank one. *Over $\mathcal{H}_i$ the ψ-stalk vanishes.* A point of Z over $D' \ge b_i$ is $q(P)$ with, say, x_1 over b_i. The inertia group of x_1 is generated by the local monodromy σ^{a_i}, so $(\sigma^{a_i}, 1, \dots, 1; \mathrm{id})$ fixes P, and its image in $\mathcal{G}/\mathcal{N}$ is $\sigma^{a_i} \neq 1$. The stalk is $\mathbb{C}[\mu_m/\mathrm{Stab}]$ with a nontrivial stabilizer, on which only characters trivial on the stabilizer occur; ψ is faithful, so the ψ-part is 0. Thus $(\pi_{Z*}\mathbb{C})_\psi = j_!\mathbb{L}_\psi$. *Monodromy.* A meridian of $\mathcal{H}_i$ at a general point $b_i + p_2 + \cdots + p_d$ moves the point near b_i once around b_i and keeps the others fixed. Its lift moves x_1 to $\sigma^{a_i} x_1$, by definition of the local monodromy, so the meridian acts on the torsor $\pi_Z^{-1}(U)$ by σ^{a_i} and on its ψ-part by $\psi(\sigma^{a_i}) = \zeta_m^{a_i}$, which is $\neq 1$ because $a_i \not\equiv 0$. □

The meridian is well defined because $\mathcal{H}_i$ is irreducible and, at a general point, smooth with local equation "position of the point near b_i" (the elementary symmetric functions of Section 8.2; Schoen proves the corresponding transversality statement for all intersections of the $\mathcal{H}_i$ [Sch88, Lemma 2.2]).

EXERCISE 8.4. Take $d = 1$, so $Z = X$, $S = Y$ and $U = Y \setminus \{b_1, \dots, b_r\}$. (a) Show directly that $(\pi_* \mathbb{C}_X)_\psi = j_! \mathbb{L}_\psi$ with meridians $\zeta_m^{a_i}$. (b) Show $H_c^0(U, \mathbb{L}_\psi) = 0$ and $H_c^2(U, \mathbb{L}_\psi) = 0$ (the latter by Poincaré duality with $H^0(U, \mathbb{L}_\psi^\vee)$). (c) Deduce from $\chi_c(U, \mathbb{L}_\psi) = \chi(U) = 2 - 2g - r$ that $\dim W_\psi = \dim H^1(X)_\psi = 2g - 2 + r$: this is the Chevalley–Weil count, recovered from step 2.

8.5. Compactly supported cohomology of a twisted torus

Step 3 needs one computation: a rank-one local system with nontrivial monodromy on a torus $(\mathbb{C}^*)^k \times \mathbb{C}^l$ has no compactly supported cohomology at all. Compact supports enter because the stalk of a pushforward of $j_!\mathbb{L}$ along a proper map is the compactly supported cohomology of the open part of the fibre. For a local system $\mathbb{L}$ on an oriented manifold M of real dimension n, Poincaré duality reads $H_c^k(M, \mathbb{L}) \cong H^{n-k}(M, \mathbb{L}^\vee)^\vee$ [Ive86].

EXAMPLE 8.10 ($H_c^*(\mathbb{C}^*, \mathbb{L}_\lambda)$ from a cell complex). Write $\mathbb{C}^* = S^1 \times \mathbb{R}$ (polar coordinates, $\log|z|$) and compactify the $\mathbb{R}$-direction to $[-\infty, \infty]$; compactly supported cohomology is cohomology relative to the two boundary circles. Relative to the boundary, the cylinder has one 1-cell $e^1 = \{1\} \times \mathbb{R}$ and one 2-cell $e^2 = (S^1 \setminus \{1\}) \times \mathbb{R}$. A cochain with values in $\mathbb{L}_\lambda$ assigns a number to each cell after trivializing $\mathbb{L}_\lambda$ on the cell; the boundary of e^2 runs along e^1 once in each direction, and the two copies are compared through the monodromy λ. So the complex is

$$0 \to \mathbb{C} \xrightarrow{1-\lambda} \mathbb{C} \to 0 \qquad \text{(degrees } 1, 2\text{)}, \qquad H_c^1 = \ker(1-\lambda), \quad H_c^2 = \operatorname{coker}(1-\lambda).$$

For $\lambda = 1$ this is $H_c^*(\mathbb{C}^*) = H^{*-1}(S^1)$: $\mathbb{C}$ in degrees 1 and 2; for $\lambda \neq 1$ everything vanishes. Poincaré duality turns this into $H^0(\mathbb{C}^*, \mathbb{L}_\lambda) = H^1(\mathbb{C}^*, \mathbb{L}_\lambda) = 0$ for $\lambda \neq 1$, which a Čech computation confirms directly. Cover $\mathbb{C}^*$ by $U' = \mathbb{C}^* \setminus \mathbb{R}_{<0}$ and $V' = \mathbb{C}^* \setminus \mathbb{R}_{>0}$; U', V' and the two components $W_\pm$ of $U' \cap V'$ are contractible, so the cover is acyclic for $\mathbb{L}_\lambda$. Trivialize $\mathbb{L}_\lambda$ on U' and on V' so that the trivializations agree on W_+; on W_- they then differ by λ. The Čech differential $\mathbb{C}^2 \to \mathbb{C}^2$ is $(a, b) \mapsto (b - a, \lambda b - a)$, of determinant $1 - \lambda$, so both H^0 and H^1 vanish for $\lambda \neq 1$, and both are $\mathbb{C}$ for $\lambda = 1$.

LEMMA 8.11 (twisted tori). *Let $M = (\mathbb{C}^*)^k \times \mathbb{C}^l$ with $k \geq 1$ and let $\mathbb{L}$ be a rank-one local system on M whose monodromy λ_i around the i-th factor $\mathbb{C}^*$ is $\neq 1$ for at least one i. Then $H_c^*(M, \mathbb{L}) = 0$. For the trivial system, $H_c^q(M, \mathbb{C})$ has dimension $\binom{k}{q-k-2l}$.*

PROOF. $\pi_1(M) = \mathbb{Z}^k$, generated by the loops of the factors, and rank-one local systems correspond to characters of π_1 (Section 5.5), so $\mathbb{L} \cong \mathbb{L}_{\lambda_1} \boxtimes \cdots \boxtimes \mathbb{L}_{\lambda_k} \boxtimes \mathbb{C}_{\mathbb{C}^l}$ is an exterior product. Give each $\mathbb{C}^*$ factor the relative cell structure of Example 8.10 and each $\mathbb{C}$ factor the one of $(D^2, \partial D^2)$, a single 2-cell. The product of these relative cell structures computes $H_c^*(M, \mathbb{L})$, and its cellular cochain complex is the tensor product of the factors' complexes, with the twisting matrices multiplied because $\mathbb{L}$ is an exterior product [Hat02, §3.B]:

$$C^\bullet = \bigotimes_{i=1}^{k} [\mathbb{C} \xrightarrow{1-\lambda_i} \mathbb{C}] \otimes \bigotimes^{l} \mathbb{C}[-2].$$

Over a field the cohomology of a tensor product of complexes is the tensor product of the cohomologies (algebraic Künneth). The factor with $\lambda_i \neq 1$ is acyclic, so $C^\bullet$ is acyclic. For the trivial system each $\mathbb{C}^*$ factor contributes $\mathbb{C}$ in degrees 1 and 2 and each $\mathbb{C}$ factor $\mathbb{C}$ in degree 2, which gives the binomial count. □

The Euler characteristic of $(\mathbb{C}^*)^{r-1} \times \mathbb{C}^{g-1}$ is 0 for every local system ($\chi(\mathbb{C}^*) = 0$), so χ alone cannot distinguish the twisted case from the trivial one, where H_c^* is large ($1, 3, 3, 1$ in degrees $3, \dots, 6$ for $(\mathbb{C}^*)^3$). The vanishing in every degree is the content of the lemma.

EXERCISE 8.5. (a) Compute $H_c^*(\mathbb{C}^*, \mathbb{L}_\zeta)$ for $\zeta = \zeta_6$ a second way: cover $\mathbb{C}^*$ by two simply connected open sets as in Example 8.10 and use Poincaré duality. (b) Write out the four-cell complex of $(\mathbb{C}^*)^2$ with monodromies $(1, \zeta)$, including signs, and check that it is

acyclic although the first factor is untwisted. (c) Show that a rank-one local system on $\mathbb{C}^l$ is trivial, so that the twist in Lemma 8.11 must sit on a $\mathbb{C}^*$ factor.

8.6. Support on one Abel–Jacobi fibre (step 3)

Let $D = b_1 + \cdots + b_r$ (the branch divisor on Y) and $\mathcal{L}_0 = K_Y(D)$, a line bundle of degree $2g-2+r = d$. Its sections are the meromorphic differentials with at most simple poles at the b_i. Since $r \geq 1$, $d \geq 2g-1$, so for $g \geq 1$ the Abel–Jacobi map $\mathrm{AJ}_d \colon S \to \mathrm{Pic}^d(Y)$ is a $\mathbb{P}^{d-g}$-bundle by Proposition 8.4. For $g = 0$ there is nothing to do: $\mathrm{Pic}^d(\mathbb{P}^1)$ is a point and $S = \mathbb{P}^d = |\mathcal{L}_0|$.

Proposition 8.12 (step 3: support). *Let $g \geq 1$. For every $\mathcal{L} \in \mathrm{Pic}^d(Y)$ with $\mathcal{L} \not\cong \mathcal{L}_0$, the evaluations at $b_1, \dots, b_r$ are linearly independent on $H^0(\mathcal{L})$, $|\mathcal{L}| \cap U \cong (\mathbb{C}^*)^{r-1} \times \mathbb{C}^{g-1}$, and $H^*_c(|\mathcal{L}| \cap U, \mathbb{L}_\psi) = 0$. Consequently $R\mathrm{AJ}_{d*}(j_!\mathbb{L}_\psi)$ is a skyscraper at $\mathcal{L}_0$, and*

$$H^*(Z, \mathbb{C})_\psi = H^*(S, j_!\mathbb{L}_\psi) \cong H^*_c(U_0, \mathbb{L}_\psi), \qquad U_0 = U \cap |\mathcal{L}_0|. \tag{8.2}$$

Proof. *Evaluations.* $\mathcal{L}(-D) = \mathcal{L} \otimes \mathcal{O}(-D)$ has degree $2g-2$ and is not K_Y, so $h^0(\mathcal{L}(-D)) = g-1$ by Proposition 8.4, while $h^0(\mathcal{L}) = d-g+1 = g+r-1$. The sections of $\mathcal{L}(-D)$ are the sections of $\mathcal{L}$ vanishing at all b_i, the kernel of the evaluation map $\mathrm{ev} \colon H^0(\mathcal{L}) \to \bigoplus_i \mathcal{L}_{b_i} \cong \mathbb{C}^r$. Its image has dimension $(g+r-1)-(g-1) = r$, so ev is onto: the r evaluations are independent linear forms. *The open part of the fibre.* Choose coordinates on $H^0(\mathcal{L}) \cong \mathbb{C}^r \oplus \mathbb{C}^{g-1}$ whose first r are $w_i = \mathrm{ev}_i$ (here $g \geq 1$ is used: $r \leq h^0(\mathcal{L})$). The divisor of a section s contains b_i exactly when $w_i(s) = 0$, so $\mathcal{H}_i \cap |\mathcal{L}|$ is the hyperplane $\{w_i = 0\}$ (as a set), and

$$|\mathcal{L}| \cap U = \{[w : v] : w_1 \cdots w_r \neq 0\} \cong (\mathbb{C}^*)^{r-1} \times \mathbb{C}^{g-1} \qquad (\text{normalize } w_r = 1).$$

The intersection is transversal at a general point: at a divisor $b_i + p_2 + \cdots + p_d$ of $|\mathcal{L}|$ with distinct points away from the other b_j, the position z of the point near b_i is a local equation of $\mathcal{H}_i$ in S (Section 8.2), and since the section vanishes simply at b_i, the implicit function theorem gives $z = w_i \cdot (\text{unit})$ on $|\mathcal{L}|$. Such divisors form a dense open subset of $\{w_i = 0\} = b_i + |\mathcal{L}(-b_i)|$: $\deg \mathcal{L}(-b_i) = 2g-3+r \geq 2g$ for $r \geq 3$, so $|\mathcal{L}(-b_i)|$ has no base points, and its general member is reduced (Bertini's theorem [Har77, III, Cor. 10.9]) and avoids any given finite set (the members through a given point form a hyperplane). Hence a loop around $w_i = 0$ in $|\mathcal{L}|$ is a meridian of $\mathcal{H}_i$ in S, on which $\mathbb{L}_\psi$ has monodromy $\zeta_m^{a_i} \neq 1$ by Proposition 8.9. The loop around $w_1 = 0$ is the loop of the first $\mathbb{C}^*$ factor, so Lemma 8.11 gives $H^*_c(|\mathcal{L}| \cap U, \mathbb{L}_\psi) = 0$. *Pushing forward.* AJ_d is proper, so by proper base change [Stacks, Lemma 20.18.1, Tag 09V5] the stalk of $R^q\mathrm{AJ}_{d*}(j_!\mathbb{L}_\psi)$ at $\mathcal{L}$ is $H^q(|\mathcal{L}|, (j_!\mathbb{L}_\psi)|_{|\mathcal{L}|}) = H^q_c(|\mathcal{L}| \cap U, \mathbb{L}_\psi)$, which vanishes for $\mathcal{L} \neq \mathcal{L}_0$. A complex of sheaves supported at one point has as cohomology its stalk there, so the Leray spectral sequence of AJ_d collapses to $H^*(S, j_!\mathbb{L}_\psi) \cong H^*(|\mathcal{L}_0|, (j_!\mathbb{L}_\psi)|_{|\mathcal{L}_0|}) = H^*_c(U_0, \mathbb{L}_\psi)$. The first equality in (8.2) is step 2. □

Intuition 8.13 (why one fibre survives). On a general fibre the ψ-local system is a flat line bundle with nontrivial holonomy around a circle of the torus $(\mathbb{C}^*)^{r-1}$, and such a bundle has no harmonic representatives at all, in any degree; this is the topological version of the statement that a charged particle on a circle threaded by a fractional flux has no zero mode. The analogy stops at the compact supports, which have no counterpart there. At $\mathcal{L}_0 = K_Y(D)$ the argument breaks for a reason one can name: the residue theorem. A differential with simple poles at the b_i has residues summing to zero, so only $r-1$ of the r evaluations are independent, the torus loses a factor, and what is left (step 4) has cohomology in exactly one degree.

Example 8.14 (the fibres in the small cases). (i) For the μ_6-cover of an elliptic curve E with exponents $(2,4,2,4)$ of Example 8.17: $g = 1$, $r = 4$, $d = 4$, $\mathrm{Pic}^4(E) \cong E$, every fibre is $|\mathcal{L}| = \mathbb{P}^3$ ($h^0 = 4$), the special one is $|\mathcal{O}_E(D)|$ (K_E is trivial), and for $\mathcal{L} \neq \mathcal{O}_E(D)$ the open part is $(\mathbb{C}^*)^3$, with no $\mathbb{C}$ factor. (ii) For a μ_3-cover of a genus-2 curve with exponents $(1,1,1)$:

$g(X)=7$, $d=5$, W_ψ of type $(3,2)$ (type $(2,3)$ in the opposite reading of the tuple), AJ_5 is a $\mathbb{P}^3$-bundle over the abelian surface $\mathrm{Pic}^5(Y)$, and the open part of a general fibre is $(\mathbb{C}^*)^2\times\mathbb{C}$. In both cases $\chi=0$ and $H^*_c=0$, while Lemma 8.6 predicts one line in $H^4(Z)_\psi$, respectively $H^5(Z)_\psi$; it has to come from the special fibre.

8.7. The special fibre is a cone over a Fermat quotient (step 4)

Sections of $\mathcal{L}_0=K_Y(D)$ are differentials η with at most simple poles at the b_i. The value of such a section at b_i, in the fibre $\mathcal{L}_{0,b_i}\cong\mathbb{C}$, is its residue $\mathrm{Res}_{b_i}\eta$, so the divisor of η contains b_i exactly when $\mathrm{Res}_{b_i}\eta=0$. Let $H^0(K_Y)$ be the space of holomorphic differentials, and

$$\mathrm{Res}\colon H^0(K_Y(D))\to R=\Big\{w\in\mathbb{C}^r:\sum_i w_i=0\Big\},\qquad \eta\mapsto(\mathrm{Res}_{b_1}\eta,\dots,\mathrm{Res}_{b_r}\eta).$$

The image lies in R by the residue theorem, the kernel is $H^0(K_Y)$, and $\dim H^0(K_Y(D))=g+r-1=g+\dim R$ (Riemann–Roch), so the map is onto. Choosing a linear splitting $R\to H^0(K_Y(D))$ identifies $H^0(K_Y(D))=H^0(K_Y)\oplus R$.

Proposition 8.15 (step 4: the cone). *(a) $|\mathcal{L}_0|=\mathbb{P}(H^0(K_Y)\oplus R)\cong\mathbb{P}^{g+r-2}$, and $\mathcal{H}_i\cap|\mathcal{L}_0|=\{w_i=0\}$, transversally at a general point. All these hyperplanes contain the vertex $\mathbb{P}(H^0(K_Y))\cong\mathbb{P}^{g-1}$, the set of divisors $D+D''$ with $D''\in|K_Y|$. (b) Projection from the vertex, $[\eta:w]\mapsto[w]$, makes $\mathcal{V}=|\mathcal{L}_0|\setminus\mathbb{P}(H^0(K_Y))$ a vector bundle of rank g over $\mathbb{P}(R)\cong\mathbb{P}^{r-2}$, and $U_0=\mathcal{V}|_{U_R}$, where $U_R\subset\mathbb{P}(R)$ is the complement of the r hyperplanes $\{w_i=0\}$, any $r-1$ of which are independent on R. Hence $\pi_1(U_0)=\pi_1(U_R)$, $H_1(U_R,\mathbb{Z})=\mathbb{Z}^r/\mathbb{Z}(1,\dots,1)$ is spanned by the meridians, $\mathbb{L}_\psi$ is pulled back from the rank-one system $\mathbb{L}'$ on U_R with meridians $\zeta_m^{a_i}=\zeta_{m'}^{a_i'}$, and $H^k_c(U_0,\mathbb{L}_\psi)\cong H^{k-2g}_c(U_R,\mathbb{L}')$. (c) The map $u\mapsto(u_1^{m'},\dots,u_r^{m'})$ sends $F=\{\sum_i u_i^{m'}=0\}\subset\mathbb{P}^{r-1}$ onto $\mathbb{P}(R)$, and $F^\circ=F|_{U_R}\to U_R$ is a G_F-torsor. With $\varsigma\colon G_F\to\mu_m$, $e_i\mapsto\sigma^{a_i}$ (e_i the generator of the i-th factor $\mu_{m'}$), $\mathbb{L}'$ is the α-isotypic part of the pushforward of $\mathbb{C}_{F^\circ}$ for $\alpha=\psi\circ\varsigma\in\{\alpha',-\alpha'\}$, and*

$$H^*_c(U_R,\mathbb{L}')=H^*(F,\mathbb{C})_\alpha=V(\alpha),\quad \text{a line in degree } r-2. \tag{8.3}$$

Proof. (a) A point of $|\mathcal{L}_0|$ is a section up to scale, i.e. a point of $\mathbb{P}(H^0(K_Y)\oplus R)$, and its divisor contains b_i iff $w_i=0$, as said above. Transversality at a general point of $\{w_i=0\}$ follows as in step 3, with $\mathcal{L}_0(-b_i)=K_Y(D-b_i)$, of degree $2g-3+r\geq 2g$, in place of $\mathcal{L}(-b_i)$. A section in $H^0(K_Y)$ is a holomorphic differential η, which as a section of $K_Y(D)$ vanishes at every b_i; its divisor as a section of $K_Y(D)$ is $D+\mathrm{div}(\eta)$, which gives the description of the vertex. (b) For fixed $[w]\neq 0$ the points $[\eta:w]$, $\eta\in H^0(K_Y)$, form an affine space $\cong H^0(K_Y)$, and globally $\mathcal{V}$ is the total space of $H^0(K_Y)\otimes\mathcal{O}_{\mathbb{P}(R)}(1)\cong\mathcal{O}(1)^{\oplus g}$. The hyperplanes $\{w_i=0\}$ contain the vertex, so U_0 is the restriction of $\mathcal{V}$ to U_R. Any $r-1$ of the coordinates w_i are independent on R (the r-th is minus their sum), so $U_R\cong\{(w_1,\dots,w_{r-1}):w_i\neq 0,\ \sum_{i<r}w_i\neq 0\}/\mathbb{C}^*$; the meridians of the r hyperplanes generate $H_1(U_R,\mathbb{Z})$ with the single relation that their sum is 0. The fibres of $U_0\to U_R$ are contractible, so $\pi_1(U_0)=\pi_1(U_R)$, every local system on U_0 is pulled back from U_R, and the meridian of $\{w_i=0\}$ in U_0 is the meridian of $\mathcal{H}_i$, with monodromy $\zeta_m^{a_i}$. A rank-one local system factors through H_1, so $\mathbb{L}'$ is determined by these monodromies; they are consistent because $\prod_i\zeta_m^{a_i}=\zeta_m^{\sum a_i}=1$. The Thom isomorphism for the rank-g complex vector bundle $U_0\to U_R$ [BT82, §6] shifts compactly supported degrees by $2g$. (c) If $\sum u_i^{m'}=0$ then $w_i=u_i^{m'}$ satisfies $\sum w_i=0$: the residue theorem has become the Fermat equation. The map is the quotient by $\mu_{m'}^r/\mu_{m'}=G_F$, of degree m'^{r-1}, and G_F acts freely exactly where no u_i vanishes, i.e. over U_R; going once around $\{w_i=0\}$ multiplies u_i by $\zeta_{m'}$, so the meridian acts on the torsor by e_i. The map ς is well defined: σ^{a_i} has order dividing m' (as $m'a_i=ma_i'$), and the diagonal $e_1\cdots e_r$ goes to $\sigma^{\sum a_i}=1$; its image is $\langle\sigma^{d'}\rangle\cong\mu_{m'}$. The α-isotypic part of the pushforward of $\mathbb{C}_{F^\circ}$ is the rank-one system on which the meridian of $\{w_i=0\}$ acts by $\alpha(e_i)=\psi(\sigma^{a_i})=\zeta_m^{a_i}=\zeta_{m'}^{a_i'}$, so it is $\mathbb{L}'$; whether α is α' or $-\alpha'$ depends on the convention for monodromy versus pull-back, and does not

matter below because Theorem 8.1(a) assumes both. Every entry of α is nonzero, and a point of F over $\{w_i = 0\}$ is fixed by the i-th factor $\mu_{m'}$, on which α is nontrivial; so the α-part of the cohomology of $F \setminus F^\circ$ vanishes, and the long exact sequence of the pair gives $H^*_c(U_R, \mathbb{L}') = H^*_c(F^\circ)_\alpha = H^*(F)_\alpha$. For α with all entries nonzero this is the line $V(\alpha)$ in the middle degree $r-2$ [Shi79, Thm. I(i), p. 176]; the other degrees of $H^*(F)$ are spanned by powers of the hyperplane class (Lefschetz hyperplane theorem; [SK79, Lemma 2.3] for the ℓ-adic statement), which is G_F-invariant. □

Putting (8.2), (b) and (8.3) together gives the chain (8.1), with degrees $d = 2g + (r-2)$ on the left and $r-2$ on the right. When $d' > 1$ the μ_m-torsor over U_R is disconnected (it is $F^\circ \times^{G_F,\varsigma} \mu_m$, with d' components); this enters only through ς. That is the situation of the $m = 12$ pieces 4 and 10, whose exponents $4, 6, 8$ give $d' = 2$, $m' = 6$, α' with entries $2, 3, 4$: on the special fibre they look exactly like μ_6-pieces (Table 7.1).

What just happened. Steps 1–4 do not produce a single cycle yet. They show that one line in the cohomology of Z, the line that pulls back to $\bigwedge^d W_\psi$, is the compactly supported cohomology of the open part of one projective space $|\mathcal{L}_0| \subset \operatorname{Sym}^d Y$, and that this open part is a rank-g vector bundle over the complement of r hyperplanes in $\mathbb{P}^{r-2}$, carrying the same local system as the Fermat covering $F^\circ \to U_R$. Counting degrees: $2g$ from the bundle, $r-2$ from the Fermat line, total d.

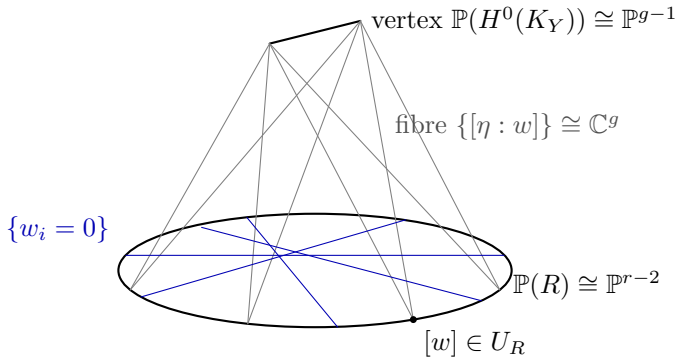


FIGURE 8.1. The special fibre $|\mathcal{L}_0| = \mathbb{P}(H^0(K_Y) \oplus R)$ as a cone: every point off the vertex lies on a unique $\mathbb{C}^g = \{[\eta : w] : \eta \in H^0(K_Y)\}$ over a point $[w]$ of $\mathbb{P}(R)$. The hyperplanes $\mathcal{H}_i \cap |\mathcal{L}_0|$ are the cones over the r hyperplanes $\{w_i = 0\}$ of $\mathbb{P}(R)$ (blue), so U_0 is the part over their complement U_R. For $g = 0$ there is no vertex; for $g = 1$ the vertex is the point D.

EXAMPLE 8.16 (genus 0: no vertex). For $Y = \mathbb{P}^1$, $K(D) = \mathcal{O}(r-2)$, $S = \operatorname{Sym}^{r-2}\mathbb{P}^1 = \mathbb{P}^{r-2} = |\mathcal{L}_0|$ and $H^0(K_Y) = 0$: the cone is all of $\mathbb{P}(R)$. Concretely, $\eta = \sum_i w_i\,\mathrm{d}x/(x - b_i)$ is regular at ∞ iff $\sum_i w_i = 0$, it then has $r-2$ zeros, and its divisor (as a section of $K(D)$) contains b_i iff $w_i = 0$; so $\operatorname{Sym}^{r-2}\mathbb{P}^1 \cong \mathbb{P}(R)$ with $\mathcal{H}_i = \{w_i = 0\}$. Chain (8.1) reads $H^{r-2}(Z)_\psi \cong H^{r-2}_c(U_R, \mathbb{L}') = V(\alpha)$ with no Thom shift. For $(3, 1, 4, 4)$ (Example 8.8): $d = 2$, $d' = 1$, $F = X_6^2$ is the Fermat sextic surface, and the ψ-line of $H^2(Z)$ corresponds to $V(\pm(3, 1, 4, 4))$, of type $(1, 1)$ since $|\alpha'| = (3 + 1 + 4 + 4)/6 = 2$.

EXAMPLE 8.17 (a μ_6-cover of an elliptic curve, exponents $(2, 4, 2, 4)$). Let $Y = E$, $m = 6$, $(a_i) = (2, 4, 2, 4)$; the cover is connected if the monodromy along a loop of E is an odd power of σ, which can be arranged since only the local monodromies are prescribed. Then $g(X) = 9$ ($2g(X) - 2 = 4 + 4 + 4 + 4$), $d = 4 \le 9$, W_ψ has type $(2, 2)$, the data are fully paired, $d' = 2$, $m' = 3$ and $\alpha' = (1, 2, 1, 2)$. Here $H^0(K_E) = \mathbb{C}\,\mathrm{d}z$, $|\mathcal{L}_0| = |\mathcal{O}_E(D)| = \mathbb{P}(\mathbb{C}\mathrm{d}z \oplus R) = \mathbb{P}^3$ with vertex the single divisor $D = b_1 + b_2 + b_3 + b_4$ (the section $\mathrm{d}z$ of $\mathcal{O}_E(D)$ vanishes exactly on D),

and the cone over $\mathbb{P}(R)=\mathbb{P}^2$ with a point as vertex is just $\mathbb{P}^3$ seen from that point; $\mathcal{V}\to\mathbb{P}^2$ is the total space of $\mathcal{O}(1)$. The four planes $\mathcal{H}_i\cap|\mathcal{L}_0|$ pass through the vertex and cut $\mathbb{P}(R)$ in four lines in general position. The chain reads $H^4(Z)_\psi\cong H^4_c(U_0,\mathbb{L}_\psi)\cong H^2_c(U_R,\mathbb{L}')=V(\alpha)\subset H^2(X_3^2)$: the Weil line of this cover is a line in the cohomology of the Fermat cubic surface, for the paired character $(1,2,1,2)$, which Chapter 9 shows is spanned by classes of lines on the surface. The μ_6-torsor over U_R has $d'=2$ components.

Exercise 8.6. For the μ_3-cover of a genus-2 curve with exponents $(1,1,1)$ of Example 8.14(ii): (a) show that $|\mathcal{L}_0|=\mathbb{P}^3$ is the join of the line $\mathbb{P}(H^0(K_Y))=D+|K_Y|$ and the line $\mathbb{P}(R)$, and that projection from $\mathbb{P}(H^0(K_Y))$ makes $\mathcal{V}$ the bundle $\mathcal{O}(1)^{\oplus 2}$ over $\mathbb{P}^1$; (b) identify U_R with $\mathbb{P}^1$ minus three points and F with the Fermat cubic curve, and check that $F^\circ\to U_R$ has degree $9=|G_F|$; (c) check the degrees in (8.1): $d=5=2\cdot 2+1$, and $V(\alpha)\subset H^1(F)$ has type $(1,0)$ or $(0,1)$. Why does Theorem 8.1(a) not apply here?

Exercise 8.7. (a) Using inclusion–exclusion for r hyperplanes in $\mathbb{P}^{r-2}$ any $r-1$ of which are independent, show $\chi(U_R)=\sum_{j=0}^{r-2}(-1)^j\binom{r}{j}(r-1-j)=(-1)^r$, consistent with (8.3) (one line in degree $r-2$, and $\chi_c(U_R,\mathbb{L}')=\chi(U_R)$). (b) Check $\chi(F^\circ)=|G_F|\,\chi(U_R)$ for the Fermat cubic surface: $\chi(X_3^2)=9$, $\chi(X_3^1)=0$, $\chi(X_3^0)=3$, so $\chi(F^\circ)=9-4\cdot 0+6\cdot 3=27=3^3$. (c) Show that $\chi\big((\mathbb{C}^*)^{r-1}\times\mathbb{C}^{g-1}\big)=0$ and explain why this alone does not prove Proposition 8.12.

Remark 8.18 (Where this is used). The chain (8.1) is the input of steps 5–7 (Chapter 9): a projective bundle Q over F maps onto $|\mathcal{L}_0|$ and lifts to Z, the pull-back of the ψ-line to Q is injective with image $\xi^g\cup\varpi^*V(\alpha)$, and an algebraic class spanning $V(\alpha)$ (full pairing, Shioda) gives an algebraic class spanning ℓ_ψ; step 8 moves it to $J(X)$. The paired condition of Theorem 8.1(b) enters only there, through α'; steps 1–4 hold for any exponents. For étale pieces ($r=0$, Chapter 10) the fibre $|\mathcal{L}_0|$ degenerates to $|K_Y|=\mathbb{P}^{g-1}$ inside $\mathrm{Sym}^{2g-2}Y$, and Schoen's 1988 argument uses its preimage directly [Sch88, §2, p. 13].

CHAPTER 9

The Fermat cone, injectivity and Shioda's linear cycles

Proof steps: (iii), steps 5–9; Schoen's theorem and the ramified pieces

Where we are. Chapter 8 reduced the Weil line $\Lambda_\psi = \bigwedge^d W_\psi$ of a cyclic cover $X \to Y$ to a line $V(\alpha)$ in the middle cohomology of a Fermat hypersurface $F = X^{r-2}_{m'}$, through the chain (8.1). That chain is a chain of isomorphisms of cohomology groups; it produces no cycle. This chapter does steps 5–9 of the proof of the cyclic-cover theorem (Theorem 8.1): it builds a smooth variety Q that maps to Z, shows that pulling back to Q loses nothing on the ψ-line (the step the whole proof leans on most), turns an algebraic class on F into an algebraic class spanning the line in $H^d(X^d)$ and then in $H^d(J(X))$, and supplies the algebraic class on F from Shioda's linear subspaces when the exponents are fully paired. It ends with the bookkeeping that matters for the Main Theorem: which result gives the Weil lines of which piece of Table 7.1, and how the statements of Schoen's theorem and of a related preprint compare with the cyclic-cover theorem (Remark 9.13). Background: Chapter 8 (notation $Z = X^d/\mathcal{N}$, q, π_Z, $S = \mathrm{Sym}^d Y$, U_0, U_R, F, G_F, ς, $\alpha = \psi \circ \varsigma$, Proposition 8.15, and the conventions for Fermat varieties in Section 8.1); Shioda's theorems on Fermat varieties [Shi79, Thms. I, II]; hard Lefschetz [Dem12, Ch. VI, (8.17), p. 313]; Lieberman's theorem [Lie68; Kle68], [Mil02, pp. 599–600]; correspondences [Mil13, §4], [Ful98, Ch. 16].

9.1. Steps 5–9 in two pages

At the end of Chapter 8 we had, for a μ_m-cover $\pi\colon X \to Y$ of a genus-g curve with r branch points and exponents a_i, a chain of isomorphisms carrying the ψ-line of $H^d(Z)$, $d = 2g+(r-2)$, to the line $V(\alpha) \subset H^{r-2}(F, \mathbb{C})$ of the Fermat hypersurface $F = \{u_1^{m'} + \cdots + u_r^{m'} = 0\} \subset \mathbb{P}^{r-1}$. Isomorphisms of cohomology groups do not move cycles. To move cycles one needs *maps of varieties*: a cycle on F is pushed or pulled along morphisms, and only then is its class known to be algebraic. Steps 5–9 provide those maps.

$$F \xleftarrow{\varpi} Q \xrightarrow{\hat{q}} Z \xleftarrow{q} X^d \xrightarrow{f} J(X). \tag{9.1}$$

Step 5 (Proposition 9.1): the special Abel–Jacobi fibre $|\mathcal{L}_0|$ is a cone with vertex $\mathbb{P}^{g-1}$ over $\mathbb{P}(R) = \mathbb{P}^{r-2}$ (Figure 8.1), and $F \to \mathbb{P}(R)$, $u \mapsto (u_i^{m'})$, is a branched cover of the base of the cone. Replacing the base by F and the cone by a projective bundle gives a smooth projective variety $Q = \mathbb{P}_F(\mathcal{O}_F^{\oplus g} \oplus \mathcal{O}_F(-m'))$, of dimension $d-g$, with a map $\varphi\colon Q \to |\mathcal{L}_0| \subset S$; Zariski's main theorem lifts it to $\hat{q}\colon Q \to Z$. *Step 6* (Theorem 9.4): $\hat{q}^*$ is injective on the line $H^d(Z, \mathbb{C})_\psi$, with image $\xi^g \cup \varpi^* V(\alpha)$, where ξ is the hyperplane class of the bundle. *Step 7*: a correspondence Γ from Q to X^d and the projector ϵ_ψ carry $\varpi^*\beta'$, for any β' spanning $V(\alpha)$, to a class x spanning the line $\ell_\psi \subset H^d(X^d)$; if β' is algebraic, so is x. *Step 8* (Proposition 9.7): the sum map f and Poincaré's formula, together with Lieberman's algebraic inverse of hard Lefschetz, move x to an algebraic class spanning $\Lambda_\psi \subset H^d(J(X))$. This proves Theorem 8.1(a): *if r is even and $V(\pm\alpha')$ are spanned by algebraic classes on F, then Λ_ψ is spanned by algebraic classes on $J(X)$. Step 9* (Proposition 9.9): if the exponents are fully paired, α' is a permutation of $(b_1, -b_1, \ldots, b_k, -b_k)$, and the linear subspace $\{u_{i_1} = \epsilon u_{j_1}, \ldots, u_{i_k} = \epsilon u_{j_k}\}$, $\epsilon^{m'} = -1$, is

a $\mathbb{P}^{k-1}$ on F, and the $V(\alpha')$-component of the class of such a subspace spans $V(\alpha')$ (after Shioda). That is Theorem 8.1(b).

The one idea of the chapter is the injectivity argument of step 6, and it is a selection rule. The group G_F acts on everything in sight, $\hat{q}$ is equivariant through ς, and the ψ-line of Z goes to the α-isotypic part of $H^*(Q)$. The only place where a class could be lost is the boundary $Q \setminus Q^\circ = F \times \mathbb{P}^{g-1}$ over the vertex, and the α-part of the cohomology of that boundary sits in degrees $(r-2)+2j$, never in the degree $d-1=(r-2)+2g-1$ where it could interfere. A parity count, not a vanishing theorem, does the work.

piece	m	exponents	m'	what the Main Theorem uses
$1,3,6,9,11$	6	$2,3,4$	6	the cyclic-cover theorem
2	6	$2,4$	3	the cyclic-cover theorem
$4,10$	12	$4,6,8$	6	the cyclic-cover theorem
$5,7,8$	$24,24,12$	étale	–	Schoen 1988 (or Patel–Zhang)

TABLE 9.1. Which result gives the Weil lines of which piece of Table 7.1. The statement of Schoen's Theorem 2.0 and Corollary 3.1 [Sch88] covers all eleven pieces. The étale pieces are Chapter 10.

Credit. The construction (the cone, the Fermat cover of its base, a smooth model and a correspondence to C^d; Schoen writes C^h, and his h is our d) is Schoen's [Sch88, §2], and its link with Fermat varieties goes through the cyclic covers of projective space branched along hyperplanes studied in [Sch89]. His Theorem 2.0 and Corollary 3.1 state the conclusion of Theorem 8.1(b) for every $m>2$ and every base genus (for Lemma 3.4 of [Ope26a] see Remark 9.13). The proof of steps 5–9 given here differs from Schoen's in the injectivity argument, which takes the place of his Lemma 2.8, and it uses from Shioda only his unconditional results: the eigenline decomposition [Shi79, Thm. I] and the inductive structure [Shi79, Thm. II and its Corollary]. Which result gives the Weil lines of which piece is in Table 9.1.

9.2. A smooth model of the cone and its lift to Z (step 5)

Why is a new variety needed at all? The fibre $|\mathcal{L}_0|$ is a smooth projective space, but the part of Z over it, $\pi_Z^{-1}(|\mathcal{L}_0|)$, is a branched cover of $|\mathcal{L}_0|$ that we do not know how to describe or resolve, and the Fermat variety F that carries the cycles maps only to the base $\mathbb{P}(R)$ of the cone, not to the cone itself. The remedy is to build the cone again, this time over F instead of $\mathbb{P}(R)$, as a projective bundle: then the space is smooth by construction, it maps to $|\mathcal{L}_0|$, and it contains a copy of F (a section) through which cycles on F enter.

Recall from Proposition 8.15 that a point of $|\mathcal{L}_0| = \mathbb{P}(H^0(K_Y)\oplus R)$ is $[\eta : w]$ with η a holomorphic differential and $w \in R = \{\sum w_i = 0\}$ the residue vector. The line bundle $\mathcal{O}_F(-m')$ has fibre over $u \in F$ the line spanned by $u^{\otimes m'}$, and $u^{\otimes m'} \mapsto (u_1^{m'}, \dots, u_r^{m'})$ is a well-defined vector of R (the Fermat equation is the residue theorem). Define

$$Q = \mathbb{P}_F(\mathcal{O}_F^{\oplus g} \oplus \mathcal{O}_F(-m')) \xrightarrow{\varpi} F, \qquad \varphi\colon Q \to |\mathcal{L}_0|, \quad (u,[\eta:\nu]) \mapsto [\eta + \nu\cdot(u_1^{m'},\dots,u_r^{m'})],$$

where $\mathbb{P}_F(E)$ is the bundle of lines in E, $\eta \in H^0(K_Y) \cong \mathbb{C}^g$, $\nu \in \mathcal{O}(-m')_u$ is read as a multiple of $(u_i^{m'})_i$, and $\xi = c_1(\mathcal{O}_Q(1))$. The group $G_F = \mu_{m'}^r/\mu_{m'}$ acts on F by scaling coordinates, $\mathcal{O}_F(-m')$ is G_F-linearized and $u^{\otimes m'}$ is G_F-invariant, so G_F acts on Q and φ is G_F-invariant. Inside Q there are three pieces:

$$Q \supset Q^\circ = \{\nu \neq 0\} \supset Q^{\circ\circ} = Q^\circ|_{F^\circ},$$
$$Q \setminus Q^\circ = \{\nu = 0\} = F \times \mathbb{P}^{g-1}, \qquad s_\infty = \{\eta = 0\} \cong F,$$

with $F^\circ \subset F$ the open set where no u_i vanishes (the G_F-torsor over U_R of Proposition 8.15(c)).

PROPOSITION 9.1 (step 5: a smooth model and a lift). *(a) Q is smooth, projective, irreducible, of dimension $(r-2)+g = d-g = \dim|\mathcal{L}_0|$; for $g=0$, $Q=F$. (b) $\varphi^{-1}(\mathbb{P}(H^0(K_Y))) = F\times\mathbb{P}^{g-1}$; Q° is the total space of $\mathcal{O}_F(m')^{\oplus g}$, which φ maps onto $\mathcal{V}=|\mathcal{L}_0|\setminus\mathbb{P}(H^0(K_Y))$, linearly on the fibres over $F\to\mathbb{P}(R)$; and $\varphi\colon Q^{\circ\circ}\to U_0$ is a G_F-torsor, connected for $r\geq 3$. In particular φ is surjective and generically finite of degree $|G_F|=m'^{r-1}$. (c) There is a morphism $\hat{q}\colon Q\to Z$ with $\pi_Z\circ\hat{q}=\varphi$, equivariant for G_F acting on Z through $\varsigma\colon G_F\to\mu_m=\mathcal{G}/\mathcal{N}$.*

PROOF. (a) A projective bundle over a smooth projective variety is smooth and projective, of dimension $\dim F+\mathrm{rk}-1=(r-2)+(g+1)-1$; F is smooth and irreducible for $r\geq 3$ (a smooth hypersurface of dimension ≥ 1). (b) $\varphi(u,[\eta:0])=[\eta:0]$ lies in the vertex, and conversely $\nu\neq 0$ gives a nonzero residue vector. On $\{\nu\neq 0\}$ normalize $\nu=u^{\otimes m'}$, i.e. $\nu=1$ in the trivialization; then η ranges over $\mathrm{Hom}(\mathcal{O}(-m'),\mathcal{O}^{\oplus g})=\mathcal{O}(m')^{\oplus g}$, and $\varphi(u,\eta)=[\eta:(u_i^{m'})]$ is the point of the fibre $\mathbb{C}^g$ of $\mathcal{V}$ over $[u_i^{m'}]\in\mathbb{P}(R)$ (Proposition 8.15(b)), linear in η. Over U_R, $F^\circ\to U_R$ is a G_F-torsor and the fibres of $Q^\circ\to F$ map isomorphically to those of $\mathcal{V}\to\mathbb{P}(R)$, so $Q^{\circ\circ}\to U_0$ is the pulled-back torsor. It is connected because F° is (a dense open subset of the irreducible F) and the fibres $\mathbb{C}^g$ are. (c) By Propositions 8.9 and 8.15, the μ_m-torsor $\pi_Z^{-1}(U_0)\to U_0$ is the pull-back from U_R of the torsor induced from F° along ς, $F^\circ\times^{G_F,\varsigma}\mu_m$: both are classified by the same meridian monodromies σ^{a_i}, and a μ_m-torsor on U_R is determined by its monodromy on $H_1(U_R,\mathbb{Z})$, which the meridians generate. The map $u\mapsto[u,1]$ from F° to the induced torsor is ς-equivariant, and pulling back along $Q^{\circ\circ}\to F^\circ$ gives a ς-equivariant map $\hat{q}^\circ\colon Q^{\circ\circ}\to\pi_Z^{-1}(U_0)$ over U_0, so far only a continuous map. It is a morphism of varieties: $Q^{\circ\circ}\to U_0$ and $\pi_Z^{-1}(U_0)\to U_0$ are finite étale (by (b) and Proposition 8.9), and by Riemann's existence theorem [SGA1, Exp. XII, Thm. 5.1] the functor from finite étale covers of U_0 to finite covering spaces of U_0 in the analytic topology is an equivalence, in particular fully faithful. (Concretely: the graph of $\hat{q}^\circ$ is open and closed in the finite étale cover $Q^{\circ\circ}\times_{U_0}\pi_Z^{-1}(U_0)$ of $Q^{\circ\circ}$, hence a union of connected components in the analytic topology, and these are Zariski open and closed.) Let $\overline{\Gamma}\subset Q\times_S Z$ be the closure of its graph. Since π_Z is finite, $Q\times_S Z\to Q$ is finite, hence so is $\overline{\Gamma}\to Q$; it is an isomorphism over the dense open $Q^{\circ\circ}$, so it is finite and birational, and $\overline{\Gamma}$ is irreducible because $Q^{\circ\circ}$ is connected. A finite birational morphism of integral schemes onto a normal one is an isomorphism (a form of Zariski's main theorem, [Stacks, Tag 0AB1]). Q is smooth, hence normal, so $\overline{\Gamma}\cong Q$ and the second projection is $\hat{q}$. Equivariance holds on the dense set $Q^{\circ\circ}$, hence everywhere. □

Intuition 9.2 (why Zariski's main theorem is the right tool). Over the open set $Q^{\circ\circ}$ the lift exists for topological reasons: both sides are torsors with the same monodromy. The danger is the boundary, where Z is branched and might have several points over one point of Q, or a whole curve. Zariski's main theorem says this cannot happen in the closure of a graph that is finite and one-to-one generically over a normal variety: normality means "no hidden branches", so the closure of a generically one-valued finite correspondence is the graph of a morphism. A way to remember why normality is the right condition: over $\mathbb{C}$, a variety is normal exactly when holomorphic functions on its smooth part that are locally bounded extend holomorphically across the singular locus, a Riemann removable-singularity property (we recall this characterization only as motivation).

EXAMPLE 9.3 (the two running examples). (i) The μ_6-cover of an elliptic curve with exponents $(2,4,2,4)$ (Example 8.17): $g=1$, $m'=3$, $F=X_3^2$ is the Fermat cubic surface, and $Q=\mathbb{P}_F(\mathcal{O}\oplus\mathcal{O}(-3))$ is a $\mathbb{P}^1$-bundle over F, of dimension $3=\dim|\mathcal{L}_0|$. The divisor $\{\nu=0\}\cong F$ is contracted by φ to the vertex point D of $|\mathcal{L}_0|=\mathbb{P}^3$, the section $s_\infty\cong F$ maps onto the plane $\mathbb{P}(R)$ by $u\mapsto(u_i^3)$ (degree $27=3^3$), and φ has degree $|G_F|=27$. (ii) Piece 9 ($m=m'=6$, $g=77$, $r=160$, $d=312$): $|\mathcal{L}_0|=\mathbb{P}^{235}$ with vertex $\mathbb{P}^{76}$ and base $\mathbb{P}^{158}$,

$F = X_6^{158}$, and Q is a $\mathbb{P}^{77}$-bundle over F of dimension $235 = d - g$; φ has degree 6^{159}, the boundary $Q \setminus Q^\circ$ is $X_6^{158} \times \mathbb{P}^{76}$, and ξ^{77} has cohomological degree 154.

EXERCISE 9.1. In Example 9.3(i): (a) show that the fibre of φ over a point of U_0 consists of 27 points permuted simply transitively by $G_F = \mu_3^4/\mu_3$; (b) show that the fibre of φ over a point $[\eta : w]$ with $[w] \in \mathbb{P}(R)$ on exactly one hyperplane $\{w_i = 0\}$ has 9 points, and explain this by the stabilizer in G_F; (c) why does the argument of Proposition 9.1(c) need $Q^{\circ\circ}$ connected, and what goes wrong for $r = 2$?

9.3. Injectivity on the ψ-line (step 6)

Pulling back along a generically finite surjective map of compact spaces of the *same* dimension is injective on cohomology (push forward and divide by the degree). Here $\hat{q}\colon Q \to Z$ goes from a space of dimension $d-g$ to one of dimension d, so $\hat{q}^*$ has a huge kernel in general; the claim is that the one line we care about survives.

THEOREM 9.4 (step 6: injectivity). *The pull-back $\hat{q}^*$ is injective on the line $H^d(Z,\mathbb{C})_\psi$, with image $\xi^g \cup \varpi^* V(\alpha)$; likewise $\hat{q}^*$ is injective on $H^d(Z,\mathbb{C})_{\bar\psi}$ with image $\xi^g \cup \varpi^* V(-\alpha)$. No assumption on the parity of r and no vanishing of the odd cohomology of F is used.*

PROOF. *Factorization.* By Proposition 8.9, $H^*(Z)_\psi = H^*(S, j_!\mathbb{L}_\psi)$, a summand of $H^*(Z) = H^*(S, \pi_{Z*}\mathbb{C})$, and $\hat{q}^*$ is induced by the composite of sheaf maps $\pi_{Z*}\mathbb{C}_Z \to \pi_{Z*}\hat{q}_*\mathbb{C}_Q = \varphi_*\mathbb{C}_Q \to R\varphi_*\mathbb{C}_Q$. On the ψ-summand it is therefore induced by a map $j_!\mathbb{L}_\psi \to R\varphi_*\mathbb{C}_Q$, and since φ maps into $|\mathcal{L}_0|$ and $\varphi^{-1}(U) = \varphi^{-1}(U_0) = Q^{\circ\circ}$, it factors as

$$H^d(S, j_!\mathbb{L}_\psi) \xrightarrow[\text{(i)}]{\sim} H_c^d(U_0, \mathbb{L}_\psi) \xrightarrow{\text{(ii)}} H_c^d(Q^{\circ\circ}, \mathbb{C}) \xrightarrow{\text{(iii)}} H^d(Q, \mathbb{C}). \tag{9.2}$$

(i) is restriction to the special fibre, an isomorphism by Proposition 8.12. (ii) is induced by $\mathbb{L}_\psi \to \varphi^\circ_*\mathbb{C}$, where $\varphi^\circ\colon Q^{\circ\circ} \to U_0$ is the finite étale G_F-torsor of Proposition 9.1(b), and the map comes from the trivialization of $\hat{q}^{\circ *}\mathbb{L}_\psi$ given by the lift. (iii) is extension by zero along the open inclusion $Q^{\circ\circ} \subset Q$. *Map (ii) is injective, into the α-part.* The sheaf $\varphi^\circ_*\mathbb{C}$ is a local system of rank $|G_F|$ with stalk the functions on a fibre; it splits under G_F into rank-one isotypic summands, one for each character. Because $\hat{q}^\circ$ is ς-equivariant, $\mathbb{L}_\psi$ maps isomorphically onto the summand on which G_F acts by $\alpha = \psi \circ \varsigma$, and that summand is a direct summand. Hence (ii) is a split injection with image $H_c^d(Q^{\circ\circ})_\alpha$. *From $Q^{\circ\circ}$ to Q°.* A point of $Q^\circ \setminus Q^{\circ\circ}$ lies over a point of F with some $u_i = 0$, which is fixed by the i-th factor $\mu_{m'} \subset G_F$, on which α is $\zeta \mapsto \zeta^{\pm a_i'} \neq 1$. On the α-part, the stalks of the pushforward of $\mathbb{C}$ along $Q^\circ \setminus Q^{\circ\circ} \to (Q^\circ \setminus Q^{\circ\circ})/G_F$ vanish, so all the groups $H_c^*(Q^\circ \setminus Q^{\circ\circ})_\alpha$ vanish, and the long exact sequence of compactly supported cohomology for the open subset $Q^{\circ\circ} \subset Q^\circ$ and its closed complement gives $H_c^*(Q^{\circ\circ})_\alpha \cong H_c^*(Q^\circ)_\alpha$. *Thom isomorphism.* Q° is a complex vector bundle of rank g over F, so $H_c^k(Q^\circ) \cong H^{k-2g}(F)$ by the Thom isomorphism [BT82, §6], G_F-equivariantly (G_F acts by bundle maps, preserving the complex orientation). In degree $k = d$: $H_c^d(Q^\circ)_\alpha = \tau \cup \varpi^* H^{r-2}(F)_\alpha = \tau \cup \varpi^* V(\alpha)$, with τ the Thom class; for $k \neq d$ the α-part vanishes, because $H^j(F)_\alpha = 0$ for $j \neq r-2$ (off the middle degree $H^*(F)$ is spanned by powers of the hyperplane class, which is invariant; Proposition 8.15(c)). *Extension by zero is injective in degree d on α-parts.* The long exact sequence of $Q^\circ \subset Q$ with closed complement $B = Q \setminus Q^\circ = F \times \mathbb{P}^{g-1}$ reads $\cdots \to H^{d-1}(Q) \to H^{d-1}(B) \to H_c^d(Q^\circ) \to H^d(Q) \to \dots$, and it is G_F-equivariant. G_F acts on B through the factor F only, so by Künneth

$$H^{d-1}(F \times \mathbb{P}^{g-1})_\alpha = \bigoplus_{j=0}^{g-1} H^{d-1-2j}(F)_\alpha \otimes H^{2j}(\mathbb{P}^{g-1}),$$

and a summand is nonzero only if $d-1-2j = r-2$, i.e. $2g+r-3-2j = r-2$, i.e. $2(g-j) = 1$: impossible. So $H^{d-1}(B)_\alpha = 0$ and $H_c^d(Q^\circ)_\alpha \to H^d(Q)$ is injective. *The image.* Extension by zero of the Thom class τ is the class of the zero section $s_\infty = \mathbb{P}_F(\mathcal{O}_F(-m')) = \{\eta = 0\}$. In Q

this section is the zero locus of the composite $\mathcal{O}_Q(-1) \to \varpi^*(\mathcal{O}^{\oplus g} \oplus \mathcal{O}(-m')) \to \varpi^*\mathcal{O}^{\oplus g}$, a section of $\mathcal{O}_Q(1)^{\oplus g}$ that vanishes transversally, so $[s_\infty] = c_g(\mathcal{O}_Q(1)^{\oplus g}) = \xi^g$. Extension by zero is $H^*(Q)$-linear, so the image of $\tau \cup \varpi^*V(\alpha)$ is $\xi^g \cup \varpi^*V(\alpha)$. Composing, $\hat{q}^*$ maps the line $H^d(Z)_\psi$ injectively onto this line. The same argument with $\bar{\psi}$, i.e. $-\alpha$, gives the second statement. □

What just happened. $\hat{q}^*$ on the ψ-line is a composite of three maps: restriction to the special fibre (an isomorphism, from Chapter 8), pull-back to a finite étale cover (injective, and it lands in the α-isotypic part), and extension by zero from the open part Q° to Q. The last map could kill a class only through H^{d-1} of the boundary $F \times \mathbb{P}^{g-1}$, whose α-part lives in degrees $(r-2)+2j$, never $d-1 = (r-2)+2g-1$. The surviving class is $\xi^g \cup \varpi^*V(\alpha)$: the Fermat line, placed on the section $s_\infty \cong F$.

Physics aside 9.5 (a selection rule). The argument is a superselection argument: decompose everything by the charge α under the finite symmetry group G_F, and observe that the boundary sector with charge α has no states in the one degree that could mix with the bulk state. The physicist's instinct "look at what the symmetry forbids" is exactly right here. The analogy stops at the degree count: there is no Hamiltonian and no energy, and "degree" is cohomological degree, whose parity bookkeeping has no dynamical meaning.

Why does the statement stress that $H^{\mathrm{odd}}(F) = 0$ is not used? For $r = 4$, F is a surface in $\mathbb{P}^3$, which has no odd cohomology, and that kills $H^{d-1}(F \times \mathbb{P}^{g-1})$ outright. For general r this is false: if $r-2$ is odd, the middle cohomology $H^{r-2}(F)$ is large and odd. The isotypic argument shows that this does not matter, since $H^{r-2}(F)$ enters only through the line $V(\alpha)$ itself, in the degree where we want it. For odd r, Theorem 8.1(a) does not apply (d is odd, so Λ_ψ has no algebraic classes to lie among), but step 6 still holds.

Exercise 9.2. (a) Write out the degree count of step 6 for piece 9: $d = 312$, $g = 77$, $r = 160$. In which degrees does $H^*(X_6^{158} \times \mathbb{P}^{76})_\alpha$ live, and why is 311 not among them? (b) Do the same for the μ_3-cover of a genus-2 curve with exponents $(1,1,1)$ of Exercise 8.6 ($r = 3$, F the Fermat cubic curve, $d = 5$), where $H^1(F) \neq 0$. (c) Show in general that $d-1-2j = r-2$ has no integer solution j for any g, r, and give a second reason, by dimension, why the α-part of H^{d-1} of the boundary vanishes.

9.4. A correspondence to the self-product (step 7)

Step 6 says that the ψ-line of Z is detected on Q by $\xi^g \cup \varpi^*V(\alpha)$. Step 7 turns this around: a class on F produces a class on X^d. Let $\tilde{Q}$ be a resolution of singularities of an irreducible component of $Q \times_Z X^d$ that dominates Q, with $\varrho_1 \colon \tilde{Q} \to Q$ generically finite and surjective and $\varrho_2 \colon \tilde{Q} \to X^d$, so that $\hat{q} \circ \varrho_1 = q \circ \varrho_2$. Put $\Gamma_* = \varrho_{2*}\varrho_1^* \colon H^k(Q) \to H^{k+2g}(X^d)$ (the action of a correspondence of degree $g = \dim X^d - \dim Q$ [Mil13, §4]), and

$$\epsilon_\psi = \frac{1}{|\mathcal{G}|} \sum_{\gamma \in \mathcal{G}} \psi(\bar{\gamma})^{-1} \gamma^* \in \mathbb{C}[\mathcal{G}],$$

$\bar{\gamma}$ the image of γ in $\mathcal{G}/\mathcal{N} \cong \mu_m$. On $H^d(X^d)$, ϵ_ψ is the projector onto the part where $\mathcal{N}$ acts trivially and $\mathcal{G}/\mathcal{N}$ by ψ, which is $q^*H^d(Z)_\psi = \ell_\psi$ (Lemma 8.6); its adjoint for the cup product is $\epsilon_{\bar{\psi}}$, since automorphisms preserve the orientation.

Lemma 9.6 (step 7). *For $0 \neq \beta' \in V(\alpha)$ the class $x = \epsilon_\psi \Gamma_* \varpi^* \beta'$ spans ℓ_ψ. If $\beta' \in \mathrm{Alg}(F) \otimes \mathbb{C}$, then $x \in \mathrm{Alg}^{d/2}(X^d) \otimes \mathbb{C}$.*

Proof. $x \in \ell_\psi$ by construction, so it suffices to pair it nontrivially with something. Take $0 \neq z' \in H^d(Z)_{\bar{\psi}}$; by Theorem 9.4, $\hat{q}^*z' = \xi^g \cup \varpi^*\gamma'$ with $0 \neq \gamma' \in V(-\alpha)$. Using the adjoint

of ϵ_ψ, the projection formula for ϱ_2, and $q\circ\varrho_2=\hat q\circ\varrho_1$,

$$\int_{X^d} x\cup q^*z' = \int_{\tilde Q}\varrho_1^*\varpi^*\beta'\cup\varrho_1^*\hat q^*z' = \deg(\varrho_1)\int_Q \varpi^*\beta'\cup\xi^g\cup\varpi^*\gamma' = \deg(\varrho_1)\int_F\beta'\cup\gamma'\neq 0.$$

The third equality holds because $\xi^g=[s_\infty]$ and $\varpi|_{s_\infty}$ is an isomorphism onto F; the last because $V(\alpha)$ and $V(-\alpha)$ are dual lines under the cup product on F (Poincaré duality is perfect and G_F-invariant, so it pairs the α-part only with the $(-\alpha)$-part). If β' is a complex combination of classes of subvarieties, so is x: ϖ^*, ϱ_1^* (pull-backs between smooth varieties), ϱ_{2*} (proper push-forward) and the automorphisms γ^* all preserve algebraic classes, as cycle class maps commute with these operations [Ful98, Ch. 19]. □

9.5. Transfer to the Jacobian (step 8)

Step 7 gives an algebraic class on X^d; the Main Theorem needs one on $J(X)$. The natural map is the sum $f\colon X^d\to J(X)$, $f(P_1,\dots,P_d)=\sum_j \mathrm{AJ}(P_j)$, and pull-back along it is easy to compute: on H^1, $f^*v=\sum_j \mathrm{pr}_j^*v$. The difficulty is going back. When ψ and $\bar\psi$ exhaust the primitive characters ($m=3,4,6$) one can use a surjective map to an abelian subvariety, but for $m=12$ the primitive part has four characters, and a different argument is needed: push forward, and undo the resulting cup product with a power of Θ algebraically.

Proposition 9.7 (step 8: transfer). *Let $g_X:=g(X)$, so that $d\le g_X$ (Remark 8.2), and let $\Theta\in H^2(J(X),\mathbb{Q})$ be the class of the theta divisor. (a) f^* maps Λ_ψ isomorphically onto ℓ_ψ. (b) (Poincaré's formula) $f_*f^*v=d!\,v\cup\Theta^{g_X-d}/(g_X-d)!$ for $v\in H^d(J(X))$. (c) The inverse of $\Theta^{g_X-d}\cup\colon H^d(J(X))\to H^{2g_X-d}(J(X))$ is induced by an algebraic correspondence. Consequently, if ℓ_ψ is spanned by an element of $\mathrm{Alg}(X^d)\otimes\mathbb{C}$, then $\Lambda_\psi\subset\mathrm{Alg}^{d/2}(J(X))\otimes\mathbb{C}$.*

Proof. (a) For $v_1,\dots,v_d\in W_\psi$, $f^*(v_1\wedge\cdots\wedge v_d)=\prod_k\sum_j\mathrm{pr}_j^*v_k$. A term in which two v_k,v_l sit in the same factor contains $v_k\cup v_l\in H^2(X)_{\psi^2}$, which is 0 because $H^2(X)$ is invariant and $\psi^2\neq1$ for $m\ge3$. The remaining terms put one v_k in each factor, and their sum is the alternating tensor, which spans ℓ_ψ and is nonzero. (b) f factors through $\mathrm{Sym}^d X\to W_d=f(X^d)\subset J(X)$, and $\mathrm{Sym}^d X\to W_d$ is birational for $d\le g_X$ [ACGH85, Ch. I §5, Ch. IV]; so $f_*1=d!\,[W_d]$, and Poincaré's formula $[W_d]=\Theta^{g_X-d}/(g_X-d)!$ [ACGH85, Ch. I §5] with the projection formula $f_*f^*v=v\cup f_*1$ gives (b). (c) Hard Lefschetz [Dem12, Ch. VI, (8.17), p. 313] makes the map an isomorphism, and Lieberman's theorem [Lie68; Kle68], as recorded in [Mil02, pp. 599–600], says that the adjoint Lefschetz operator, written $L^\vee$ here (often Λ in the literature), on an abelian variety is algebraic, i.e. the Lefschetz standard conjecture $B(A)$ holds; it follows that the inverse of L^{g_X-d} on H^d is a polynomial in L and $L^\vee$, hence algebraic [Kle68], [Lie68]. Patel and Zhang use the same step [PZ27], and Schoen's transfer to the generalized Prym is the same argument [Sch88, proof of Cor. 3.1, pp. 24–25]. For the consequence: if $f^*v=x$ is algebraic, then $v=\frac{(g_X-d)!}{d!}\,L^{-(g_X-d)}(f_*x)$ is obtained from x by algebraic operations. □

Example 9.8 (Poincaré's formula on a genus-2 Jacobian). Let $g_X=2$ with symplectic basis a_1,b_1,a_2,b_2 of $H^1(X)=H^1(J)$, $\int_X a_ib_j=\delta_{ij}$, so $\Theta=a_1b_1+a_2b_2$ and $\Theta^2/2=a_1b_1a_2b_2$ is the orientation class. For $d=1$, $f\colon X\to J$ is the Abel–Jacobi embedding, $W_1=f(X)$, and the formula says $f_*1=\Theta$: the curve is Poincaré dual to Θ. Test it by integrating: $\int_X f^*\Theta=\int_X(a_1b_1+a_2b_2)=2$ and $\int_J\Theta\cup\Theta=2$; and for $v=a_1$, $w=b_1$, $\int_X a_1b_1=1=\int_J a_1\cup\Theta\cup b_1$. For $d=2=g_X$ the formula says $f_*1=2\cdot[J]$: the map $X^2\to J$ has degree 2 (it factors through $\mathrm{Sym}^2X\to J$, birational, Example 8.5, and $X^2\to\mathrm{Sym}^2X$ of degree 2).

Exercise 9.3. (a) For $g_X=2$, $d=1$, verify $f_*f^*v=v\cup\Theta$ for all four basis vectors $v\in H^1(J)$ by pairing with $H^1(J)$, keeping track of signs. (b) Show that for $d>g_X$ Poincaré's formula cannot hold as stated (what is f_*1 in degree 0?), and explain why the inequality $d\le g_X$, automatic for the eigenspaces of Theorem 8.1, enters only here. (c) Check $d\le g_X$ for piece 10 of Table 7.1.

9.6. Shioda's linear cycles for fully paired characters (step 9)

Steps 5–8 prove Theorem 8.1(a): everything reduces to the question whether the Fermat lines $V(\pm\alpha')$ are spanned by algebraic classes. For a general character this is an instance of the Hodge conjecture for Fermat varieties, which is known only in special cases [Shi79, pp. 179–182], [Sil21, Thms. 2.6–2.8]. For fully paired characters there is a completely explicit answer: linear subspaces. On $F = X^{r-2}_{m'}$, $\sum u_i^{m'} = 0$, a pair of coordinates with $u_i = \epsilon u_j$, $\epsilon^{m'} = -1$, contributes $u_j^{m'}(\epsilon^{m'} + 1) = 0$; so a pairing of the $r = 2k$ coordinates into k pairs, with a choice of ϵ for each, gives the linear subspace

$$\Pi = \{u_{i_1} = \epsilon_1 u_{j_1},\ \dots,\ u_{i_k} = \epsilon_k u_{j_k}\} \cong \mathbb{P}^{k-1} \subset F, \qquad \epsilon_l^{m'} = -1,$$

of dimension $k - 1 = (r - 2)/2$, the right dimension for a class in the middle degree $r - 2$.

Proposition 9.9 (step 9: Shioda's linear cycles). *Let $\alpha' \in \mathfrak{A}^{r-2}_{m'}$, $r = 2k$, be* fully paired*: there is a pairing $\{\{i_l, j_l\}\}$ with $a'_{i_l} + a'_{j_l} \equiv 0$ for all l. Then $V(\alpha')$ is spanned by the $V(\alpha')$-component of the class of a linear subspace Π of that pairing, so $\alpha' \in \mathfrak{C}^{r-2}_{m'}$. The same holds for $t\alpha'$ for every $t \in (\mathbb{Z}/m')^\times$, in particular for $-\alpha'$. Conversely, a character for which the class of a linear subspace Π has a nonzero component is neutral on every pair of Π.*

Proof. *Forward.* This is Shioda's: his Corollary to Theorem II(i) [Shi79, p. 179] says that if r, s are odd and $(\beta', \gamma') \in \mathfrak{C}^{r-1}_m \times \mathfrak{C}^{s-1}_m$, then $\beta' * \gamma' \in \mathfrak{C}^n_m$. Applied inductively with $(r, s) = (1, 2j - 1)$, starting from $\mathfrak{C}^0_{m'} = \{(b, -b)\}$, the point classes of $X^0_{m'}$ [Shi79, (1.10), p. 177], it gives $\alpha' \in \mathfrak{C}$; in the inductive step $V(\beta' * \gamma')$ is the image of $V(\beta') \otimes V(\gamma')$ under the map of [Shi79, (2.10)–(2.14)], induced by the join construction, and the join of a point and a linear subspace is a linear subspace [Shi79, Thm. II, (2.7), p. 178]. So $V(\alpha')$ lies in the span of the $V(\alpha')$-components of classes of linear subspaces of the pairing followed by the induction; since each component lies in the line $V(\alpha')$, one of them spans it. Both statements hold for every m', with no condition on $\gcd(b_l, m')$ and allowing $b_l = m'/2$. Multiplication by a unit t preserves full pairing, so the same applies to $t\alpha'$.

Converse. Let $H = \{g \in G_F : \zeta_{i_l} = \zeta_{j_l} \text{ for all } l\}$; it maps Π to itself, so $h^*[\Pi] = [\Pi]$ for $h \in H$. Write the projection to $V(\alpha)$ as $e_\alpha = |G_F|^{-1} \sum_g \alpha(g)^{-1} g^*$. Summing over cosets of H gives the factor $\sum_{h \in H} \alpha(h)^{-1}$, which is 0 unless $\alpha|_H = 1$, i.e. $a_{i_l} + a_{j_l} \equiv 0$ for all l.

Stability. The algebraic classes form a G_F-stable subspace and the $V(\alpha)$ are distinct lines, so the projection $e_\alpha x$ of an algebraic class x lies in $\mathrm{Alg}(F) \otimes \mathbb{C}$: $\alpha \in \mathfrak{C}$ as soon as some algebraic class has a nonzero $V(\alpha)$-component. The $V(\alpha)$ are defined over $\mathbb{Q}(\zeta_{m'})$, and $\sigma_t \in \mathrm{Gal}(\mathbb{Q}(\zeta_{m'})/\mathbb{Q})$ acting on coefficients maps $V(\alpha)$ to $V(t\alpha)$, because $\sigma_t(\alpha(g)) = \alpha(g)^t$; as $\mathrm{Alg}(F)$ is defined over $\mathbb{Q}$, $\mathfrak{C}$ is stable under $(\mathbb{Z}/m')^\times$ (cf. [Shi79, p. 176]). □

Example 9.10 (lines on the Fermat sextic surface). For $r = 4$, $m' = 6$, $F = X^2_6$ is the Fermat sextic surface and the Π are lines: 3 pairings times 6^2 choices of (ϵ_1, ϵ_2) give 108 lines. Each meets $22 = 4 \cdot 6 - 2$ others and has self-intersection $2 - 6 = -4$. Of the 105 characters in $\mathfrak{A}^2_6$, 85 are Hodge characters and 61 are fully paired. Concretely, for the paired character $(2, 4, 3, 3)$ and the line $L = \{u_1 = \epsilon u_2,\ u_3 = \epsilon u_4\}$, the sum over the 216 elements of G_F gives $\int e_\alpha[L] \cup [L] = -1/6$ for the pairing $\{12, 34\}$ and 0 for the two others, on which $(2, 4, 3, 3)$ is not neutral. The character $(1, 3, 4, 4)$ is a Hodge character ($|(1, 3, 4, 4)| = |(5, 3, 2, 2)| = 2$) but no two of its entries sum to 0 mod 6, and the same computation gives 0 for all three pairings: no line sees it. Its line is still algebraic, by the Lefschetz $(1, 1)$ theorem, since it consists of Hodge classes on a surface; but lines do not reach it, which is why the fully paired hypothesis of Theorem 8.1(b) is a real restriction.

Example 9.11 (the cubic surface of the elliptic example, and piece 9). For the μ_6-cover of E with exponents $(2, 4, 2, 4)$ (Example 8.17), $\alpha' = (1, 2, 1, 2)$ on the cubic surface X^2_3. It is neutral on the pairings $\{12, 34\}$ and $\{14, 23\}$ and not on $\{13, 24\}$, and the sum over G_F gives $-1/3$, $-1/3$ and 0 for the three corresponding lines. So $V(\alpha')$ is spanned by the $V(\alpha')$-component of the class of one of the 27 lines, e.g. $\{u_1 = -u_2,\ u_3 = -u_4\}$, and Lemma 9.6

and Proposition 9.7 turn it into an algebraic class spanning the Weil line $\Lambda_\psi \subset H^4(J(X))$, $g(X) = 9$. For piece 9, $\alpha' = (2^{78}, 4^{78}, 3^4)$ mod 6 (exponents as value$^{\text{multiplicity}}$): pair each 2 with a 4 and the four 3's among themselves; the linear subspace is a $\mathbb{P}^{79}$ on X_6^{158}, and its class, placed on the section s_∞ of the $\mathbb{P}^{77}$-bundle Q and pushed through Γ, ϵ_ψ and f, gives an algebraic class spanning the ψ-line of piece 9 in $H^{312}(J(X_9))$. Hodge types fit: $|\alpha'| = (2\cdot 78 + 3\cdot 4 + 4\cdot 78)/6 = 80 = r/2$, so $V(\alpha')$ has type $(79, 79)$ and $\xi^{77} \cup \varpi^* V(\alpha')$ type $(156, 156) = (d/2, d/2)$.

Exercise 9.4. (a) Show that the Fermat surface X_m^2 contains the $3m^2$ standard lines, and that a given standard line meets exactly $4m - 2$ of the others (count separately the lines with the same pairing and those with a different pairing). (b) Check that $L^2 = 2 - m$ by adjunction: $L \cong \mathbb{P}^1$, $K_{X_m^2} = \mathcal{O}(m - 4)$.

Exercise 9.5. (a) For $m' = 3$ and $\alpha' = (1, 2, 1, 2)$, compute $\int e_{\alpha'}[L] \cup [L]$ for the line $L = \{u_1 = \epsilon u_2,\ u_3 = \epsilon u_4\}$ directly from the formula $|G|^{-1} \sum_g \alpha'(g)(L \cdot gL)$, using $L \cdot gL \in \{-1, 1, 0\}$ according to whether $gL = L$, gL meets L in a point, or $gL \cap L = \emptyset$. (b) Show that for $\alpha = (1, 3, 4, 4)$ mod 6 the same sum vanishes for every standard line, by exhibiting for each pairing an element of the stabilizer of L on which α is nontrivial. (c) Is $(1, 1, 4, 4, 4, 4)$ mod 6 fully paired? Is it a Hodge character of X_6^4?

Step 9 assembled. If the multiset $\{a_i\}$ is fully paired, so is α' (Proposition 9.12 below), and so is $t\alpha'$ for every $t \in (\mathbb{Z}/m)^\times$, since multiplication by a unit preserves the pairs (if m is even, t is odd and fixes $m/2$). By Proposition 9.9, $\pm\alpha' \in \mathfrak{C}_{m'}^{r-2}$ for ψ and for every Galois conjugate ψ^t, so Theorem 8.1(a) applies to each ψ^t. Each pair contributes exactly 1 to $\sum_i \langle ta_i/m\rangle$, so W_{ψ^t} has Hodge numbers $(g - 1 + r/2,\ g - 1 + r/2) = (d/2, d/2)$. Finally $\bigwedge^d_{\mathbb{Q}(\zeta_m)} H^1(X, \mathbb{Q})_{[\psi]}$ is a $\mathbb{Q}$-subspace whose complexification is $\bigoplus_t \Lambda_{\psi^t} \subset \mathrm{Alg} \otimes \mathbb{C}$, and a rational subspace whose complexification is spanned by algebraic classes is spanned by rational algebraic classes, since $(\mathrm{Alg}_\mathbb{Q} \otimes \mathbb{C}) \cap H_\mathbb{Q} = \mathrm{Alg}_\mathbb{Q}$. This proves Theorem 8.1(b).

9.7. Proper-ideal exponents: the gcd reduction

Three of the eight ramified pieces have exponents that generate a proper ideal of $\mathbb{Z}/m$: piece 2 (exponents $2, 4$ mod 6) and pieces 4, 10 (exponents $4, 6, 8$ mod 12). For such covers the μ_m-torsor over the open part of the special fibre is disconnected, and one might fear that the Fermat picture breaks. It does not: everything passes to the reduced data.

Proposition 9.12 (the gcd reduction). *Let $d' = \gcd(m, a_1, \dots, a_r)$, $m' = m/d'$, $a_i' = a_i/d' \bmod m'$. (a) The local system $\mathbb{L}'$ on U_R has meridians $\zeta_m^{a_i} = \zeta_{m'}^{a_i'}$, so steps 4–8 run with $F = X_{m'}^{r-2}$ and $\alpha \in \{\pm\alpha'\}$; the μ_m-torsor $\pi_Z^{-1}(U_0)$ has d' components and enters only through $\varsigma\colon G_F \to \mu_m$, whose image is $\langle\sigma^{d'}\rangle \cong \mu_{m'}$. (b) $\{a_i\}$ is fully paired mod m if and only if $\{a_i'\}$ is fully paired mod m', and the reduction mod m' of $t \in (\mathbb{Z}/m)^\times$ is a unit. (c) $|t\alpha'| = r/2$ for every unit t when the data are fully paired, so $V(\pm t\alpha')$ are of type $(\frac{r-2}{2}, \frac{r-2}{2})$.*

Proof. (a) $\zeta_m^{a_i} = e^{2\pi i d' a_i'/(d'm')} = \zeta_{m'}^{a_i'}$, and the rest is Proposition 8.15 and Proposition 9.1, which never used $d' = 1$: the lift of step 5 is built from $F^\circ \times^{G_F,\varsigma} \mu_m$, whatever its number of components (each component is a copy of the quotient of F° by $\ker\varsigma$, and there are $[\mu_m : \mathrm{im}\,\varsigma] = d'$ of them). The cover X itself stays connected: loops of Y (the handle monodromy) generate together with the σ^{a_i}, but they do not reach U_0, because $\pi_1(U_0) = \pi_1(U_R)$ is generated by meridians. (b) $a_i + a_j \equiv 0$ mod m iff $d'(a_i' + a_j') \equiv 0$ mod $d'm'$ iff $a_i' + a_j' \equiv 0$ mod m'. (c) Each pair $\{b, -b\}$ with $b \not\equiv 0$ contributes $\langle tb\rangle/m' + \langle -tb\rangle/m' = 1$. □

For pieces 4 and 10, $d' = 2$, $m' = 6$ and α' has entries $2, 3, 4$: the special-fibre problem is literally the μ_6-problem of the other pieces. For piece 4 the multiset $(4^{324}, 6^6, 8^{324})$ mod 12 becomes $(2^{324}, 3^6, 4^{324})$ mod 6, with $|\alpha'| = (2\cdot 324 + 3\cdot 6 + 4\cdot 324)/6 = 327 = r/2$. So the proof treats pieces 4 and 10 with exactly the same Fermat variety type as pieces $1, 3, 6, 9, 11$. Schoen's 1988 theorem needs no gcd condition either.

EXERCISE 9.6. For piece 4: (a) compute d', m' and α'; (b) list the units $t \in (\mathbb{Z}/12)^\times$ and their reductions mod 6, and check that $t\alpha'$ is fully paired for each; (c) show that the μ_{12}-torsor over U_R has two components, and identify $\ker \varsigma$; (d) why is it harmless that $t = 1$ and $t = 7$ give the same character α' mod 6 but different characters ψ^t of μ_{12}?

9.8. Schoen's 1988 theorem

Schoen's Theorem 2.0 [Sch88, p. 11] says: if the $(\mathbb{Z}/m)^\times$-orbit of the branching data of (C, σ) is *simple* (fully paired, in our language; "after a permutation of the indices, $\alpha_{2i} = -\alpha_{2i-1}$", p. 11), then the space $U \subset H^d(C^d, \mathbb{Q})$ is generated by algebraic cycles; Corollary 3.1 [Sch88, p. 24] transfers this to the generalized Prym variety. The statement has no condition on $\gcd(\alpha_i, m)$ and none on the genus of the base. All eight ramified pieces lie inside it. Its proof obtains the injectivity of a pull-back, the analogue of our step 6, from his Lemma 2.8.

REMARK 9.13 (which result gives which piece). (a) *Schoen 1988*: the statement of Theorem 2.0 and Corollary 3.1 contains the eight ramified pieces, which are fully paired, and the étale pieces $5, 7, 8$ ($r = 0$: the pairing condition is empty). The transfer of Corollary 3.1 uses Poincaré's formula and Lieberman's operator as in Proposition 9.7. The inequality $d \leq g_X$ that Poincaré's formula needs is automatic for every cyclic cover with $m \geq 3$ (Remark 8.2), so the transfer needs no extra hypothesis. The proof explained here uses Schoen's theorem for the étale pieces (Chapter 10) and the cyclic-cover theorem for the eight ramified pieces. (b) *A related preprint*: Lemma 3.4 of the preprint [Ope26a] claims, for every connected cyclic cover of curves and every faithful character, that the top exterior power of the eigenspace lies in the span of the images of the cohomology of CM abelian varieties under algebraic correspondences. Its proof uses the construction of steps 1–5, for which [Ope26a] names Schoen's work [Sch88] as a direct precedent, and its injectivity step uses Deligne's weight argument, whereas step 6 uses extension by zero. The statement of that lemma, combined with Shioda's theorem for abelian varieties dominated by Fermat curves of degree $m \leq 20$, would give the cyclic-cover theorem (b) for $m \leq 20$; for $m = 12$ this uses Shioda's computer-assisted verification of his condition (P_{12}) [Shi79, pp. 181–182].

REMARK 9.14 (Where this is used). Theorem 8.1(b) gives the Weil lines $\Lambda_j \subset H^{d_j}(J(X_j))$ of the eight ramified pieces as complex combinations of algebraic classes, step (iii). Chapter 10 does the same for the étale pieces by Schoen's $r = 0$ argument, Proposition 7.10 pulls all eleven lines back to $J(C)$, and Chapter 11 combines them with the integral relation $\sum_j c_j v_j = e_\rho$ into the Weil classes of the G_{32} family. For all eight ramified pieces this chapter's argument is the one the proof relies on.

CHAPTER 10

The étale pieces: generalized Pryms, Schoen 1988 and Patel–Zhang

Proof steps: (iv), (v)

Where we are. This chapter explains step (iv): the Weil lines of the three étale pieces $5, 7, 8$ of Table 7.1 are complex combinations of algebraic classes. Step (v), the pull-back of these lines to $J(C)$, was done in Proposition 7.10 and is only recalled. The input is a published theorem, Schoen's Corollary 3.1 [Sch88] in its simplest case, where the cyclic cover has no branch points; Patel and Zhang [PZ27] prove a more general statement by a different route and serve as an alternative reference. Nothing new is proved here: the chapter explains what these theorems say, why the étale case is the easy one, what Schoen's cycle is, and checks that the three pieces satisfy the hypotheses. The hypothesis check uses the enumeration of the pieces in Section 7.5. Needs: Chapter 7 (pieces, Definition 7.1 and Propositions 7.4 and 7.10); the mechanism of Chapter 8 helps but is not required; Jacobians and the Abel–Jacobi map [Mil08, I, §18], Weil classes (Definition 1.1), and Lieberman's theorem [Lie68; Kle68]. Notation: $X \to Y$ is a connected étale Galois cover with group $\mu_m = \langle\sigma\rangle$ of smooth projective complex curves, $g = g(Y) \geq 2$, $g_X = g(X)$, and $d = 2g - 2$. Schoen and Patel–Zhang write h for d; in this companion h is a Hermitian form.

10.1. Step (iv) in two pages

Three of the eleven pieces of the relation $e_\rho = \sum_j c_j v_j$ (Proposition 7.8) are étale: the cyclic cover $X_j = C/\ker\psi_j \to Y_j = C/H_j$ has no branch points. They cannot be avoided, because the ramified pieces span a lattice of rank $13 < 14$ (Section 7.5). For an étale μ_m-cover every non-trivial character χ of μ_m occurs in $H^1(X, \mathbb{C})$ with the same multiplicity $d = 2g - 2$ and with Hodge numbers $(g-1, g-1)$ (Exercise 7.4), so the top exterior power

$$\Lambda_\chi = \textstyle\bigwedge^d H^1(X, \mathbb{C})_\chi \subset H^d(J(X), \mathbb{C})$$

is a line of Hodge type $(g-1, g-1)$, a Weil-type Hodge class. Step (iv) needs $\Lambda_{\psi_j} \subset \mathrm{Alg}(J(X_j)) \otimes \mathbb{C}$ for $j = 5, 7, 8$, where Alg is the span of classes of algebraic cycles. The data are:

j	m	$g(Y_j)$	$d_j = 2g(Y_j) - 2$	$g(X_j) = m(g(Y_j) - 1) + 1$	c_j
5	24	2341	4680	56161	$+1$
7	24	4213	8424	101089	-1
8	12	4681	9360	56161	$+1$

The one idea is Schoen's [Sch88, p. 13]. On the base curve, effective divisors of degree $d = 2g - 2$ form the symmetric product $\mathrm{Sym}^d Y$, which maps to $\mathrm{Pic}^d(Y)$ by sending a divisor to its line bundle. The fibre over a line bundle L is the projective space $|L|$ of divisors of sections of L. Riemann–Roch makes these fibres $\mathbb{P}^{g-2}$, except over the canonical bundle K_Y, where the fibre jumps to $|K_Y| \cong \mathbb{P}^{g-1}$ (Proposition 10.7). The étale cover $X \to Y$ induces an étale μ_m-cover $Z \to \mathrm{Sym}^d Y$; over the simply connected $|K_Y|$ it splits into m disjoint copies $Q, \sigma Q, \ldots, \sigma^{m-1}Q$ of $\mathbb{P}^{g-1}$. The combination $z_\chi = \sum_t \chi(\sigma^t)^{-1}[\sigma^t Q]$ is an algebraic class with $\mathbb{C}$-coefficients on which μ_m acts by χ, it is non-zero because $Q \cdot z_\chi = Q \cdot Q = (-1)^{g-1}$, and it

spans the χ-part of the middle cohomology of Z, which is a line (Proposition 10.8). Carrying z_χ to $J(X)$ and undoing a power of the Lefschetz operator algebraically (Lieberman) gives Λ_χ (Theorem 10.10). The same mechanism, with branch points, is the cyclic-cover theorem (Theorem 8.1) of Chapters 8 and 9; there the special fibre is a cone over a quotient of a Fermat variety, and for $r = 0$ the cone shrinks to its vertex $|K_Y|$, so the Fermat input disappears.

Why the étale case is automatic. Schoen's theorem requires the branching exponents to be "simple" (fully paired, Definition 7.6), and he notes that this "is automatically satisfied if $r = 0$ or 1" [Sch88, p. 12], where his $2r$ is the number of branch points. Balance (Hodge numbers $(g-1, g-1)$ on every eigenspace) is automatic as well [Sch88, Lemma 1.6a]. The only hypotheses left are $m > 2$, $g \geq 2$ and connectedness, and for pieces $5, 7, 8$ they follow from the enumeration of the pieces (Proposition 10.12).

Sources. Schoen's Theorem 2.0 and Corollary 3.1, whose étale case is proved on p. 13 of [Sch88], are the primary reference for the étale pieces, and Patel–Zhang [PZ27] is an alternative (Theorem 10.11). For the eight ramified pieces the proof uses the cyclic-cover theorem (Chapters 8 and 9); for fully paired data, which Schoen calls simple, his Theorem 2.0 and Corollary 3.1 state the same conclusion.

10.2. Étale cyclic covers and their generalized Pryms

An étale μ_m-cover of a curve is the algebraic version of a flat μ_m-bundle: a connected cover $X \to Y$ of degree m with μ_m acting freely and $Y = X/\mu_m$, classified by a surjection $\pi_1(Y) \to \mu_m$, or by a line bundle η with $\eta^{\otimes m} \cong \mathcal{O}_Y$ of order exactly m. Riemann–Hurwitz ([Stacks, Tag 0C1F], which holds for any finite separable cover of curves) has no ramification term, so $2g_X - 2 = m(2g-2)$, i.e. $g_X = m(g-1)+1$. For $g = 0$ there are no non-trivial étale covers and for $g = 1$ the cover is again of genus 1, so the interesting case is $g \geq 2$.

Proposition 10.1 (the eigenspaces of an étale cover). *Let $X \to Y$ be an étale μ_m-cover, $g = g(Y) \geq 1$. Then $H^1(X, \mathbb{C})^{\mu_m} \cong H^1(Y, \mathbb{C})$, and every non-trivial character χ of μ_m occurs in $H^1(X, \mathbb{C})$ with multiplicity $d = 2g-2$, with Hodge numbers $\dim H^{1,0}(X)_\chi = \dim H^{0,1}(X)_\chi = g-1$.*

Proof. The invariant part is Lemma 7.3. For $\tau \neq 1$ in μ_m the Lefschetz number is $2 - \operatorname{tr}(\tau^* | H^1) = 0$, because τ has no fixed point and a map with non-zero Lefschetz number has one [Hat02, Thm. 2C.3]; so $\operatorname{tr}(\tau^* | H^1) = 2$, and the multiplicity of $\chi \neq 1$ is $\frac{1}{m}(2g_X + 2\sum_{\tau \neq 1} \overline{\chi(\tau)}) = \frac{2g_X - 2}{m} = 2g-2$. For the Hodge numbers, the holomorphic Lefschetz formula (Atiyah–Bott, as used in [Sch88, Lemma 1.6a]) gives $\operatorname{tr}(\tau^* | H^0(\mathcal{O}_X)) - \operatorname{tr}(\tau^* | H^{0,1}) = 0$ for $\tau \neq 1$, so the character of $H^{0,1}$ takes the value 1 off the identity and is therefore $1 + (g-1) \cdot \mathrm{reg}$; every $\chi \neq 1$ occurs $g-1$ times in $H^{0,1}$ and, by complex conjugation, $g-1$ times in $H^{1,0}$. Exercise 7.4 derives the same from the cyclic Chevalley–Weil formula. □

The proposition is the reason the étale pieces enter step (ii) with $\dim W_j = 2g(Y_j) - 2$ and balanced Hodge numbers. Patel and Zhang give a proof of the multiplicity without fixed-point formulas (Exercise 10.2).

Definition 10.2 (generalized Prym varieties). Let $X \to Y$ be an étale μ_m-cover and $\mathrm{Nm}\colon J(X) \to J(Y)$ the norm map. The *generalized Prym variety* is $\mathrm{Prym}(X/Y) = (\ker \mathrm{Nm})^0$, the connected component of the identity [PZ27]. Schoen's *primitive Prym* $\mathrm{Prym}^{\mathrm{prim}}(X/Y) \subseteq \mathrm{Prym}(X/Y)$ is the image of the endomorphism $\prod_{1 \neq S \leq \mu_m} \sum_{\tau \in S}(1-\tau)$ of $J(X)$ [Sch88, p. 24], the abelian subvariety whose tangent space is the sum of the eigenspaces of the primitive (faithful) characters of μ_m.

The endomorphism $\sum_{\tau \in S}(1-\tau) = |S|\,(1 - e_S)$, with e_S the averaging projector of the subgroup S, kills exactly the part on which S has invariants; the product over all $S \neq 1$ therefore keeps exactly the characters that are non-trivial on every non-trivial subgroup, the faithful ones. On first cohomology, which for an abelian variety V/L is $\mathrm{Hom}(L, \mathbb{Z})$ [Mil08, I,

Thm. 2.3],

$$H^1(\mathrm{Prym},\mathbb{C}) = \bigoplus_{\chi\neq 1} H^1(X,\mathbb{C})_\chi, \qquad H^1(\mathrm{Prym}^{\mathrm{prim}},\mathbb{C}) = \bigoplus_{\chi \text{ primitive}} H^1(X,\mathbb{C})_\chi,$$

so $\dim \mathrm{Prym} = g_X - g = (m-1)(g-1)$ and $\dim \mathrm{Prym}^{\mathrm{prim}} = \varphi(m)(g-1) = \varphi(m)\,d/2$, with φ Euler's function. The two agree when m is prime. Pull-back $\pi^*\colon J(Y)\to J(X)$ and Nm satisfy $\mathrm{Nm}\circ\pi^* = [m]$ (Exercise 7.10), so $J(Y)\times \mathrm{Prym}\to J(X)$, $(a,b)\mapsto \pi^*a + b$, has finite kernel between varieties of the same dimension: $J(X)\sim J(Y)\times \mathrm{Prym}(X/Y)$, an instance of Poincaré reducibility [Mil08, I, Prop. 10.1]. For a piece (H,ψ), ψ induces a faithful character of $H/\ker\psi \cong \mu_m$ (Definition 7.1), so its eigenspace lies in the primitive Prym; that is why Schoen's primitive version suffices for the proof, although Patel–Zhang treat all non-trivial characters.

Example 10.3 (an étale triple cover of a genus-2 curve). Let Y have genus 2 and let $X\to Y$ be étale with group μ_3 (one of the $(3^4-1)/2 = 40$ subgroups of order 3 of $J(Y)[3]\cong(\mathbb{Z}/3)^4$). Then $g_X = 3\cdot 1+1 = 4$, $d=2$, and $H^1(X,\mathbb{C})$ has character multiplicities $(4,2,2)$ for $(1,\chi,\bar\chi)$, with Hodge numbers $(1,1)$ on each non-trivial eigenspace. The Prym is a surface ($m=3$ is prime, so both Pryms agree, $\dim = 2\cdot 1 = 2$) on which σ acts with eigenvalues ζ_3,ζ_3^2 on the tangent plane: an abelian surface of Weil type for $\mathbb{Q}(\sqrt{-3})$ (Definition 1.1) with $\dim_{\mathbb{Q}(\zeta_3)} H^1 = 2$. Its Weil classes $\bigwedge^2_{\mathbb{Q}(\zeta_3)} H^1$ form a 2-dimensional $\mathbb{Q}$-space of classes of type $(1,1)$ in H^2, algebraic by the Lefschetz $(1,1)$ theorem; the toy is therefore a case where the answer is known in advance and Schoen's construction can be watched (Example 10.9). The étale double cover of the same curve (Exercise 7.10) has $g_X = 3$ and an elliptic curve as Prym; for $m=2$ the field is $\mathbb{Q}$ and there is no Weil structure, which is why Schoen's Corollary 3.1 asks for $m>2$. Schoen's own fourfold example, an étale triple cover of a genus-3 curve with $g_X = 7$ and a Prym fourfold, is [Sch88, Thm. 3.2 and its proof, pp. 26–27].

Exercise 10.1. (a) For pieces $5,7,8$ compute $\dim \mathrm{Prym}(X_j/Y_j) = (m-1)(g(Y_j)-1)$ and $\dim \mathrm{Prym}^{\mathrm{prim}} = \varphi(m)(g(Y_j)-1)$, and check $\dim \mathrm{Prym} = g(X_j) - g(Y_j)$. (b) Show that $\dim \mathrm{Prym}^{\mathrm{prim}}$ is the number of primitive characters times $d/2$, and that $\mathrm{Prym}^{\mathrm{prim}} = \mathrm{Prym}$ if and only if m is prime.

Exercise 10.2. (Patel–Zhang's proof of the multiplicity.) Let L_χ be the rank-one local system on Y whose monodromy is the composite of the classifying map $\pi_1(Y)\to\mu_m$ of the cover with χ, so that $H^1(X,\mathbb{C})_\chi \cong H^1(Y,L_\chi)$ for a suitable convention. Use that the Euler characteristic of a rank-one local system equals $\chi_{\mathrm{top}}(Y) = 2-2g$ and that $H^0(Y,L_\chi) = H^2(Y,L_\chi) = 0$ for $\chi\neq 1$ to show $\dim H^1(Y,L_\chi) = 2g-2$, without any fixed-point formula.

Exercise 10.3. Let $X\to Y$ be an étale Galois cover with an arbitrary finite group G, $g(Y)=g$. Show that $H^{1,0}(X)\cong \mathbb{C}\oplus\mathbb{C}[G]^{\oplus(g-1)}$ as a G-representation, i.e. each irreducible ρ' occurs $(g-1)\dim\rho'$ times, plus one extra trivial summand. Deduce that every isotypic multiplicity space $M_{\rho'}$, $\rho'\neq 1$, of an étale G-cover is balanced. (Use the holomorphic Lefschetz formula as in Proposition 10.1.)

10.3. The Weil Hodge structure of an étale cover

The Prym carries an action of the field $\mathbb{Q}(\mu_m)$ on its primitive part, and with it the Weil construction of Definition 1.1, here over the cyclotomic field $\mathbb{Q}(\mu_m)$ of degree $\varphi(m)$ instead of a quadratic one, as in [Del82, §4].

Definition 10.4 (Weil structures U' and U_{Weil}). Let $X\to Y$ be étale with group μ_m, $m>2$, $d=2g-2$. The group algebra acts on $V = H^1(\mathrm{Prym}^{\mathrm{prim}},\mathbb{Q})$ through its primitive factor $\mathbb{Q}(\mu_m)$, and V is a $\mathbb{Q}(\mu_m)$-vector space of dimension d. Schoen's Weil structure is

[Sch88, p. 24]

$$U' = \bigwedge^d_{\mathbb{Q}(\mu_m)} V \subseteq \bigwedge^d_{\mathbb{Q}} V = H^d(\mathrm{Prym}^{\mathrm{prim}}, \mathbb{Q}), \qquad U' \otimes \mathbb{C} = \bigoplus_{\chi \text{ primitive}} \bigwedge^d H^1(X, \mathbb{C})_\chi,$$

a $\mathbb{Q}$-space of dimension $\varphi(m)$. We write $U'' \subseteq H^d(J(X), \mathbb{Q})$ for its pull-back along the projection $J(X) \to \mathrm{Prym}^{\mathrm{prim}}$; $U'' \otimes \mathbb{C}$ is spanned by the same lines Λ_χ, χ primitive, now inside $H^d(J(X), \mathbb{C})$. Patel and Zhang's version uses all non-trivial characters: $U_{\mathrm{Weil}} = \bigwedge^d_{\mathbb{Q}[\mu_m]_{\mathrm{nt}}} H^1(\mathrm{Prym}, \mathbb{Q})$, of $\mathbb{Q}$-dimension $m-1$, where $\mathbb{Q}[\mu_m]_{\mathrm{nt}}$ is the product of the fields attached to the non-trivial rational representations [PZ27].

Why the complexification is a sum of lines: $V \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_\chi V_\chi$ over the $\varphi(m)$ embeddings of $\mathbb{Q}(\mu_m)$ into $\mathbb{C}$, i.e. over the primitive characters, each $V_\chi = H^1(X, \mathbb{C})_\chi$ of dimension d, and $\bigwedge^d_{\mathbb{Q}(\mu_m)} V \otimes \mathbb{C} = \bigoplus_\chi \bigwedge^d V_\chi$; the mixed wedge products $v_{\chi_1} \wedge \cdots \wedge v_{\chi_d}$ with not all χ_i equal are not in $U' \otimes \mathbb{C}$ [Del82, Lemma 4.3], a statement about any étale $\mathbb{Q}$-algebra, in particular $\mathbb{Q}(\mu_m)$. Each $\Lambda_\chi = \bigwedge^d V_\chi$ has Hodge type $(g-1, g-1)$ by Proposition 10.1: a basis of V_χ consists of $g-1$ vectors of type $(1,0)$ and $g-1$ of type $(0,1)$, and their wedge product has type $(g-1, g-1)$. So U' consists of Hodge classes, as in Deligne's criterion [Del82, Prop. 4.4], and the étale hypothesis is what makes every primitive χ balanced at once. For a ramified cover this fails in general, which is why the ramified pieces need the balance of Definition 7.6.

Intuition 10.5 (a Weil class is a charge-neutral top form). A Hodge decomposition is a $U(1)$-action: $z \in U(1)$ acts on $H^{p,q}$ by z^{p-q}, so a class of type (p,p) is one of charge zero. The μ_m-symmetry splits $H^1(X)$ into sectors V_χ, and Λ_χ is the top form of one sector, a "filled Fermi sea" of d one-particle states. Its $U(1)$-charge is the number of holomorphic minus the number of antiholomorphic directions in V_χ; it vanishes exactly when the sector is balanced. For a free action every sector is balanced, because the holomorphic Lefschetz number of a fixed-point-free symmetry vanishes. The analogy stops at the conclusion: charge neutrality (Hodge type) is necessary for algebraicity, and the content of Schoen's theorem is the construction of actual cycles.

LEMMA 10.6 (single lines). *Suppose U'' is spanned over $\mathbb{Q}$ by classes of algebraic cycles. Then every line Λ_χ, χ primitive, lies in* $\mathrm{Alg}(J(X)) \otimes \mathbb{C}$.

PROOF. $U'' \otimes \mathbb{C}$ is spanned over $\mathbb{C}$ by algebraic classes and contains Λ_χ as a subspace. Which convention for the action is used (τ^* or $(\tau^{-1})^*$, as in Chapter 5) does not matter, since the set of primitive characters is closed under $\chi \mapsto \bar\chi$. □

The lines cannot be separated by automorphisms of $J(X)$. One might try to cut Λ_χ out of $U'' \otimes \mathbb{C}$ with the projector $\frac{1}{m}\sum_\tau \chi(\tau)^{-1}\tau^*$ built from the automorphisms of $J(X)$. That fails: σ acts on $\Lambda_\chi = \bigwedge^d V_\chi$ by $\chi(\sigma)^d$, not by $\chi(\sigma)$, and for the three pieces m divides d ($4680 = 24 \cdot 195$, $8424 = 24 \cdot 351$, $9360 = 12 \cdot 780$), so σ acts trivially on all of U''. The lines are separated by the $\mathbb{Q}(\mu_m)$-structure, not by σ^*; Schoen's explicit eigenvectors live upstairs, on the cover Z of Section 10.4, where the deck group acts on a single factor of X^d and z_χ has eigenvalue $\chi(\sigma)$.

A single line Λ_χ is in general not spanned by one rational algebraic class: the Galois group of $\mathbb{Q}(\mu_m)/\mathbb{Q}$ permutes the lines, so a rational vector of U'' with only a χ-component would have all components (Exercise 10.4). Step (iv) needs only the complex statement, because the whole proof works in $\mathrm{Alg} \otimes \mathbb{C}$ until Chapter 11 returns to rational classes.

EXERCISE 10.4. (a) Show that a non-zero $x \in U''$ cannot lie in a single line Λ_χ (use that $\mathrm{Gal}(\mathbb{Q}(\mu_m)/\mathbb{Q})$ acts transitively on the primitive characters and that x is fixed by it). (b) For the toy Example 10.3 write $U'' \otimes \mathbb{C} = \Lambda_\chi \oplus \Lambda_{\bar\chi}$ and give a rational basis of U'' in terms of a basis vector of Λ_χ and its complex conjugate.

10.4. Schoen's cycle: the canonical fibre of the Abel–Jacobi map

Fix a base point and let $\phi\colon \operatorname{Sym}^d Y \to \operatorname{Pic}^d(Y) \cong J(Y)$ send an effective divisor D of degree $d = 2g-2$ to its line bundle $\mathcal{O}(D)$; $\operatorname{Pic}^d(Y) \cong J(Y)$ by the theorems of Abel and Jacobi [Mil08, I, Thms. 18.3–18.4 and p. 73]. The fibre over L is the linear system $|L| = \mathbb{P}(H^0(Y,L))$ [Mil08, III, §5].

Proposition 10.7 (the fibres in degree $2g-2$). *Let $g \geq 2$. The map ϕ is surjective; its fibre over $L \ncong K_Y$ is $\mathbb{P}^{g-2}$, and its fibre over K_Y is $|K_Y| \cong \mathbb{P}^{g-1}$.*

Proof. Riemann–Roch for L of degree $2g-2$ reads $h^0(L) - h^0(K_Y \otimes L^{-1}) = g-1$. The line bundle $K_Y \otimes L^{-1}$ has degree 0, so it has a non-zero section only if it is trivial: $h^0(K_Y \otimes L^{-1}) = 1$ for $L \cong K_Y$ and 0 otherwise. Hence $h^0(L) = g$ or $g-1 \geq 1$; the fibre is non-empty and of dimension $h^0(L) - 1$. □

So ϕ is a $\mathbb{P}^{g-2}$-bundle away from one point, with a jump there. This is different from Chapter 8, where $d = 2g-2+r \geq 2g-1$ makes the Abel–Jacobi map a projective bundle and the special fibre $|L_0|$ is singled out by the local system rather than by its dimension. Macdonald's formula for the Betti numbers of symmetric products [Mac62] gives the same picture: the Poincaré polynomial of $\operatorname{Sym}^{2g-2} Y$ is $(1+s)^{2g}(1+s^2+\cdots+s^{2g-4}) + s^{2g-2}$, that of $H^\bullet(J(Y)) \otimes H^\bullet(\mathbb{P}^{g-2})$ plus one class of degree $2g-2$, the class of the jumping fibre (Exercise 10.5).

The étale cover of the symmetric product. The cover $X \to Y$ is classified by a character $\pi_1(Y) \to \mu_m$, which factors through $H_1(Y,\mathbb{Z}) = H_1(J(Y),\mathbb{Z})$; so it is pulled back from an étale μ_m-cover (an isogeny) $A \to J(Y)$, and since ϕ also induces an isomorphism on H_1 (for $d \geq 2$), one can pull back along ϕ [PZ27]:

$$Z = \operatorname{Sym}^d Y \times_{J(Y)} A \longrightarrow \operatorname{Sym}^d Y,$$

an étale μ_m-cover. Schoen constructs the same cover as the quotient $X^d/\mathcal{N}$, the space called Z in Chapter 8 for ramified covers, of the self-product by the subgroup $\mathcal{N} \subset \mu_m^d \rtimes S_d$ generated by the permutations and the tuples $(\tau_1,\dots,\tau_d)$ with $\tau_1\cdots\tau_d = 1$ [Sch88, pp. 5, 12]; Patel and Zhang identify the two descriptions. The fibre product description makes the key point visible: over the point $[K_Y]$ the isogeny $A \to J(Y)$ has m preimages, so the preimage of $|K_Y|$ in Z is m disjoint copies

$$Q_t = \sigma^t Q \cong \mathbb{P}^{g-1}, \qquad t \in \mathbb{Z}/m,$$

each mapping isomorphically to $|K_Y|$ (equivalently: $|K_Y|$ is simply connected, so the restricted cover is trivial). Each Q_t has dimension $g-1 = d/2$ in the d-dimensional Z: a middle-dimensional cycle.

Proposition 10.8 (the χ-part of $H^d(Z)$). *For $\chi \neq 1$, $H^k(Z,\mathbb{C})_\chi = 0$ for $k \neq d$ and $H^d(Z,\mathbb{C})_\chi$ is one-dimensional, spanned by*

$$z_\chi = \sum_{t \in \mathbb{Z}/m} \chi(\sigma^t)^{-1}\,[Q_t], \qquad \sigma_* z_\chi = \chi(\sigma) z_\chi, \qquad [Q]\cdot z_\chi = [Q]\cdot[Q] = (-1)^{g-1}.$$

The argument. This is [Sch88, p. 13 and Lemma 2.6] and, for the vanishing and the dimension, [PZ27]; we follow the latter's line, which is the $r=0$ case of the support argument of Proposition 8.12. The push-forward of the constant sheaf along the isogeny splits as $\bigoplus_\chi L_\chi$ into rank-one local systems on $J(Y)$, so $H^\bullet(Z,\mathbb{C})_\chi = H^\bullet(\operatorname{Sym}^d Y, \phi^* L_\chi) = H^\bullet(J(Y), R\phi_*\mathbb{C} \otimes L_\chi)$. The stalks of the higher direct images are the cohomology of the fibres. For $i \leq g-2$ the powers ξ^i of the class ξ of the divisor $\{D \geq y_0\}$, which restricts to the hyperplane class on every fibre, trivialize $R^{2i}\phi_*\mathbb{C} \cong \mathbb{C}$; $R^{2g-2}\phi_*\mathbb{C}$ is a skyscraper $\mathbb{C}$ at $[K_Y]$, the top class of the jumping fibre; odd ones vanish. A non-trivial rank-one local system on a torus has no cohomology at all (Künneth over the circle factors, one of which has non-trivial monodromy; compare Lemma 8.11), so in the Leray spectral sequence [Stacks, Tag 01F2]

$E_2^{p,q} = H^p(J(Y), R^q\phi_*\mathbb{C} \otimes L_\chi)$ only $E_2^{0,d}$, the stalk of L_χ at $[K_Y]$, survives: $H^\bullet(Z)_\chi$ is one-dimensional, in degree d. Since the Q_t are disjoint, $[Q]\cdot[Q_t] = 0$ for $t \neq 0$, and $[Q]\cdot z_\chi = [Q]\cdot[Q]$ is the top Chern class of the normal bundle of Q in Z, which is that of $|K_Y|$ in $\mathrm{Sym}^d Y$ because the cover is étale. Schoen computes it: $|K_Y|$ plus a fixed point y_0 is a whole fibre of the projective bundle $\mathrm{Sym}^{d+1} Y \to \mathrm{Pic}^{d+1}(Y)$, whose normal bundle is trivial of rank g, and the normal bundle of $\mathrm{Sym}^d Y$ in $\mathrm{Sym}^{d+1} Y$ restricts to $\mathcal{O}(1)$; so the normal bundle of $|K_Y|$ in $\mathrm{Sym}^d Y$ has total Chern class $(1+H)^{-1}$ and top Chern class $(-1)^{g-1}H^{g-1}$ [Sch88, proof of Lemma 2.6]. Hence $z_\chi \neq 0$, and it spans the line. □

Example 10.9 (the genus-2 toy, by hand). For $g = 2$, $d = 2$, $\phi\colon \mathrm{Sym}^2 Y \to J(Y)$ is an isomorphism away from $[K_Y]$ (fibres $\mathbb{P}^0$) and contracts the curve $|K_Y| \cong \mathbb{P}^1$, the pencil of pairs swapped by the hyperelliptic involution: $\mathrm{Sym}^2 Y$ is $J(Y)$ blown up at one point, with $b_2 = 6+1 = 7$, which is what Macdonald's formula gives. For the triple cover of Example 10.3, Z is the abelian surface A (triple cover of $J(Y)$) blown up at the three preimages of $[K_Y]$, and Q_0, Q_1, Q_2 are the three exceptional curves, each with $Q_t \cdot Q_t = -1 = (-1)^{g-1}$. Since σ acts on A by translations, it acts trivially on $H^\bullet(A)$; on $H^2(Z) = H^2(A) \oplus \mathbb{Q}[Q_0] \oplus \mathbb{Q}[Q_1] \oplus \mathbb{Q}[Q_2]$ it permutes the $[Q_t]$ cyclically. So $b_2(Z) = 9 = 7 + 2$, the non-trivial characters occur once each, on $z_\chi = [Q_0] + \chi(\sigma)^{-1}[Q_1] + \chi(\sigma)^{-2}[Q_2]$ and on $z_{\bar\chi}$, exactly as Proposition 10.8 says; $z_\chi + z_{\bar\chi} = 2[Q_0] - [Q_1] - [Q_2]$ is rational. The Euler characteristics agree as well: $e(Z) = 3\,e(\mathrm{Sym}^2 Y) = 3$, and $H^\bullet(Z)$ is $H^\bullet(\mathrm{Sym}^2 Y)$ (Euler characteristic 1) plus one even class per non-trivial character.

Exercise 10.5. (a) For $g = 3$, describe the fibres of $\mathrm{Sym}^4 Y \to J(Y)$ and the class that the jumping fibre $|K_Y| \cong \mathbb{P}^2$ adds to H^4. (b) Expand Macdonald's generating function $\sum_n P(\mathrm{Sym}^n Y; s)\, t^n = (1+st)^{2g}/((1-t)(1-s^2t))$ to get $P(\mathrm{Sym}^2 Y; s)$ for $g = 2$ and compare with the blow-up description. (c) Show $e(\mathrm{Sym}^{2g-2} Y) = 1$ for every $g \geq 2$, both from the generating function and from Proposition 10.7.

Exercise 10.6. (a) Check $\sigma_* z_\chi = \chi(\sigma) z_\chi$ and $[Q] = \frac{1}{m}\sum_\chi z_\chi$, the sum over all characters with $z_1 = \sum_t [Q_t]$. (b) Using $[Q]\cdot z_\chi = (-1)^{g-1}$, show that the m classes $[Q_t]$ are linearly independent in $H^d(Z, \mathbb{Q})$. (c) Compute c_{g-1} of a bundle with total Chern class $(1+H)^{-1}$ on $\mathbb{P}^{g-1}$ and confirm the value -1 for $g = 2$ against the exceptional curve.

10.5. Schoen's theorem, the transfer to $J(X)$, and Patel–Zhang

The cycles z_χ live on Z, a quotient of X^d; the classes Λ_χ live on $J(X)$. Schoen's Corollary 3.1 transports one to the other. We state it for complex curves in the notation of this chapter; Schoen's Theorem 2.0 is formulated in ℓ-adic cohomology over any algebraically closed field of characteristic prime to m, and the comparison isomorphism gives the Betti version.

Theorem 10.10 (Schoen 1988, Theorem 2.0 and Corollary 3.1). *Let X be a smooth connected projective complex curve with a faithful action of μ_m, $m > 2$, and let $X \to Y = X/\mu_m$ have even primitive multiplicity d and simple (fully paired) branching exponents. Then $U' \subseteq H^d(\mathrm{Prym}^{\mathrm{prim}}(X/Y), \mathbb{Q})$, and with it $U'' \subseteq H^d(J(X), \mathbb{Q})$, is generated by classes of algebraic cycles of codimension $d/2$ [Sch88, pp. 11, 24–25]. If $X \to Y$ is étale, the exponent condition is empty, $d = 2g(Y) - 2$ is even, and the cycles are those of Proposition 10.8 [Sch88, p. 13]. (Only the étale case is used here.)*

The transfer [Sch88, proof of Cor. 3.1, p. 25]. Let $\Xi\colon X^d \to J(X)$, $(x_1, \dots, x_d) \mapsto \sum_i \mathrm{AJ}(x_i)$, be the sum map. On H^1, Ξ^* is the diagonal map, and the pull-back of Λ_χ to X^d is the antisymmetrization of $V_\chi^{\otimes d}$; Schoen shows that $\Xi^* U''$ is the Hodge structure $U \subseteq H^d(X^d, \mathbb{Q})$ of his §1, and Theorem 2.0 says that U is spanned by the pull-backs to X^d of the cycles on $Z = X^d/\mathcal{N}$ (Proposition 10.8 for $r = 0$). Going back from X^d to $J(X)$ is the transfer of Proposition 9.7, with Ξ in the role of the sum map there: Poincaré's formula for the image of Ξ and Lieberman's theorem, which makes the inverse of a power of the Lefschetz

operator algebraic, show that Λ_χ is algebraic as soon as its pull-back to X^d is. That transfer needs $d \le g_X$, which holds automatically for $m > 2$: the $\varphi(m) \ge 2$ primitive eigenspaces, each of dimension d, are independent in $H^1(X,\mathbb{C})$, so $\varphi(m)\,d \le 2g_X$ (Remark 8.2). For an étale cover it is also visible directly, $g_X = m(g-1)+1 > 2g-2 = d$, and for the three pieces $4680 \le 56161$, $8424 \le 101089$, $9360 \le 56161$.

THEOREM 10.11 (Patel–Zhang). *Let $X \to Y$ be an étale Galois cover of smooth projective complex curves with finite abelian group G, $g(Y) \ge 2$, $d = 2g(Y)-2$. Then $H^1(\mathrm{Prym}(X/Y),\mathbb{Q})$ is a free $\mathbb{Q}[G]_{\mathrm{nt}}$-module of rank d, and $U_{\mathrm{Weil}} = \bigwedge^d_{\mathbb{Q}[G]_{\mathrm{nt}}} H^1(\mathrm{Prym},\mathbb{Q})$, whose complexification is $\bigoplus_{\chi\neq 1}\bigwedge^d H^1(X,\mathbb{C})_\chi$, is spanned by classes of algebraic cycles [PZ27].*

Patel and Zhang present the cover $Z \to \mathrm{Sym}^d Y$ as unramified geometric class field theory (the fibre product with an isogeny above), prove Proposition 10.8 for every abelian G by the Leray argument sketched there, and transfer to the Prym with Macdonald's description of $H^\bullet(\mathrm{Sym}^d X)$, Poincaré's formula and the algebraicity of the inverse Lefschetz operator. They attribute the cyclic case with primitive characters to Schoen [Sch88]. For the proof only the primitive character ψ_j of a cyclic group matters, so Theorem 10.10 suffices, and [PZ27] is an alternative reference.

What just happened. For an étale μ_m-cover over a base of genus $g \ge 2$, each non-trivial character has multiplicity $d = 2g-2$ and balanced Hodge numbers, so $\Lambda_\chi = \bigwedge^d H^1(X)_\chi$ is a Hodge class, for every χ at once. Schoen's cycle is the preimage of the jumping fibre $|K_Y| \cong \mathbb{P}^{g-1}$ of $\mathrm{Sym}^d Y \to J(Y)$ in the étale cover Z: m disjoint copies Q_t, whose χ-combinations z_χ are non-zero (self-intersection $(-1)^{g-1}$) and span the χ-lines of $H^d(Z)$. Poincaré's formula and Lieberman's algebraic inverse Lefschetz carry them to $U'' \subseteq H^d(J(X))$, whose complexification contains every primitive Λ_χ.

10.6. Pieces 5, 7 and 8

PROPOSITION 10.12 (the étale pieces). *For $j \in \{5,7,8\}$ the cover $X_j \to Y_j$ of Table 7.1 is a connected étale μ_m-cover with $m = 24, 24, 12$ and $g(Y_j) = 2341, 4213, 4681$; ψ_j is a primitive character; $d_j = 2g(Y_j)-2 = 4680, 8424, 9360$ is even. Hence, by Theorem 10.10 and Lemma 10.6, $\Lambda_{\psi_j} = \bigwedge^{d_j} H^1(X_j,\mathbb{C})_{\psi_j} \subset \mathrm{Alg}(J(X_j)) \otimes \mathbb{C}$, and by Proposition 7.10, $\Lambda_j = \bigwedge^{\mathrm{top}} W_j \in \mathrm{Alg}(J(C)) \otimes \mathbb{C}$.*

PROOF. Connectedness of $X_j = C/\ker\psi_j$ and faithfulness of the induced action of $H_j/\ker\psi_j \cong \mu_m$ hold for every piece (Definition 7.1); a generator acts on X_j through ψ_j, which is faithful on $H_j/\ker\psi_j$, i.e. primitive. The values $r_{\psi_j} = 0$, m and $g(Y_j)$ come from the enumeration of the pieces (Section 7.5), a computation from the character table and the subgroup lattice of G_{32}: r_ψ and $g(Y)$ come from the coset actions of the thirteen local monodromies $\gamma_1,\dots,\gamma_{13}$ (Proposition 7.5); for an étale piece, ψ is trivial on every local stabilizer in H. $m > 2$ and d_j even are then arithmetic. The conclusion is Theorem 10.10 applied to the étale cover $X_j \to Y_j$, then Lemma 10.6 for the single line Λ_{ψ_j}, then Proposition 7.10. □

The numbers:

j	$\lvert H_j\rvert$	m	$\varphi(m)$	$g(Y_j)$	d_j	$g(X_j)$	$\dim \mathrm{Prym}$	$\dim \mathrm{Prym}^{\mathrm{prim}}$
5	216	24	8	2341	4680	56161	53820	18720
7	120	24	8	4213	8424	101089	96876	33696
8	108	12	4	4681	9360	56161	51480	18720

For each j, the piece of the enumeration with the table's $|H|$, m, $g(Y)$ and vector has $d_j = \sum_{\rho'} v_j(\rho')\dim M_{\rho'}$, with Hodge numbers $(g(Y_j)-1, g(Y_j)-1)$, and by the Lefschetz formula of Proposition 10.1 every non-trivial character has multiplicity d_j on a cover of genus $g(X_j)$. Over all 69 étale pieces of the enumeration $\min g(Y) = 2341$, so every étale piece

satisfies the hypotheses, not only these three. Schoen's cycles for piece 5 are the 24 copies of $\mathbb{P}^{2340}$ over the canonical fibre of $\mathrm{Sym}^{4680} Y_5$; U' has $\mathbb{Q}$-dimension $\varphi(24) = 8$, and Λ_{ψ_5} is one of its eight complex lines.

Exercise 10.7. (a) Verify $g(X_j) = m(g(Y_j) - 1) + 1$ for $j = 5, 7, 8$ and explain why pieces 5 and 8 have the same $g(X) = 56161$ although m and $g(Y)$ differ. (b) For piece 8 list the primitive characters of μ_{12} and the non-primitive non-trivial ones; which of them does Schoen's theorem cover, and which only Patel–Zhang's? (c) What is the codimension of Schoen's cycle $\Xi_* Q$ in $J(X_8)$, and in which cohomological degree does Λ_8 live?

Remark 10.13 (Where this is used). Proposition 10.12 supplies the three étale lines $\Lambda_5, \Lambda_7, \Lambda_8 \in \mathrm{Alg}(J(C)) \otimes \mathbb{C}$; together with the eight ramified lines of Chapters 8 and 9 they are the input of the contraction in Chapter 11, where Λ_5 and Λ_8 sit on the $c_j = +1$ side and Λ_7 on the $c_j = -1$ side. The review chapter Chapter 13 lists step (iv) as resting on Schoen 1988, with Patel–Zhang as an alternative and the piece data from the enumeration of Chapter 7.

CHAPTER 11

Assembling the pieces: $\mathbb{C}[G]$-correspondences and the contraction

Proof steps: (vi); χ- and $\bar{\chi}$-lines to rational classes

Where we are. This chapter explains step (vi), the step that turns the eleven-piece relation $e_\rho = \sum_j c_j v_j$ of Proposition 7.8 into the algebraicity of the Weil classes of $\mathcal{B}_C$. The input is the output of steps (iii)–(v): for each of the eleven pieces, the line $\Lambda_j = \bigwedge^{\mathrm{top}} W_j$ lies in $\mathrm{Alg}(P) \otimes \mathbb{C}$, where $P = J(C)$ (Proposition 7.10). The output is that the Weil classes $\mathrm{HW}(A_C)$ of the sixfold A_C, with $H^1(A_C, \mathbb{Q}) = \mathcal{B}_C$, are algebraic, for every member C of the Hurwitz component $\mathcal{H}$ of Example 5.19; steps (vii)–(viii) (Chapter 12) take it from there. Needs: Chapter 3 (Wedderburn Theorem 3.20, matrix units Definition 3.21 and Proposition 3.22, density Proposition 3.26, V_ρ Proposition 3.27, $k_{\min}$ Definition 3.17), Chapter 7 (pieces Definition 7.1, the relation Proposition 7.8, Table 7.1, the pull-back Proposition 7.10), Chapter 5 ($\mathcal{B}_C$, Definition 5.17); the cohomology ring of an abelian variety [Mil08, I, Thm. 2.3], the description of abelian varieties as polarizable complex tori [Mil08, I, Thm. 2.9], and the compatibility of cycle classes with pull-back, push-forward and products [Ful98, Ch. 19]. Notation: $G = G_{32}$, $Z = Z(G) \cong \mu_6$, κ_0 the central character of ρ, $g = g(C) = 505441$; $\mathrm{Alg}(X) \subseteq H^{2\bullet}(X, \mathbb{Q})$ is the $\mathbb{Q}$-span of classes of algebraic cycles and $\mathrm{Alg}(X) \otimes \mathbb{C}$ its complex span.

11.1. Step (vi) in two pages

After step (v) we hold eleven algebraic lines, one for each piece $W_j \subset H^1(C, \mathbb{C})$, and we want a twelfth: the line

$$\Lambda_u = \textstyle\bigwedge^6 (u \otimes M_\rho) \subset H^6(P, \mathbb{C}), \qquad u \in \rho \setminus \{0\},$$

where $u \otimes M_\rho = \{m(u) : m \in M_\rho\}$ is a copy of the six-dimensional multiplicity space inside $H^1(C, \mathbb{C})$ (Proposition 3.22). Each W_j is a sum of multiplicity spaces, $W_j = \bigoplus_{\rho'} \rho'^{(H_j, \psi_j)} \otimes M_{\rho'}$ with $\dim \rho'^{(H_j, \psi_j)} = v_j(\rho')$, and the relation $e_\rho = \sum_j c_j v_j$ says that the multiplicity vectors on the two sides of

$$\underbrace{\textstyle\bigoplus_{c_j = +1} W_j}_{U_+,\ 7 \text{ pieces}} \qquad \text{and} \qquad \underbrace{(u \otimes M_\rho) \oplus \textstyle\bigoplus_{c_j = -1} W_j}_{\mathcal{T} = (u \otimes M_\rho) \oplus U_-,\ 1+4 \text{ pieces}}$$

agree. Put each piece in its own copy of P (a *slot*): U_+ lives in $H^1(X_+)$ with $X_+ = P^7$, and $\mathcal{T}$ in $H^1(P_0 \times X_-)$ with $X_- = P^4$ and one more copy P_0. Since the multiplicity vectors agree, U_+ and $\mathcal{T}$ are isomorphic in the way that matters: for every ρ' the ρ'-parts have the same size, so a linear map $A_{\rho'}$ between direct sums of copies of ρ', tensored with the identity of $M_{\rho'}$, carries the one onto the other. By Wedderburn's theorem the family $(A_{\rho'})_{\rho'}$ is a single matrix $\Psi \in \mathrm{Mat}_{5 \times 7}(\mathbb{C}[G])$ (Theorem 3.20), and Ψ acts on cohomology, with complex coefficients, through the action of G on P (Remark 11.2).

The step then has three moves.

(1) Transport. The induced map on top exterior powers, $\bigwedge^D \Psi^*$ with $D = D_+ = \dim U_+ = 28374$ (in this chapter D stands for D_+, and also for the degree in general lemmas; it is not

the branch divisor of Chapter 8), sends the algebraic line $\Lambda_+ = \bigwedge^D U_+$ (a cup product of the seven Λ_j, one per slot) to a nonzero multiple of

$$\bigwedge^D \mathcal{T} = \Lambda_u \boxtimes \Lambda_-, \qquad \Lambda_- = \bigwedge^{D_-} U_-, \quad D_- = 28368.$$

The subtle point is that $\bigwedge^D \Psi^*$ is not linear in Ψ; it is a polynomial of degree D. For rational Ψ it is a rational multiple of pull-back by a homomorphism of abelian varieties, which preserves algebraic classes, and rational matrices are Zariski dense among complex ones, so every value of the polynomial lies in the complex span of its rational values (Proposition 11.5). Hence $\Lambda_u \boxtimes \Lambda_- \in \mathrm{Alg}(P_0 \times X_-) \otimes \mathbb{C}$.

(2) Contraction. The four minus pieces have to be removed from the product. The central character κ_0 is not real, so U_- is isotropic for the polarization and pairs perfectly with its complex conjugate $\bar{U}_-$; this makes $\int_{X_-} \Lambda_- \cup \bar{\Lambda}_- \cup \Theta^{g_- - D_-} \neq 0$ ($g_- = \dim X_-$). The class $\mu = \bar{\Lambda}_- \cup \Theta^{g_- - D_-}$ is again in $\mathrm{Alg} \otimes \mathbb{C}$, and pairing $\Lambda_u \boxtimes \Lambda_-$ with μ over X_- leaves a nonzero multiple of Λ_u (Proposition 11.11). So $\Lambda_u \in \mathrm{Alg}(P) \otimes \mathbb{C}$ (Theorem 11.8).

(3) Back to $\mathbb{Q}$. For a rational vector $u \in V_\rho$, evaluation $f \mapsto f(u)$ embeds $\mathcal{B}_C$ in $H^1(C, \mathbb{Q})$ as H^1 of a quotient B_u of P, and the complexified Weil classes of B_u are spanned by the line $\Lambda_{u'}$ and its complex conjugate. A rational subspace whose complexification is spanned by algebraic classes consists of algebraic classes, so $\mathrm{HW}(B_u)$ is algebraic, and B_u is K-isogenous to A_C (Proposition 11.15).

object	where	dimension
$U_+ = \bigoplus_{c_j=+1} W_j$ (pieces $2,4,5,6,8,9,11$)	$H^1(X_+)$, $X_+ = P^7$	$D_+ = 28374$
$U_- = \bigoplus_{c_j=-1} W_j$ (pieces $1,3,7,10$)	$H^1(X_-)$, $X_- = P^4$	$D_- = 28368$
$u \otimes M_\rho$	$H^1(P_0)$	6
Ψ	$\mathrm{Mat}_{5\times 7}(\mathbb{C}[G])$	$35 \cdot 155520$ coefficients

Only single complex lines are ever handled, never $\mathbb{Q}$- or K-rational Weil structures of the pieces; this is why a relation with coefficient 1 at e_ρ is needed, and why the discriminant of the pieces plays no role (Section 11.7). Factorizations of determinant lines across direct sums and contractions of paired determinants, for abelian covers, are among the tools of the preprint [Ope26a].

11.2. Correspondences between abelian varieties and complex combinations

11.2.1. Homomorphisms act on cohomology by exterior powers. A homomorphism $f\colon A \to B$ of abelian varieties acts on H^1 by pull-back, and since $H^\bullet(A, \mathbb{Q}) = \bigwedge^\bullet H^1(A, \mathbb{Q})$ as rings [Mil08, I, Thm. 2.3] and f^* is a ring homomorphism, it acts on H^k by $\bigwedge^k f^*|_{H^1}$. Pull-back along any morphism, push-forward along a proper one, cup product and exterior product of classes all preserve Alg, because cycle class maps are compatible with these operations [Ful98, Ch. 19]; in particular f^* preserves Alg. Complex conjugation on $H^\bullet(X, \mathbb{C}) = H^\bullet(X, \mathbb{Q}) \otimes \mathbb{C}$ preserves $\mathrm{Alg}(X) \otimes \mathbb{C}$, since $\mathrm{Alg}(X)$ is rational.

Homomorphisms can be added: $(f + f')(a) = f(a) + f'(a)$. On H^1 this is additive, $(f + f')^* = f^* + f'^*$, because the group law $\mathfrak{m}\colon B \times B \to B$ satisfies $\mathfrak{m}^* e = e \otimes 1 + 1 \otimes e$ on H^1 and $f + f' = \mathfrak{m} \circ (f, f')$. On H^k it is not additive: $(f + f')^* = \bigwedge^k (f^* + f'^*)$, whereas the sum of cycles $\Gamma_f + \Gamma_{f'}$ acts as $\bigwedge^k f^* + \bigwedge^k f'^*$. The simplest instance is $[2] = \mathrm{id} + \mathrm{id}$, which acts on H^k by 2^k, while $2[\Delta]$ acts by 2. So there are two different ways to add correspondences between abelian varieties, the sum of cycles, which is linear on all of $H^\bullet$, and the sum in $\mathrm{Hom}(A, B)$, which is linear only on H^1. Group-algebra elements are sums of the second kind.

Physics aside 11.1 (second quantization). $H^\bullet(A) = \bigwedge^\bullet H^1(A)$ is literally a fermionic Fock space on the one-particle space $H^1(A)$. A homomorphism acts on it by the multiplicative second quantization $\Gamma(L) = \bigoplus_k \bigwedge^k L$ of its one-particle operator $L = f^*|_{H^1}$, and a sum of cycles acts by a sum of such operators. The map $L \mapsto \Gamma(L)$ is not linear: on the k-particle

sector it is homogeneous of degree k in L (its matrix entries are the $k \times k$ minors of L), just as $\Gamma(\mathrm{e}^{tX}) = \mathrm{e}^{t\,\mathrm{d}\Gamma(X)}$ is not linear in t. Step (vi) needs $\Gamma(\Psi)$ on the D-particle sector for a one-particle operator Ψ with complex coefficients. The analogy stops at the question that matters here: whether an operator on Fock space is induced by an algebraic cycle has no counterpart in physics.

11.2.2. The group algebra acting on products of Jacobians. The group G acts on C, hence on $P = J(C)$: $g \in G$ induces the automorphism g_P of P given by push-forward of divisor classes. Under $H^1(P,\mathbb{Q}) = H^1(C,\mathbb{Q})$, the identification by pull-back along an Abel–Jacobi map $C \to P$, one has $g_P^* = g^*$, because g_P composed with the Abel–Jacobi map and the Abel–Jacobi map composed with g differ by a translation, which acts trivially on cohomology. Pull-back reverses products, $(gh)^* = h^*g^*$, so we use, as in Chapter 5, the left action $g \cdot x = (g^{-1})^* x$ of G on $H^1(C)$, extended linearly to $\mathbb{C}[G]$; the decomposition $H^1(C,\mathbb{C}) = \bigoplus_{\rho'} \rho' \otimes M_{\rho'}$ and Proposition 3.22 refer to this action. For $y = \sum_g n_g g \in \mathbb{Z}[G]$ put $y^\iota = \sum_g n_g g^{-1}$ and let $[y] = \sum_g n_g g_P$ be the corresponding sum of homomorphisms $P \to P$. Pull-back along a sum of homomorphisms is additive on H^1 (below), so $[y^\iota]^* = \sum_g n_g (g^{-1})^*$ is the action $x \mapsto y \cdot x$.

Matrices act in the same way. For $\Psi \in \mathrm{Mat}_{b\times a}(\mathbb{C}[G])$ we write $\Psi^*\colon H^1(P^a,\mathbb{C}) = H^1(C,\mathbb{C})^{\oplus a} \to H^1(P^b,\mathbb{C}) = H^1(C,\mathbb{C})^{\oplus b}$ for the operator

$$(\Psi^* x)_i = \sum_{j=1}^{a} \Psi_{ij} \cdot x_j, \qquad i = 1,\dots,b,$$

so a column of a classes is multiplied by the $b \times a$ matrix, without transposition, and $(\Psi\Psi')^* = \Psi^* \circ \Psi'^*$. For an integral $y \in \mathrm{Mat}_{b\times a}(\mathbb{Z}[G])$ the operator y^* is the pull-back by a homomorphism in the opposite direction: $f_y\colon P^b \to P^a$, $f_y(p)_j = \sum_{i=1}^{b} [y^\iota_{ij}](p_i)$, satisfies $f_y^* = y^*$, because $\mathrm{pr}_j \circ f_y = \sum_i [y^\iota_{ij}] \circ \mathrm{pr}_i$ gives $f_y^* \mathrm{pr}_j^* x_j = \sum_i \mathrm{pr}_i^*(y_{ij} \cdot x_j)$. In step (vi), $\Psi \in \mathrm{Mat}_{5\times 7}(\mathbb{C}[G])$ acts as $\Psi^*\colon H^1(X_+,\mathbb{C}) \to H^1(P_0 \times X_-,\mathbb{C})$, and when it is integral, $\Psi^* = f_\Psi^*$ for the homomorphism $f_\Psi\colon P_0 \times X_- \to X_+$. The isomorphism $\mathrm{Mat}_{b\times a}(\mathbb{Q}[G]) \otimes \mathbb{C} = \mathrm{Mat}_{b\times a}(\mathbb{C}[G])$ is that of Proposition 3.26.

Remark 11.2 (conventions). The notation Ψ^* is read through the left action of $\mathbb{C}[G]$, so that products of matrices act as composites and the multiplicity spaces $M_{\rho'}$ are left untouched (Proposition 3.22). The price is the anti-involution $y \mapsto y^\iota$ in the passage to homomorphisms: an integral Ψ is realized by f_Ψ, built from the entries Ψ^ι_{ij}, and f_Ψ goes from P^b to P^a. For a single group element, viewed as a 1×1 matrix, the operator is $g \cdot = (g^{-1})^*$, not g^*. Since $y \mapsto y^\iota$ maps $\mathrm{Mat}(\mathbb{Z}[G])$ and $\mathrm{Mat}(\mathbb{Q}[G])$ onto themselves, the density argument below is unaffected.

Lemma 11.3 (exterior powers are polynomial). *Let V, W be finite-dimensional complex vector spaces and $D \geq 0$. In bases of V, W and the induced bases $e_{i_1} \wedge \cdots \wedge e_{i_D}$ $(i_1 < \cdots < i_D)$, the matrix entries of $\bigwedge^D L\colon \bigwedge^D V \to \bigwedge^D W$ are the $D \times D$ minors of the matrix of L. Hence $\Phi\colon \mathrm{Hom}(V,W) \to \mathrm{Hom}(\bigwedge^D V, \bigwedge^D W)$, $L \mapsto \bigwedge^D L$, is a homogeneous polynomial map of degree D; it satisfies $\bigwedge^D(LL') = \bigwedge^D L \bigwedge^D L'$ and $\bigwedge^D(cL) = c^D \bigwedge^D L$; and it maps rational matrices to rational matrices.*

Proof. $\bigwedge^D L(e_{j_1} \wedge \cdots \wedge e_{j_D}) = Le_{j_1} \wedge \cdots \wedge Le_{j_D}$, and expanding $Le_{j_k} = \sum_i L_{ij_k} e_i$ multilinearly, the coefficient of $e_{i_1} \wedge \cdots \wedge e_{i_D}$ is $\sum_\sigma \mathrm{sgn}(\sigma) \prod_k L_{i_{\sigma(k)} j_k}$, the minor with rows $i_1,\dots,i_D$ and columns $j_1,\dots,j_D$. Functoriality is the definition $\bigwedge^D L(v_1 \wedge \cdots \wedge v_D) = Lv_1 \wedge \cdots \wedge Lv_D$. □

A homogeneous polynomial map of degree D can be recovered from its values by *polarization*: with $\tilde\Phi$ the symmetric multilinear map with $\tilde\Phi(X,\dots,X)=\Phi(X)$,

$$\begin{aligned}\tilde\Phi(X_1,\dots,X_D) &= \frac{1}{D!}\sum_{S\subseteq\{1,\dots,D\}}(-1)^{D-|S|}\,\Phi\Big(\sum_{i\in S}X_i\Big),\\ \Phi\Big(\sum_k z_kX_k\Big) &= \sum_{k_1,\dots,k_D} z_{k_1}\cdots z_{k_D}\,\tilde\Phi(X_{k_1},\dots,X_{k_D}).\end{aligned}\tag{11.1}$$

If the X_k are rational, the right-hand side writes $\Phi(\sum_k z_kX_k)$ explicitly as a complex combination of values of Φ at rational points. This is a constructive form of the density argument of Proposition 3.26; for $D=2$ it reads $\Phi(\sum z_kX_k)=\sum_k z_k^2\Phi(X_k)+\sum_{k<l}z_kz_l\big(\Phi(X_k+X_l)-\Phi(X_k)-\Phi(X_l)\big)$.

Example 11.4 (μ_3 acting on $E\times E$). Let $E=\mathbb{C}/\mathbb{Z}[\omega]$ with the automorphism $t=[\omega]$ of order 3, and $A=E\times E$ with $\mu_3=\langle t\rangle$ acting diagonally. In the basis $dz_1,dz_2,d\bar z_1,d\bar z_2$ of $H^1(A,\mathbb{C})$, t^* multiplies dz_i by ω and $d\bar z_i$ by $\bar\omega$. Since μ_3 is abelian, we let group-algebra elements act here simply by pull-back, $t^l\mapsto(t^l)^*$; this differs from the convention of Remark 11.2 by $t\mapsto t^{-1}$, which only exchanges χ and $\bar\chi$. A matrix $x=(x_{ij})\in\mathrm{Mat}_2(\mathbb{C}[\mu_3])$, $x_{ij}=\sum_l a_{ijl}t^l$, acts on H^1 by $\mathrm{diag}(P,Q)$ with

$$P_{ij}=\chi(x_{ij})=\sum_l a_{ijl}\,\omega^l,\qquad Q_{ij}=\bar\chi(x_{ij})=\sum_l a_{ijl}\,\bar\omega^l.$$

For rational x one has $Q=\bar P$, but for complex x the matrices P and Q are independent: the map $\mathbb{C}[\mu_3]\to\mathbb{C}^3$, $a\mapsto(\text{trivial},\chi,\bar\chi)$, has determinant $-3\sqrt{-3}\neq0$ on the basis $1,t,t^2$. Exterior square: $H^2(A,\mathbb{C})$ has the basis $dz_1\wedge dz_2$ (type $(2,0)$), the four $dz_i\wedge d\bar z_j$ (type $(1,1)$, all algebraic since E has complex multiplication and $\mathrm{NS}(A)\otimes\mathbb{C}=H^{1,1}$ has rank 4), and $d\bar z_1\wedge d\bar z_2$; on these $\bigwedge^2x^*$ is $\det P$, $P\otimes Q$ and $\det Q$. Now take the idempotents $e_\chi=\frac13(1+\bar\omega t+\omega t^2)$, $e_{\bar\chi}=\overline{e_\chi}$ and

$$\Psi=\begin{pmatrix}e_\chi&0\\0&e_{\bar\chi}\end{pmatrix},\qquad P=\begin{pmatrix}1&0\\0&0\end{pmatrix},\quad Q=\begin{pmatrix}0&0\\0&1\end{pmatrix}.$$

Then $\bigwedge^2\Psi^*$ is the projector onto the line $\mathbb{C}\,dz_1\wedge d\bar z_2$, which lies in $\mathrm{Alg}(A)\otimes\mathbb{C}$ but is not defined over $\mathbb{Q}$: its complex conjugate is the different line $\mathbb{C}\,dz_2\wedge d\bar z_1$. Formula (11.1) for $D=2$ writes this projector explicitly: Ψ has 6 nonzero coefficients on the integral basis $E_{ij}t^l$, and $\bigwedge^2\Psi^*$ equals the stated combination of $\bigwedge^2$ of $6+15$ integral matrices, each $\bigwedge^2X^*$ being the pull-back by a homomorphism $A\to A$. Note also that $\bigwedge^2$ is not additive ($\bigwedge^2(1+t)\neq\bigwedge^2(1)+\bigwedge^2(t)$; on $dz_1\wedge dz_2$, $1+t$ acts by $(1+\omega)^2=\omega$). This is step (vi) in miniature: a complex combination of correspondences, which is no correspondence at all, has an exterior power that is a complex combination of correspondences and isolates a single non-rational line of algebraic classes.

Exercise 11.1. In Example 11.4: (a) compute $\bigwedge^2x^*$ on the six basis vectors of $H^2(A,\mathbb{C})$ for $x=\mathrm{diag}(1+t,1)$, and check that it maps $H^{1,1}$ into itself. (b) Show that $\Psi=e_\chi\cdot\mathbf{1}_2$ gives the projector onto the transcendental line $\mathbb{C}\,dz_1\wedge dz_2$ of type $(2,0)$, and explain why this does not contradict Proposition 11.5.

Exercise 11.2. (a) Show that $\bigwedge^D(\Psi\circ\Psi')=\bigwedge^D\Psi\circ\bigwedge^D\Psi'$ for composable matrices over $\mathbb{C}[G]$ acting on H^1, and conclude that the set of Ψ for which $\bigwedge^D\Psi^*$ maps $\mathrm{Alg}\otimes\mathbb{C}$ into $\mathrm{Alg}\otimes\mathbb{C}$ is closed under composition. (b) For homomorphisms $f,f'\colon A\to B$ of abelian varieties of dimension n, compare the actions of $f+f'$ and of the cycle $\Gamma_f+\Gamma_{f'}$ on H^{2n}, and evaluate both for $f=f'=\mathrm{id}_E$ on an elliptic curve.

11.3. Why complex combinations preserve complexified algebraic classes

Proposition 11.5 (density). *Let $a,b,D\geq1$ and, for $\Psi\in\mathrm{Mat}_{b\times a}(\mathbb{C}[G])$, let $\Phi(\Psi)=\bigwedge^D\Psi^*\colon H^D(P^a,\mathbb{C})\to H^D(P^b,\mathbb{C})$. Then $\Phi(\Psi)$ maps $\mathrm{Alg}(P^a)\otimes\mathbb{C}$ into $\mathrm{Alg}(P^b)\otimes\mathbb{C}$. More*

precisely, $\Phi(\Psi)$ lies in the complex span of the maps f_y^*, $y \in \mathrm{Mat}_{b\times a}(\mathbb{Z}[G])$, pull-backs by homomorphisms $P^b \to P^a$.

PROOF. The map $\Psi \mapsto \Psi^*|_{H^1}$ is $\mathbb{C}$-linear, and $\Phi(\Psi)$ is $\bigwedge^D$ of it [Mil08, I, Thm. 2.3], so Φ is a homogeneous polynomial map of degree D in the $ab|G|$ coefficients of Ψ (Lemma 11.3). Let $x \in \mathrm{Mat}_{b\times a}(\mathbb{Q}[G])$ and $n \geq 1$ with $y = nx$ integral. Then $\Phi(x) = n^{-D}\Phi(y) = n^{-D} f_y^*$ on H^D, by homogeneity and because f_y^* acts on H^D as $\bigwedge^D$ of its action on H^1; and f_y^* preserves Alg. So $\Phi(x)$ maps $\mathrm{Alg}(P^a)\otimes\mathbb{C}$ into $\mathrm{Alg}(P^b)\otimes\mathbb{C}$ for every rational x, and Proposition 3.26(b) (rational points are Zariski dense, so the image of the polynomial map Φ lies in the span of its rational values) gives the same for every Ψ, together with the span statement. □

In other words, every linear form vanishing on $\Phi(\mathrm{Mat}(\mathbb{Q}[G]))$ vanishes on $\Phi(\Psi)$, which is the proof of Proposition 3.26(b). Formula (11.1) even gives the coefficients: writing $\Psi = \sum_k z_k X_k$ in the integral basis $X_k = E_{ij}\,g$, $\Phi(\Psi)$ is a combination of the f^* for $f = f_{X_{k_1}+\cdots+X_{k_r}}$. For $D = 28374$ this is of no practical use, but it shows that nothing transcendental is hidden in the density argument. What the proposition does *not* say is that $\Phi(\Psi)$ maps $\mathrm{Alg}(P^a)$ into $\mathrm{Alg}(P^b)$: the rational structure is lost, as the non-rational line of Example 11.4 shows, and it is recovered only at the very end (Section 11.6).

EXERCISE 11.3. (a) Let $V_\mathbb{Q}$ be a finite-dimensional $\mathbb{Q}$-vector space and $V = V_\mathbb{Q}\otimes\mathbb{C}$. Show that a polynomial function on V (a polynomial in the coordinates of some, equivalently every, basis) that vanishes on $V_\mathbb{Q}$ is zero; the point is that the notion does not depend on a basis of $V_\mathbb{Q}$. (b) Write $\det\begin{pmatrix} 1 & \mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix}$ as a complex combination of determinants of rational 2×2 matrices, using (11.1).

11.4. The correspondence Ψ and the bookkeeping

We now build Ψ for the eleven pieces. Number the slots of $X_+ = P^7$ by the pieces with $c_j = +1$ and those of $X_- = P^4$ by the pieces with $c_j = -1$, and let P_0 be one more copy of P. Then $H^1(X_+,\mathbb{C}) = H^1(P,\mathbb{C})^{\oplus 7} = \bigoplus_{\rho'}(\rho'^{\oplus 7})\otimes M_{\rho'}$, and similarly with $\rho'^{\oplus 5}$ for $P_0\times X_-$. The subspaces of the brief decompose as

$$U_+ = \bigoplus_{\rho'} \mathfrak{u}_+(\rho')\otimes M_{\rho'}, \qquad \mathfrak{u}_+(\rho') = \bigoplus_s \rho'^{(H_s,\psi_s)} \subseteq \rho'^{\oplus 7},$$
$$\mathcal{T} = \bigoplus_{\rho'} \mathfrak{t}(\rho')\otimes M_{\rho'}, \qquad \mathfrak{t}(\rho') = \delta_{\rho'\rho}\,\mathbb{C}u \oplus \bigoplus_t \rho'^{(H_t,\psi_t)} \subseteq \rho'^{\oplus 5},$$

with s running over the plus slots, t over the minus slots, and $\mathbb{C}u$ in slot P_0. Write $n_\pm(\rho') = \sum_{c_j=\pm 1} v_j(\rho')$. Then $\dim \mathfrak{u}_+(\rho') = n_+(\rho')$ and $\dim \mathfrak{t}(\rho') = \delta_{\rho'\rho} + n_-(\rho')$, and the relation $\sum_j c_j v_j = e_\rho$ is precisely the statement $n_+(\rho') = \delta_{\rho'\rho} + n_-(\rho')$ for all ρ'.

EXAMPLE 11.6 (the bookkeeping for G_{32}). For the fourteen irreducibles $\rho_1 = \rho, \rho_2, \dots, \rho_{14}$ with central character κ_0, numbered as in Chapter 7, the Chevalley–Weil data and the piece vectors of Proposition 7.8 give

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$\dim\rho_k$	4	4	20	20	20	20	20	36	36	60	60	60	64	80
$\dim M_{\rho_k}$	6	42	102	102	138	174	174	162	270	342	378	450	384	552
$n_+(\rho_k)$	1	1	3	2	4	4	5	5	7	9	9	11	10	14
$n_-(\rho_k)$	0	1	3	2	4	4	5	5	7	9	9	11	10	14

so $n_+ = e_\rho + n_-$ entry by entry. Then $D_+ = \dim U_+ = \sum_{\rho'} n_+(\rho')\dim M_{\rho'} = 28374$ and $D_- = \dim U_- = \sum n_-(\rho')\dim M_{\rho'} = 28368$, in agreement with the column $d = 2g(Y_j)-2+r$ of Table 7.1 (which comes from coset actions, not from the character table); the difference is $6 = \dim M_\rho$. All multiplicity spaces are balanced, with $h^{1,0} = h^{0,1} = \frac{1}{2}\dim M_{\rho'}$ (Example 5.13). Each $\mathfrak{u}_+(\rho')$ sits in $\rho'^{\oplus 7}$ and each $\mathfrak{t}(\rho')$ in $\rho'^{\oplus 5}$ ($n_+(\rho') \leq 5\dim\rho'$ in every column). The

degrees satisfy $\sum(\dim\rho')^2 = 25920 = |G|/6$, as they must for the block of $\mathbb{C}[G]$ with central character κ_0.

Definition 11.7 (the correspondence Ψ). For each irreducible ρ' of G choose a linear map $A_{\rho'}\colon \rho'^{\oplus 7} \to \rho'^{\oplus 5}$ that maps $\mathfrak{u}_+(\rho')$ isomorphically onto $\mathfrak{t}(\rho')$ (possible since the dimensions agree; for ρ' with the wrong central character take $A_{\rho'} = 0$). Write $A_{\rho'}$ as a 5×7 block matrix with blocks $(A_{\rho'})_{ts} \in \mathrm{End}(\rho') = \mathrm{Mat}_{d'}(\mathbb{C})$ and put

$$\Psi_{ts} = \sum_{\rho'} \sum_{a,b=1}^{\dim\rho'} \big((A_{\rho'})_{ts}\big)_{ab}\, E^{\rho'}_{ab} \in \mathbb{C}[G], \qquad \Psi = (\Psi_{ts}) \in \mathrm{Mat}_{5\times 7}(\mathbb{C}[G]).$$

By Proposition 3.22, $E^{\rho'}_{ab}$ acts on $\rho''\otimes M_{\rho''}$ as $e_{ab}\otimes\mathrm{id}$ if $\rho'' = \rho'$ and as 0 otherwise, so Ψ^* acts on $H^1(X_+,\mathbb{C}) = \bigoplus \rho'^{\oplus 7}\otimes M_{\rho'}$ as $\bigoplus_{\rho'} A_{\rho'}\otimes \mathrm{id}_{M_{\rho'}}$, and

$$\Psi^*(U_+) = \bigoplus_{\rho'} A_{\rho'}(\mathfrak{u}_+(\rho'))\otimes M_{\rho'} = \bigoplus_{\rho'} \mathfrak{t}(\rho')\otimes M_{\rho'} = \mathcal{T},$$

isomorphically. The multiplicity spaces, which carry the Hodge structures and depend on the curve, are never touched; only the representation factors are rearranged. This is why Ψ can be chosen once for all members of $\mathcal{H}$.

Theorem 11.8 (the Weil line on P). *Assume that $\Lambda_j = \bigwedge^{\mathrm{top}} W_j \in \mathrm{Alg}(P)\otimes\mathbb{C}$ for the eleven pieces (steps (iii)–(v), Proposition 7.10). Then:*
(a) $\Lambda_u \boxtimes \Lambda_- := \mathrm{pr}_{P_0}^*\Lambda_u \cup \mathrm{pr}_{X_-}^*\Lambda_-$ lies in $\mathrm{Alg}(P_0\times X_-)\otimes\mathbb{C}$;
(b) $\Lambda_u = \bigwedge^6(u\otimes M_\rho)$ lies in $\mathrm{Alg}(P)\otimes\mathbb{C}$ for every $u\in\rho\setminus\{0\}$.

Proof of (a). In each plus slot s, $\mathrm{pr}_s^*\Lambda_{j(s)} = \bigwedge^{\mathrm{top}}$ of the copy of $W_{j(s)}$ in that slot, and the cup product of these seven classes is, up to a sign from reordering, the top exterior power of their direct sum: $\Lambda_+ := \bigwedge^D U_+ = \pm\bigcup_s \mathrm{pr}_s^*\Lambda_{j(s)}$, which lies in $\mathrm{Alg}(X_+)\otimes\mathbb{C}$ because pull-backs and cup products preserve it. Choose bases $u_1,\dots,u_D$ of U_+ and $\tau_1,\dots,\tau_D$ of $\mathcal{T}$. Since Ψ^* maps U_+ isomorphically onto $\mathcal{T}$,

$$\textstyle\bigwedge^D\Psi^*(u_1\wedge\cdots\wedge u_D) = \Psi^*u_1\wedge\cdots\wedge\Psi^*u_D = \det\big(\Psi^*|_{U_+}\big)\,\tau_1\wedge\cdots\wedge\tau_D, \qquad \det\big(\Psi^*|_{U_+}\big)\neq 0,$$

the determinant taken in these bases. The space $\mathcal{T}$ is the direct sum of $u\otimes M_\rho \subset H^1(P_0)$ and $U_-\subset H^1(X_-)$, so by Künneth $\tau_1\wedge\cdots\wedge\tau_D = \pm\,\Lambda_u\boxtimes\Lambda_-$ for suitable generators of the two lines. By Proposition 11.5, $\bigwedge^D\Psi^*$ maps Λ_+ into $\mathrm{Alg}(P_0\times X_-)\otimes\mathbb{C}$. Part (b) is proved in Section 11.5. □

Intuition 11.9 (what the relation buys). A relation with coefficients ±1 cannot be realized by maps alone: a correspondence can map pieces *onto* other pieces but cannot subtract them. The minus pieces are therefore moved to the target side, $U_+ \cong (u\otimes M_\rho)\oplus U_-$, and the price is that the transported class carries the passenger Λ_-, which has to be removed by a pairing. Coefficients ±1 mean one slot per piece; a coefficient ±2 would mean two slots, and a relation for ke_ρ would produce k copies of $u\otimes M_\rho$ and hence the product of k Weil lines instead of one (Section 11.7).

Example 11.10 (the exact model for $\mathrm{SL}(2,3)$). The construction can be carried out in exact arithmetic over $\mathbb{Q}(\zeta_{12})$ for $G = G_4 \cong \mathrm{SL}(2,3)$. The model is an étale G-cover C of a curve of genus 2, so $g(C) = 1+24(2-1) = 25$ and, by Chevalley–Weil, $H^1(C) = 2\cdot\mathbf{1}\oplus\mathbb{C}[G]\otimes\mathbb{C}^2$; every non-trivial ρ' has $\dim M_{\rho'} = 2\dim\rho'$, in particular $\dim M_\rho = 4$ for the target $\rho = \chi_5$ of Example 3.19, which is 2-dimensional, with non-real character and not monomial. The relation is the three-term one of that example,

$$e_{\chi_5} = v_{Z,\mathrm{sgn}} - v_{C_4,\psi_4} - v_{C_6,\psi_6},$$

so $X_+ = P$ (one slot), $X_- = P^2$, and $\Psi\in\mathrm{Mat}_{3\times1}(\mathbb{C}[G])$. The dimensions are $D_+ = \dim W_{Z,\mathrm{sgn}} = 2\cdot(4+4+4) = 24$ and $D_- = 12+8 = 20$, with $D_+ - D_- = 4 = \dim M_\rho$.

In the model, Ψ^* maps U_+ onto $\mathcal{T}$, and $U_- \oplus \bar{U}_-$ is non-degenerate (Lemma 11.12, applied slot by slot as explained after it), so the contraction of Proposition 11.11 applies, with $g_- = \dim X_- = 50$. The example shows that the bookkeeping is realisable as described; it says nothing about steps (iii)–(iv).

Exercise 11.4. (a) From the table of Example 11.6, compute $\sum_{\rho'}(n_+(\rho') - n_-(\rho'))\dim M_{\rho'}$ and $\sum_{\rho'}(n_+(\rho') - n_-(\rho'))\dim \rho'$. Show that the second number equals $\sum_j c_j[G : H_j]$, and check it against the column $|H_j|$ of Table 7.1. (b) In Example 11.10, verify $D_+ = 24$ and $D_- = 20$ from the vectors $(0,0,0,2,2,2,0)$, $(0,0,0,1,1,1,0)$, $(0,0,0,1,0,1,0)$ and $\dim M_{\rho'} = 2\dim \rho'$.

11.5. The contraction

It remains to strip Λ_- from $\Lambda_u \boxtimes \Lambda_-$. The tool is elementary.

Proposition 11.11 (contraction). *Let Y, X be smooth projective varieties, $n = \dim X$, $\alpha \in H^a(Y,\mathbb{C})$, $\beta \in H^b(X,\mathbb{C})$ with $\alpha\boxtimes\beta \in \mathrm{Alg}(Y\times X)\otimes\mathbb{C}$, and suppose there is $\mu \in \mathrm{Alg}(X)\otimes\mathbb{C}$ of degree $2n-b$ with $\int_X \beta\cup\mu \neq 0$. Then $\alpha \in \mathrm{Alg}(Y)\otimes\mathbb{C}$.*

Proof. By the projection formula and Künneth, $\mathrm{pr}_{Y*}(\mathrm{pr}_Y^*\alpha \cup \mathrm{pr}_X^*(\beta\cup\mu)) = \left(\int_X \beta\cup\mu\right)\alpha$, since $\beta\cup\mu$ has top degree $2n$ (integration over the fibre of pr_Y). Reordering $\mathrm{pr}_X^*\beta \cup \mathrm{pr}_X^*\mu$ next to $\mathrm{pr}_Y^*\alpha$ costs at most a sign. The left side is obtained from $\alpha\boxtimes\beta$ by cup product with the class $\mathrm{pr}_X^*\mu \in \mathrm{Alg}\otimes\mathbb{C}$ and proper push-forward, both of which preserve $\mathrm{Alg}\otimes\mathbb{C}$. Divide by the nonzero number. □

So everything rests on finding an algebraic μ that pairs non-trivially with Λ_-. On a polarized abelian variety the natural candidate is the complex conjugate class times a power of the polarization; the issue is non-vanishing, and here the non-real central character enters. Let $\Theta \in H^2(P,\mathbb{Q}) = \bigwedge^2 H^1(P,\mathbb{Q})$ be the class of the theta divisor. In a symplectic basis $a_1, b_1, \dots, a_g, b_g$ of $H^1(C,\mathbb{Q})$ for the cup product pairing $\langle x, y\rangle = \int_C x\cup y$ one has $\Theta = \sum_i a_i \wedge b_i$ (up to a sign convention that plays no role here), and this bivector is the same for every symplectic basis. The pairing $\langle\,,\rangle$ is G-invariant, since automorphisms of C preserve the orientation. On $X_- = P^4$ we use the product polarization $\Theta_- = \sum_t \mathrm{pr}_t^*\Theta$ and the orthogonal sum of the four pairings; $g_- = \dim X_- = 4g = 2021764$.

Lemma 11.12 (isotropy). *Let (H,ψ) be a piece, so $\psi|_Z = \kappa_0$, and $W = W_{H,\psi} \subset H^1(C,\mathbb{C})$. Then W is isotropic for $\langle\,,\rangle$, $\bar{W} = W_{H,\bar{\psi}}$, $W \cap \bar{W} = 0$, and $\langle\,,\rangle$ is non-degenerate on $W \oplus \bar{W}$. Consequently $\widetilde{U}_- = U_- \oplus \bar{U}_-$ is a non-degenerate subspace of $H^1(X_-,\mathbb{C})$ of dimension $2D_-$, stable under complex conjugation.*

Proof. Decompose $H^1(C,\mathbb{C})$ into isotypic parts under H. The pairing is H-invariant, so its restriction to the ψ-part times the χ-part (χ an irreducible character of H) is an H-invariant element of the dual of (ψ-part)$\otimes$(χ-part); such an element can be nonzero only if $\psi\otimes\chi$ contains the trivial representation, i.e. only if $\chi = \psi^{-1} = \bar{\psi}$ (Lemma 3.1). Since κ_0 takes a non-real value, so does ψ, hence $\psi \neq \bar{\psi}$: W pairs trivially with itself. Complex conjugation commutes with the action of H (which is defined over $\mathbb{Q}$), so it maps the ψ-eigenspace onto the $\bar{\psi}$-eigenspace: $\bar{W} = W_{H,\bar{\psi}}$, and $W \cap \bar{W} = 0$ because $\psi \neq \bar{\psi}$.

Non-degeneracy on $W \oplus \bar{W}$ follows from the Hodge–Riemann relations. Since H acts by holomorphic automorphisms, $W = (W \cap H^{1,0}) \oplus (W \cap H^{0,1})$, and likewise for $\bar{W}$. So $W \oplus \bar{W}$ is stable under complex conjugation and under the Weil operator $\mathcal{J}$ (multiplication by i on $H^{1,0}$ and by $-$i on $H^{0,1}$, a real operator); hence $W \oplus \bar{W} = R \otimes \mathbb{C}$ for the real subspace $R = (W \oplus \bar{W}) \cap H^1(C,\mathbb{R})$, and $\mathcal{J}R = R$. Every $x \in H^1(C,\mathbb{R})$ is $x = \eta + \bar{\eta}$ with $\eta \in H^{1,0}$, and $\mathcal{J}x = \mathrm{i}\eta - \mathrm{i}\bar{\eta}$; since the product of two $(1,0)$-forms on a curve vanishes, $\langle x, \mathcal{J}x\rangle = -2\mathrm{i}\int_C \eta\wedge\bar{\eta}$, which is negative for $\eta \neq 0$ (locally $\eta = f\,dz$ and $\mathrm{i}\,\eta\wedge\bar{\eta} = 2\,|f|^2\,dx\wedge dy$). So for $0 \neq x \in R$ the vector $\mathcal{J}x \in R$ has $\langle x, \mathcal{J}x\rangle \neq 0$: the pairing is non-degenerate on R, hence on $R\otimes\mathbb{C} = W \oplus \bar{W}$. (Alternatively: $H^1(C,\mathbb{C})$ is the direct sum of its H-isotypic parts, pairwise orthogonal except

for dual pairs, and $\langle\,,\rangle$ is non-degenerate, so the pairing $W \times W_{H,\bar\psi} \to \mathbb{C}$ is perfect.) Across slots the pairings are orthogonal by construction. □

In the SL(2, 3) model of Example 11.10 the central character is real, and it is the non-real characters ψ_4, ψ_6 of the minus pieces that make the lemma apply, slot by slot.

Proof of Theorem 11.8(b). Let $\widetilde U_-^{\perp}$ be the orthogonal complement of $\widetilde U_-$ in $H^1(X_-, \mathbb{C})$, non-degenerate of dimension $2k$, $k = g_- - D_- = 1993396$. Choosing symplectic bases of $\widetilde U_-$ and of $\widetilde U_-^{\perp}$ (complex ones; non-degenerate alternating forms have them over any field), their union is a symplectic basis of $H^1(X_-, \mathbb{C})$, so $\Theta_- = \Theta_{\widetilde U} + \Theta_\perp$ with $\Theta_{\widetilde U} \in \bigwedge^2 \widetilde U_-$ and $\Theta_\perp = \sum_{i=1}^k a_i' \wedge b_i' \in \bigwedge^2 \widetilde U_-^{\perp}$. The class $\Lambda_- \cup \bar\Lambda_-$ is a nonzero element of the top power $\bigwedge^{2D_-} \widetilde U_-$, because $U_- \oplus \bar U_- = \widetilde U_-$ is direct; so its product with $\Theta_{\widetilde U}$ vanishes, and expanding Θ_-^k binomially (the classes are even and commute),

$$\Lambda_- \cup \bar\Lambda_- \cup \Theta_-^k = \Lambda_- \cup \bar\Lambda_- \cup \Theta_\perp^k = k!\, \Lambda_- \cup \bar\Lambda_- \cup a_1' \wedge b_1' \wedge \cdots \wedge a_k' \wedge b_k' \neq 0,$$

a product of top forms of two complementary subspaces, hence a nonzero element of $\bigwedge^{2g_-} H^1(X_-) = H^{2g_-}(X_-, \mathbb{C}) \cong \mathbb{C}$. Put $\mu = \bar\Lambda_- \cup \Theta_-^k$. Since $\Lambda_- = \pm \bigcup_t \mathrm{pr}_t^* \Lambda_{j(t)}$ lies in $\mathrm{Alg}(X_-) \otimes \mathbb{C}$, so does its complex conjugate, and Θ_- is the class of a divisor; hence $\mu \in \mathrm{Alg}(X_-) \otimes \mathbb{C}$ and $\int_{X_-} \Lambda_- \cup \mu \neq 0$. By part (a) and Proposition 11.11 with $Y = P_0$, $X = X_-$, $\Lambda_u \in \mathrm{Alg}(P) \otimes \mathbb{C}$. □

Intuition 11.13 (a partial inner product). In Fock-space language, $\Lambda_u \boxtimes \Lambda_-$ is a product state of 6 modes in the factor P_0 and D_- modes in X_-, and the contraction is the partial inner product with the state μ on X_-, leaving a multiple of the P_0 state. Two things are needed: that μ is algebraic, which holds because it is built from Λ_- itself by conjugation and multiplication with the polarization, and that the overlap is nonzero, which holds because Λ_- is "particle-like" (isotropic) and $\bar\Lambda_-$ is its "antiparticle", the polarization filling the remaining modes. If κ_0 were real, U_- could meet $\bar U_-$ and the overlap could vanish.

Exercise 11.5. Let a finite abelian group Z act on a complex vector space with a Z-invariant non-degenerate bilinear form $\langle\,,\rangle$, and let V_κ be the κ-eigenspace for a character κ. (a) Show that $\langle V_\kappa, V_{\kappa'}\rangle = 0$ unless $\kappa\kappa' = 1$, and that V_κ is isotropic if κ is not real-valued. (b) For $Z = \mu_3$ acting on $H^1(E, \mathbb{C})$ of $E = \mathbb{C}/\mathbb{Z}[\omega]$, check (a) on dz and $d\bar z$. (c) Give an example with κ real ($\kappa = \mathrm{sgn}$ of μ_2) where V_κ is not isotropic.

Exercise 11.6. On $X = E^2$ with $\Theta = \Theta_1 + \Theta_2$, $\Theta_i = \frac{\mathrm{i}}{2\,\mathrm{Im}\,\tau} dz_i \wedge d\bar z_i$ for $E = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ (normalized so that $\int_E \Theta_i = 1$), let $U = \mathbb{C}\,dz_1$, $\Lambda = dz_1$, $D = 1$. Compute $\int_X \Lambda \cup \bar\Lambda \cup \Theta^{g-D}$ with $g = 2$ and show it is nonzero. What goes wrong if one takes instead $U = \mathbb{C}(dz_1 + d\bar z_1)$, a subspace that meets its conjugate?

11.6. From single lines to rational Weil classes

Theorem 11.8 produces one complex line Λ_u for each $u \in \rho \setminus \{0\}$; the theorem to be proved is about the rational Weil classes of a sixfold. Two pieces of linear algebra close the gap.

Lemma 11.14. *Let X be smooth projective and $S \subseteq H^{2p}(X, \mathbb{Q})$ a $\mathbb{Q}$-subspace with $S \otimes \mathbb{C} \subseteq \mathrm{Alg}(X) \otimes \mathbb{C}$. Then $S \subseteq \mathrm{Alg}(X)$. That is, $(\mathrm{Alg}(X) \otimes \mathbb{C}) \cap H^{2p}(X, \mathbb{Q}) = \mathrm{Alg}^p(X)$.*

The proof is Exercise 11.7.

Proposition 11.15 (rational Weil classes). *Assume Theorem 11.8(b). Let $u \in V_\rho$ be a nonzero rational vector, and write $u = u' + u''$ in $V_\rho \otimes \mathbb{C} = \rho \oplus \bar\rho$ (Proposition 3.27), so $u'' = \bar u'$. Evaluation $\mathrm{ev}_u \colon \mathcal{B}_C \to H^1(C, \mathbb{Q})$, $f \mapsto f(u)$, is injective, and its image is $p^* H^1(B_u, \mathbb{Q})$ for an abelian variety B_u and a surjective homomorphism $p \colon P \to B_u$; give B_u the K-action transported from $\mathcal{B}_C$. Then $\mathrm{HW}(B_u)$ is algebraic, and so is $\mathrm{HW}(A_C)$.*

Proof. Injectivity is Proposition 3.27(e). The image $W = \mathrm{ev}_u(\mathcal{B}_C)$ is a sub-Hodge structure of $H^1(C,\mathbb{Q}) = H^1(P,\mathbb{Q})$, because $\mathcal{B}_C$ carries the Hodge structure induced from $H^1(C)$ and ev_u is a morphism of Hodge structures. We realize it by a quotient of P, using that abelian varieties are the polarizable complex tori, with homomorphisms the $\mathbb{C}$-linear maps that preserve the lattices [Mil08, I, Thms. 2.8–2.9]. Write $P = T/L$, with T the tangent space, so that $H^1(P,\mathbb{Z}) = \mathrm{Hom}(L,\mathbb{Z})$ [Mil08, I, Thm. 2.3], and let E be a Riemann form on L. Let $N_{\mathbb{Q}} \subseteq L\otimes\mathbb{Q}$ be the annihilator of W. Since $W\otimes\mathbb{R}$ is stable under the complex structure, so is $N_{\mathbb{R}} = N_{\mathbb{Q}}\otimes\mathbb{R} \subseteq T$; the Hermitian form of E is positive definite on $N_{\mathbb{R}}$, so E is non-degenerate there and $T = N_{\mathbb{R}} \oplus N^{\perp}$, where the E-orthogonal complement $N^{\perp}$ is a complex subspace defined over $\mathbb{Q}$. The projection $\pi\colon T \to N^{\perp}$ along $N_{\mathbb{R}}$ is $\mathbb{C}$-linear and defined over $\mathbb{Q}$, so $M\pi(L) \subseteq L\cap N^{\perp}$ for some integer $M \geq 1$. The torus $B_u = N^{\perp}/(L\cap N^{\perp})$ is polarized by the restriction of E, hence an abelian subvariety of P, with inclusion $s\colon B_u \to P$; and $M\pi$ induces a homomorphism $p\colon P \to B_u$ with $p\circ s = [M]$. On $L\otimes\mathbb{Q}$ the map $M\pi$ is onto $N^{\perp}\cap(L\otimes\mathbb{Q})$ with kernel $N_{\mathbb{Q}}$, so p^* identifies $H^1(B_u,\mathbb{Q})$ with the linear forms vanishing on $N_{\mathbb{Q}}$, that is, with W. Complexify: $\mathcal{B}_C\otimes\mathbb{C} = \mathrm{Hom}_G(\rho, H^1(C,\mathbb{C})) \oplus \mathrm{Hom}_G(\bar\rho, H^1(C,\mathbb{C})) = M_\rho \oplus M_{\bar\rho}$, and $f(u) = f|_\rho(u') + f|_{\bar\rho}(u'')$, so

$$p^*H^1(B_u,\mathbb{C}) = u'\otimes M_\rho \;\oplus\; u''\otimes M_{\bar\rho},$$

and $a\in K$ acts on the two summands by $\iota(a)$ and $\bar\iota(a)$. The Weil classes are $\mathrm{HW}(B_u) = \bigwedge^6_K H^1(B_u,\mathbb{Q})$, whose complexification is $\bigwedge^6$ of the ι-eigenspace plus $\bigwedge^6$ of the $\bar\iota$-eigenspace (Definition 1.1; [Del82, proof of Prop. 4.4]). The first is $\Lambda_{u'}$; the second is $\bigwedge^6(u''\otimes M_{\bar\rho}) = \overline{\Lambda_{u'}}$ (Exercise 11.8). Both lie in $\mathrm{Alg}(P)\otimes\mathbb{C}$, by Theorem 11.8(b) for $u'\neq 0$ (note $u'\neq 0$, since $u' = 0$ would force $u = 0$) and by conjugation. So $p^*\mathrm{HW}(B_u)\otimes\mathbb{C} \subseteq \mathrm{Alg}(P)\otimes\mathbb{C}$, and Lemma 11.14 gives $p^*\mathrm{HW}(B_u) \subseteq \mathrm{Alg}(P)$. Then $s^*p^* = [M]^*$ acts on $H^6(B_u)$ by M^6, and $\mathrm{HW}(B_u) = M^{-6}s^*p^*\mathrm{HW}(B_u)$ consists of algebraic classes since s^* preserves them. Finally, $(p^*)^{-1}\circ\mathrm{ev}_u$ is a K-linear isomorphism of rational Hodge structures from $\mathcal{B}_C$ onto $H^1(B_u,\mathbb{Q})$; an integer multiple of it maps a lattice of A_C into $H^1(B_u,\mathbb{Z})$, so it comes from an isogeny between B_u and A_C that commutes with K [Mil08, I, Thm. 2.9], and Proposition 12.11 transfers algebraicity. □

What just happened. The relation of Proposition 7.8 says that seven pieces and four pieces plus $u\otimes M_\rho$ have the same multiplicity vectors. Wedderburn's matrix units turn the matching into one matrix $\Psi \in \mathrm{Mat}_{5\times 7}(\mathbb{C}[G])$, which maps U_+ onto $\mathcal{T} = (u\otimes M_\rho)\oplus U_-$. Its exterior power $\bigwedge^D\Psi^*$ is a polynomial in Ψ, equal to n^{-D} times a pull-back by a homomorphism for rational Ψ, so by density it preserves $\mathrm{Alg}\otimes\mathbb{C}$ and carries the algebraic Λ_+ to $\Lambda_u \boxtimes \Lambda_-$. Because κ_0 is not real, U_- is isotropic and $\mu = \bar\Lambda_- \cup \Theta^{g-D_-}$ pairs non-trivially with Λ_-; the contraction leaves Λ_u. A rational u gives a quotient B_u of P whose complexified Weil classes are $\Lambda_{u'}$ and $\overline{\Lambda_{u'}}$, and a rational space spanned over $\mathbb{C}$ by algebraic classes is algebraic.

Exercise 11.7. (a) Prove Lemma 11.14: choose a $\mathbb{Q}$-basis of $H^{2p}(X,\mathbb{Q})$ that extends a basis of $\mathrm{Alg}^p(X)$ and compare coordinates. (b) Why does the lemma say nothing about the line $\Lambda_{u'}$ by itself?

Exercise 11.8. With the notation of Proposition 11.15, show that complex conjugation on $H^1(C,\mathbb{C})$ maps $u'\otimes M_\rho$ onto $u''\otimes M_{\bar\rho}$, and deduce $\bigwedge^6(u''\otimes M_{\bar\rho}) = \overline{\Lambda_{u'}}$. (Use that f is rational for $f\in\mathcal{B}_C$, and that $\mathcal{B}_C\otimes\mathbb{C}\to M_\rho$, $f\mapsto f|_\rho$, is surjective.)

11.7. Scope: single lines and the discriminant

What steps (iii)–(iv) deliver are single lines. The cyclic-cover theorem (Theorem 8.1) gives each line Λ_{ψ^t}, $t\in(\mathbb{Z}/m)^\times$, directly in $\mathrm{Alg}\otimes\mathbb{C}$; Schoen's theorem and Patel–Zhang's give a rational Weil structure of the cyclic piece, a $\mathbb{Q}(\zeta_m)$-determinant of the generalized Prym, whose complexification is the direct *sum* of the lines over the Galois conjugates of ψ, so that each

summand lies in $\mathrm{Alg}\otimes\mathbb{C}$ (Chapter 10). Step (vi) uses only these single lines. One could instead work K-rationally throughout, which would keep track of discriminants: the K-determinant of a piece complexifies to the *tensor product* of the lines over the $\mathrm{Gal}(\mathbb{Q}(\zeta_m)/K)$-orbit of ψ, a line whose multiplicity vector is $|\mathrm{orbit}|\cdot v_{H,\psi}$. The orbit sizes are $1,2,3,4,4$ for $m=6,12,18,24,30$; the conjugates ψ^t ($t\equiv 1 \bmod 6$) have the same vector because all characters of G_{32} with central character κ_0 take values in K. In the language of Definition 3.17, the lattice of single-line vectors (143 distinct vectors among the 334 pieces) has $k_{\min}=1$, while the lattice of K-rational vectors has $k_{\min}=4$ (by a Smith normal form computation). A K-rational version of step (vi) would therefore reach at best $\bigwedge^6$ of four copies of $u\otimes M_\rho$, the product of four Weil lines, not one.

Remark 11.16 (the discriminant). How can a class-2 Weil structure come out of cyclic pieces whose own Weil structures need not have discriminant 2? The discriminant is an invariant of a K-rational Hermitian structure; it constrains combinations of K-rational determinants, which are tensor products over Galois orbits as above. We know of no discriminant-type invariant of a single complex line in $\mathrm{Alg}\otimes\mathbb{C}$ that is preserved by correspondences with complex coefficients; the toy Example 11.4 shows such a correspondence isolating a line that is not defined over $\mathbb{Q}$.

The same bookkeeping explains why the value of $k_{\min}$ matters: a relation for $2e_\rho$, which is all that the étale pieces and the ramified pieces satisfying the prime condition of Section 7.5 give, makes step (vi) produce $\Lambda_u\boxtimes\Lambda_{u'}$ on $P_0\times P_0'$ instead of Λ_u.

Remark on the proof 11.1: $k=2$. We do not know whether $\Lambda_u\in\mathrm{Alg}\otimes\mathbb{C}$ can be extracted from $\Lambda_u\boxtimes\Lambda_{u'}\in\mathrm{Alg}(P\times P)\otimes\mathbb{C}$. The contraction of Proposition 11.11 would need an algebraic μ pairing non-trivially with $\Lambda_{u'}$, that is, a class of the very kind being sought.

Remark 11.17 (Where this is used). Proposition 11.15 is the end of the pointwise argument: for every member C of $\mathcal{H}$ the Weil classes of A_C are algebraic. Step (vii) (Chapter 12) spreads this from the image of the period map, which is open by the dominance of Chapter 6, to the whole connected Shimura variety, and step (viii) reaches every non-split sixfold by Landherr and Proposition 12.11. The review Chapter 13 lists what step (vi) depends on, including its reliance on single lines and on a relation with $k=1$.

Part 4

Every sixfold, and what the proof rests on

CHAPTER 12

From one family to every sixfold: globalization and isogenies

Proof steps: (vii), (viii)

Where we are. This chapter explains the last two steps of the proof, step (vii) and step (viii). They take two inputs from earlier chapters. From step (vi) (Chapter 11): for every member C of the Hurwitz component $\mathcal{H}$ (Example 5.19) the Weil classes of the sixfold A_C are algebraic. From step (i)(b) (Theorem 6.6): the period map of $\mathcal{H}$ has rank 9, the dimension of the domain X^+ of Definition 2.14. The output is the Main Theorem: every (A, ι, λ) satisfying its hypotheses (a)–(c), among them $\det h_\lambda \equiv -2$, has algebraic Weil classes. Neither step involves G_{32}, covers or cycles; they are soft arguments with Shimura varieties, countable unions and isogenies, and the standard facts they rest on are collected in Remark on the proof 12.1. These two steps contain no computation. Needs: Definition 2.14 and Propositions 2.6 and 2.15 and Section 2.7; arithmetic quotients of bounded symmetric domains, the theorems of Baily–Borel and Borel [Mil05, §3]; abelian varieties as polarizable complex tori [Mil08, I, §2].

12.1. Globalization and isogenies in two pages

After step (vi) the Weil classes are known to be algebraic on the sixfolds A_C of one family, $C \in \mathcal{H}$. The theorem is about every Weil sixfold of class 2, and these form, up to isogeny, the nine-dimensional quotient $S' = \Gamma' \backslash X^+$ of the domain of Proposition 2.15. Two steps close the gap.

Step (vii): from the family to every point of S'. Three facts combine. First, S' is an algebraic variety and the period map $\mathcal{P}$ from (a finite cover of) $\mathcal{H}$ to S' is an algebraic morphism (Borel). Since $\mathcal{P}$ has rank $9 = \dim S'$ (Chapter 6), its image contains a dense Zariski-open subset $U' \subset S'$. Second, S' carries a family of abelian sixfolds $\mathcal{A} \to S'$, and their Weil classes form a local system of rank 2 whose monodromy is finite: an integral unitary matrix acts on $\bigwedge^6_K$ by its determinant, which is a unit of norm 1 in $\mathbb{Z}[\omega]$, hence one of the six roots of unity. On a finite cover the Weil classes are therefore single-valued flat sections $\mathfrak{w}$. Third, for a flat section $\mathfrak{w}$ the set of points where $\mathfrak{w}$ is algebraic is a countable union of closed algebraic subsets, because algebraic cycles come in algebraic families (Hilbert schemes), of which there are countably many (Proposition 12.8). This set contains U'. A countable union of proper closed subvarieties cannot contain an open set (Baire's theorem), so one of the closed subsets is everything. The picture to keep in mind is in Example 12.6: the lines $w = qz$ in $\mathbb{C}^2$ with $q \in \mathbb{Q}(\mathrm{i})$ are dense but contain no ball. Density alone would prove nothing, and this is why the rank 9 of Chapter 6 matters: a family of smaller rank would give algebraicity only on the Zariski closure of its image, a proper subvariety.

Step (viii): from S' to every (A, ι, λ). Let (A, ι, λ) satisfy the hypotheses (a)–(c) of the Main Theorem. Its Hermitian form h_λ has dimension 6, signature $(3, 3)$ and determinant $\equiv -2$, like the form h of the family, so Landherr's theorem gives a K-linear isometry $f\colon H^1(A, \mathbb{Q}) \to V$ (Section 2.7). Transporting the Hodge structure of A by f gives a point x_A of X^+, because positivity of the Riemann form is transported too. The fibre of $\mathcal{A}$ over the image of x_A has the same rational Hodge structure with K-action as A, so it is K-isogenous to A, and algebraicity of Weil classes does not see isogenies (Proposition 12.11).

ingredient	what it gives	where
Borel, Baily–Borel	S' and $\mathcal{P}$ algebraic, a universal family	Proposition 12.2
rank 9 + Chevalley	image of $\mathcal{P}$ contains a dense open U'	Theorem 6.6
$\det \in \mu_6$	Weil classes trivial on a finite cover S''	Lemma 12.3
Hilbert schemes + Baire	algebraic on $U' \Rightarrow$ algebraic on S'	Propositions 12.8 and 12.9
Landherr + positivity	every (A, ι, λ) gives $x_A \in X^+$	Section 2.7
Riemann + isogeny invariance	$A \sim_K \mathcal{A}_{[x_A]}$, algebraicity transfers	Proposition 12.11 and Theorem 12.13

12.2. The arithmetic group, the period map and the universal family

Fix a base point $t_0 \in \mathcal{H}$ and write $V = \mathcal{B}_{C_{t_0}}$, a six-dimensional K-vector space with the Hermitian form h of Proposition 5.18, normalized by $E = \mathrm{Tr}_{K/\mathbb{Q}}(\theta h)$, where E is the polarization form with its own sign (so that the family's complex structures satisfy $E(x, Jx) > 0$ and lie in X^+ rather than in its mirror image X^-). The integral structure is $V_\mathbb{Z}$, the image of $\mathrm{Hom}_{\mathbb{Z}[G]}(L_\rho, H^1(C_{t_0}, \mathbb{Z}))$ for a G-stable lattice $L_\rho \subset V_\rho$; after rescaling E by a positive rational number, which changes neither X^+ nor the groups below, E is integral on $V_\mathbb{Z}$, of some type $\delta = (d_1, \dots, d_6)$.

Definition 12.1 (the arithmetic groups). $\Gamma = \mathrm{U}(V_\mathbb{Z}, h)$ is the group of K-linear $\gamma\colon V \to V$ with $\gamma V_\mathbb{Z} = V_\mathbb{Z}$ and $h(\gamma x, \gamma y) = h(x, y)$. For $N \geq 1$, $\Gamma_{\mathrm{Sp}}(N) = \{\gamma \in \mathrm{Sp}(V_\mathbb{Z}, E) : (\gamma - 1)V_\mathbb{Z} \subseteq NV_\mathbb{Z}\}$, and

$$\Gamma' = \mathrm{U}(V_\mathbb{Z}, h) \cap \Gamma_{\mathrm{Sp}}(N), \qquad N \geq 3 \text{ fixed}, \qquad S' = \Gamma' \backslash X^+.$$

Since $E = \mathrm{Tr}(\theta h)$, every $\gamma \in \Gamma$ preserves E, so $\Gamma \subseteq \mathrm{Sp}(V_\mathbb{Z}, E)$, and Γ' is the kernel of the reduction map from Γ to the finite group $\mathrm{Aut}(V_\mathbb{Z}/NV_\mathbb{Z})$: it has finite index in Γ. The point of $N \geq 3$ is that $\Gamma_{\mathrm{Sp}}(N)$, hence Γ', has no elements of finite order other than 1 (Exercise 12.1). For $N = 2$ this fails already for -1, which lies in $\Gamma_{\mathrm{Sp}}(2)$ and has order 2. Torsion-freeness is what makes the quotient S' a manifold and what Borel's theorem needs.

Exercise 12.1 (no torsion at level $N \geq 3$). Let $\gamma \in \mathrm{GL}_n(\mathbb{Z})$ with $\gamma \equiv 1 \pmod N$, $N \geq 3$, and $\gamma^k = 1$ for some $k \geq 1$. Show that $\gamma = 1$. (Reduce to $\gamma^p = 1$ with p prime; write $\gamma = 1 + q^r X$ with q a prime dividing N, $r \geq 1$, and $r \geq 2$ if $q = 2$, chosen so that $q \nmid X$; expand $(1 + q^r X)^p$ modulo q^{r+1} or q^{r+2}.) Where does the argument break for $N = 2$?

The monodromy of the family over $\mathcal{H}$ preserves $H^1(C_t, \mathbb{Z})$ and the cup product and commutes with G, so it acts on $\mathrm{Hom}_{\mathbb{Z}[G]}(L_\rho, H^1(C_t, \mathbb{Z}))$ preserving E and the K-action, i.e. through Γ, up to one ambiguity. A point of $\mathcal{H}$ determines its G-cover only up to isomorphism of G-covers, and the automorphisms of a connected G-cover $C \to \mathbb{P}^1$ that commute with G are the elements of the centre $Z(G)$; so transport along a path in $\mathcal{H}$ is defined only up to the action of $Z(G)$. An element $z \in Z(G)$ acts on $\mathrm{Hom}_G(V_\rho, H^1(C_t, \mathbb{Q}))$ by $f \mapsto f \circ \rho(z)^{\pm 1}$, that is, by the scalar $\kappa_0(z)^{\pm 1} \in \mu_6 \subset K$, with κ_0 the central character of ρ (Chapter 7). These scalars lie in Γ, act trivially on X^+ (every $J \in X^+$ commutes with K), and act on the Weil classes $\bigwedge^6_K V$ by $\kappa_0(z)^{\pm 6} = 1$. So the monodromy is a well-defined homomorphism $\pi_1(\mathcal{H}, t_0) \to \Gamma/\mu$, with $\mu = \kappa_0(Z(G)) \subseteq \mu_6$; the period point of C_t in $\Gamma\backslash X^+$ is well defined; and the Weil classes form an honest local system over $\mathcal{H}$.

Moreover $\Gamma' \cap \mu = 1$: if $\zeta^k \neq 1$ and $(\zeta^k - 1)V_\mathbb{Z} \subseteq NV_\mathbb{Z}$, then $(\zeta^k - 1)/N$ preserves the lattice $V_\mathbb{Z}$, hence is an algebraic integer, but its norm is $\mathrm{Nm}(\zeta^k - 1)/N^2 \leq 4/9$.

The subgroup of $\pi_1(\mathcal{H}, t_0)$ mapping into the image of Γ' in Γ/μ therefore has finite index, and on the corresponding connected finite étale cover $\mathcal{H}' \to \mathcal{H}$ the monodromy lifts uniquely

to Γ'. The period map [Sch73, (3.23)–(3.26)] is then a map $\mathcal{P}\colon \mathcal{H}' \to S'$. The spaces of this chapter and the maps between them are

$$\begin{array}{ccccc} \mathcal{H}' & \xrightarrow{\ \mathcal{P}\ } & S' = \Gamma'\backslash X^+ & \longrightarrow & \mathcal{A}_{6,\delta,N} \\ {\scriptstyle\text{finite étale}}\downarrow & & \uparrow{\scriptstyle\pi} & & \\ \mathcal{H} & & S_0'' & & \end{array}$$

The universal family $\mathcal{A} \to S'$ of Proposition 12.2(c) is the pull-back of the universal abelian scheme along $S' \to \mathcal{A}_{6,\delta,N}$, the cover $\pi\colon S_0'' \to S'$ is the finite étale cover of Lemma 12.3 on which the Weil classes become single-valued, and $\mathcal{A}'' = \mathcal{A} \times_{S'} S_0''$. The left column has dimension 13 (Definition 5.6), everything else dimension 9.

PROPOSITION 12.2 (Borel and the universal family). *In the situation above:*

(a) S' is a smooth connected quasi-projective complex variety of dimension 9, and $\mathcal{P}\colon \mathcal{H}' \to S'$ is a morphism of algebraic varieties.

(b) The image of $\mathcal{P}$ contains a dense Zariski-open subset U' of S'.

(c) There is a projective abelian scheme $f\colon \mathcal{A} \to S'$ whose fibre over the image $[x]$ of $x \in X^+$ has $H^1(\mathcal{A}_{[x]}, \mathbb{Z}) = V_{\mathbb{Z}}$ with the Hodge structure x; the constant action of K on V is an action on each fibre by elements of $\mathrm{End}(\mathcal{A}_{[x]}) \otimes \mathbb{Q}$. For $t \in \mathcal{H}'$ the fibre over $\mathcal{P}(t)$ is K-isogenous to A_{C_t}.

THE ARGUMENT. (a) X^+ is a bounded symmetric domain of dimension 9 (Proposition 2.15). The group Γ' has finite index in $\mathrm{U}(V,h)(\mathbb{Q}) \cap \mathrm{GL}(V_{\mathbb{Z}})$, so it is an arithmetic subgroup of $\mathrm{U}(V,h)(\mathbb{Q})$, and it is torsion-free; its image in the automorphism group of X^+ is torsion-free too (if γ^k acts trivially, it is a scalar in Γ', hence a root of unity, hence 1). The theorems of Baily–Borel and Borel need this image to be an arithmetic subgroup of $\mathrm{Hol}(X^+)^+$ in the sense of [Mil05, §3, definition before Prop. 3.6]: there are an algebraic group G over $\mathbb{Q}$, a surjective homomorphism $G(\mathbb{R})^+ \to \mathrm{Hol}(X^+)^+$ with compact kernel, and an arithmetic subgroup Γ_0 of $G(\mathbb{Q})$ such that $\Gamma_0 \cap G(\mathbb{R})^+$ maps onto the group in question. Take $G = \mathrm{U}(V,h)$ and $\Gamma_0 = \Gamma'$; here $G(\mathbb{R})^+ = G(\mathbb{R})$, because $G(\mathbb{R}) \cong \mathrm{U}(3,3)$ is connected. The group $G(\mathbb{R})$ acts on X^+ by $J \mapsto gJg^{-1}$ (Proposition 2.15(d)); the subgroup $\mathrm{SU}(3,3)$ maps onto $\mathrm{Hol}(X^+)^+$ with compact kernel, because $(\mathrm{SU}(V,h), X^+)$ is a connected Shimura datum [Mil05, App. B, case (A), with $F_0 = \mathbb{Q}$, $r = s = 3$] and [Mil05, Prop. 4.8] applies; and $\mathrm{U}(3,3)$ is the product of $\mathrm{SU}(3,3)$ with the central circle, which acts trivially. So S' is a smooth quasi-projective variety by Baily–Borel [Mil05, Prop. 3.1, Thm. 3.12]; it is connected because X^+ is. The period map is holomorphic [Sch73, (3.27)]; $\mathcal{H}'$ is smooth and quasi-projective, being finite étale over $\mathcal{H}$, which is finite étale over the smooth affine variety $\mathcal{U}_{13}$ of 13-point subsets of $\mathbb{C}$ (Definition 5.6; $\mathcal{U}_{13}$ is the complement of the discriminant hypersurface in the space $\mathbb{C}^{13}$ of monic polynomials of degree 13, and a finite étale cover of a quasi-projective variety, even one given only as a covering space, is again a quasi-projective variety, by the Riemann existence theorem of Grauert and Remmert). Borel's extension theorem [Mil05, Thm. 3.14], [Bor72] makes every holomorphic map from a smooth quasi-projective variety to S' algebraic.

(b) By Theorem 6.6 the map $\mathcal{H} \to \Gamma\backslash X^+$ has generic rank 9, and so does $\mathcal{P}$, which differs from it by finite étale maps. By Chevalley's theorem the image of a morphism is constructible, a finite union of locally closed sets [Har77, Ex. II.3.18–3.19]. A constructible set of dimension 9 in the irreducible 9-dimensional S' contains a dense open subset of S' (Exercise 12.2). This is where the computer-assisted dominance theorem enters.

(c) X^+ is a complex submanifold of the Siegel space of $(V_{\mathbb{R}}, E)$, the complex structures preserving E with $E(\cdot, J\cdot) > 0$, and $\Gamma' \subseteq \Gamma_{\mathrm{Sp}}(N)$, so $x \mapsto x$ induces a holomorphic map $S' \to \Gamma_{\mathrm{Sp}}(N)\backslash\{\text{Siegel space}\} = \mathcal{A}_{6,\delta,N}(\mathbb{C})$, the moduli space of abelian sixfolds with a polarization of type δ and a level-N structure. It is algebraic by Borel's theorem, since $\Gamma_{\mathrm{Sp}}(N)$ is torsion-free. For $N \geq 3$, $\mathcal{A}_{6,\delta,N}$ is a fine moduli space carrying a projective universal abelian scheme [MFK94, Ch. 7], and $\mathcal{A}$ is its pull-back. An element a of the order

$\{a \in K : aV_{\mathbb{Z}} \subseteq V_{\mathbb{Z}}\}$ preserves the lattice and commutes with every $x \in X^+$, so it is an endomorphism of each fibre; only this fibrewise action is used below. Finally, transport along a path in $\mathcal{H}'$ from t_0 to t identifies $\mathrm{Hom}_{\mathbb{Z}[G]}(L_\rho, H^1(C_t,\mathbb{Z}))$ with $V_{\mathbb{Z}}$, K-linearly and up to Γ', carrying the Hodge structure of $\mathcal{B}_{C_t}$ to $\mathcal{P}(t)$; so the fibre over $\mathcal{P}(t)$ is the abelian variety with integral Hodge structure $\mathrm{Hom}_{\mathbb{Z}[G]}(L_\rho, H^1(C_t,\mathbb{Z}))$ and K-action, a model of A_{C_t}, which is only defined up to K-isogeny (for two lattices in the same rational Hodge structure the isogeny is constructed as in the proof of Theorem 12.13). □

The numbers behind (b): $\mathcal{H}'$ has dimension 13, the dimension of $\mathcal{U}_{13}$. A Möbius transformation $\gamma \in \mathrm{PGL}_2(\mathbb{C})$ near the identity moves a configuration of branch points to a nearby one without changing the cover up to isomorphism (Definition 5.6 and the paragraph after it), so the period map is constant along the orbits of this local action, which have dimension $3 = \dim \mathrm{PGL}_2(\mathbb{C})$, and the curves of $\mathcal{H}$ depend on $13 - 3 = 10$ moduli. The domain has dimension $\dim_{\mathbb{R}} \mathrm{U}(3,3)/2 - \dim_{\mathbb{R}}(\mathrm{U}(3)\times\mathrm{U}(3))/2 = (36-18)/2 = 9$. So the generic fibre of $\mathcal{P}$ has dimension $13 - 9 = 4$, made of the three Möbius directions and one more: modulo $\mathrm{PGL}_2(\mathbb{C})$, the generic fibre has dimension one. In (c), $\Gamma'\backslash X^+$ for an arbitrary torsion-free $\Gamma' \subset \mathrm{U}(V_{\mathbb{Z}}, h)$ is not literally a moduli space; the route through $\Gamma_{\mathrm{Sp}}(N)$ and $\mathcal{A}_{6,\delta,N}$ avoids this.

Exercise 12.2 (images are constructible, not open). (a) Show that the image of $\mathbb{C}^2 \to \mathbb{C}^2$, $(x,y) \mapsto (x, xy)$, is $\{x \neq 0\} \cup \{(0,0)\}$, which is neither open nor closed, but contains the dense open set $\{x \neq 0\}$. (b) Let $Z \subseteq S$ be constructible in an irreducible variety S with $\dim \overline{Z} = \dim S$. Show that Z contains a dense open subset of S. (c) Show, with the holomorphic inverse function theorem, that a holomorphic map of rank 9 at one point into a 9-dimensional manifold has an image with non-empty interior; compare with Remark 12.10.

12.3. The Weil classes as a local system with finite monodromy

For $s \in S'$ put $\mathrm{HW}_s = \bigwedge^6_K H^1(\mathcal{A}_s,\mathbb{Q}) \subset H^6(\mathcal{A}_s,\mathbb{Q})$, the two-dimensional space of Weil classes (Definition 1.1), a $\mathbb{Q}$-subspace of dimension 2 in a space of dimension $\binom{12}{6} = 924$. On X^+ the lattice $V_{\mathbb{Z}}$ and the action of K are constant, so $\bigwedge^6_K V$ is a constant subspace of $\bigwedge^6_{\mathbb{Q}} V$; on S' the Weil classes form a rank-2 local system $\mathbb{HW} \subset R^6 f_* \mathbb{Q}$, whose monodromy is the action of Γ' on $\bigwedge^6_K V$.

Lemma 12.3 (the monodromy of the Weil classes is finite). *Every $\gamma \in \mathrm{U}(V_{\mathbb{Z}}, h)$ acts on the one-dimensional K-space $\bigwedge^6_K V$ by multiplication with $\det_K \gamma$, and $\det_K \gamma \in \mu_6$, the group of sixth roots of unity. Hence the monodromy of $\mathbb{HW}$ factors through a cyclic group of order dividing 6; on the finite étale cover $S'' \to S'$ defined by its kernel, $\mathbb{HW}$ is trivial, and every connected component S''_0 of S'' maps onto S'.*

Proof. On a top exterior power over K a K-linear map acts by its determinant. The eigenvalues of γ on $V \otimes \mathbb{C}$ are algebraic integers, because γ preserves the lattice $V_{\mathbb{Z}}$ and so has an integral characteristic polynomial; $\det_K \gamma$ is the product of the six eigenvalues on the summand $V \otimes_{K,\iota} \mathbb{C}$ of $V \otimes_{\mathbb{Q}} \mathbb{C} = V \otimes_{K,\iota} \mathbb{C} \oplus V \otimes_{K,\bar\iota} \mathbb{C}$, an algebraic integer lying in K, hence in $\mathbb{Z}[\omega]$. Taking the determinant of $h(\gamma x, \gamma y) = h(x,y)$ in a basis gives $\det_K \gamma \cdot \overline{\det_K \gamma} = 1$. The elements of $\mathbb{Z}[\omega]$ of norm 1 are its units, and these are exactly the six powers of $\zeta = e^{2\pi \mathrm{i}/6}$. The monodromy representation of $\mathbb{HW}$ is therefore $\pi_1(S') \to \Gamma' \to \mu_6$, its kernel has finite index, and the corresponding cover S'' is finite étale, with $\mathbb{HW}$ trivial on it. A finite étale map is open and closed, so the image of a connected component is open, closed and non-empty in the connected S', hence all of S'. □

Example 12.4 (a monodromy of order six). For the diagonal model $h_2 = \langle 2,1,1,-1,-1,-1\rangle$ of Example 2.8 on $\mathbb{Z}[\omega]^6$, the matrix $g = \mathrm{diag}(\zeta,1,1,1,1,1)$ is unitary, since $\zeta\bar\zeta = 1$, and preserves the lattice; it acts on $\bigwedge^6_K$ by $\det g = \zeta$, of order 6. The scalar $\zeta \cdot 1_6$, on the other hand, acts by $\zeta^6 = 1$: it is a monodromy that moves every class in

H^1 and fixes the Weil classes. Since $\zeta - 1 = \omega$ is a unit, neither matrix is congruent to 1 modulo any integer $N \geq 2$, so neither lies in Γ'.

Physics aside 12.5 (a discrete holonomy). A flat section of $\mathbb{HW}$ transported around a loop in S' comes back multiplied by $\det_K \gamma \in \mu_6$, a phase in a finite group, much as the wave function of a charge transported around a flux tube of a discrete $\mathbb{Z}_6$ gauge field. Passing to the cover S'' is the analogue of working on the covering space on which the holonomy is unwound. The analogy stops at the source of the phase: here it is not a field strength but the arithmetic of $\mathbb{Z}[\omega]$, whose only units are the six roots of unity, and over $\mathbb{Q}$ instead of $\mathbb{Z}[\omega]$ the "holonomy" could be any element of norm 1, of infinite order (Exercise 12.3(c)).

Exercise 12.3 (determinants of integral unitary matrices). (a) Show that $\det_K \gamma \in \mu_6$ for $\gamma \in \mathrm{U}(\mathbb{Z}[\omega]^6, h_2)$ directly, and give an element with $\det_K \gamma = -1$. (b) Show that the scalar $u \cdot 1_6$, $u \in \mu_6$, acts trivially on $\bigwedge^6_K$ but by u^3 on $\bigwedge^3_K$. (c) Why does the argument need the lattice? Give a unitary $\gamma \in \mathrm{U}(V, h_2)(\mathbb{Q})$ with $\det_K \gamma$ of infinite order, using the element $(2+\theta)/(2-\theta)$ of norm 1.

Exercise 12.4 (why a connected component). (a) Give a finite étale cover of a connected S that is not connected. (b) Let $S'' = S \sqcup S$ be the trivial double cover, and let $Z = S \sqcup \emptyset$. Show that Z is a closed algebraic subset with non-empty interior that is not all of S'', so that the last step of Proposition 12.8 fails on S''. (c) Show that it holds on each component, and that the conclusion on one component S''_0 already gives it on S', because $S''_0 \to S'$ is onto.

12.4. Countable unions, Baire, and the globalization

Before the general statement, here is the topological fact on which everything turns, in two complex dimensions.

Example 12.6 (dense, but no interior). For $q \in \mathbb{Q}(\mathrm{i})$ let $L_q = \{(z, w) \in \mathbb{C}^2 : w = qz\}$, a line through the origin. The union $L = \bigcup_q L_q$ is a countable union of closed sets, each with empty interior. L is dense in $\mathbb{C}^2$: given (z, w) with $z \neq 0$, choose $q_n \in \mathbb{Q}(\mathrm{i})$ with $q_n \to w/z$, then $(z, q_n z) \in L$ tends to (z, w). But L contains no ball. The point $(1, \sqrt{2})$ is not in L, because $\sqrt{2} \notin \mathbb{Q}(\mathrm{i})$ ($x^2 - 2$ stays irreducible over $\mathbb{Q}(\mathrm{i})$), and every ball contains points (z, w), $z \neq 0$, with $w/z \notin \mathbb{Q}(\mathrm{i})$, since $\mathbb{Q}(\mathrm{i})$ is countable. Baire's category theorem says this in general: in a complete metric space, or an open subset of $\mathbb{C}^n$, a countable union of closed sets with empty interior has empty interior.

Intuition 12.7 (why "open" and not "dense"). The algebraicity locus of a flat class is a countable union of algebraic subvarieties. Such a union can be dense without being everything, exactly like L; this is the situation of a Hecke orbit, a countable set that is dense in the Shimura variety [Mil22, §5, Remark 3], and it is why algebraicity at the points of a countable set, however dense, does not by itself propagate. What does propagate is algebraicity on a set with interior: one of the countably many subvarieties must then contain an open set, and an irreducible variety has no proper closed subvariety with interior. Step (vii) is built to meet this requirement, and the rank 9 of Chapter 6 is what supplies the open set.

Proposition 12.8 (the algebraicity locus of a flat class). *Let $f\colon \mathcal{A} \to S$ be a projective abelian scheme over a smooth connected quasi-projective variety S, and $\mathfrak{w}$ a flat section of $R^{2r} f_* \mathbb{Q}$ over S. Then $\{s \in S : \mathfrak{w}_s$ is algebraic$\}$ is a countable union of closed algebraic subsets of S. If it contains a non-empty open subset of S in the analytic topology, it is all of S.*

Proof. This is the Hilbert-scheme argument of [CS14, proof of Prop. 40] and [Voi13, §1.2 and Remark 4.3], which those sources apply to the locus of Hodge classes under the

assumption of the Hodge conjecture; we spell it out for one flat section. Fix a relatively ample line bundle on $\mathcal{A}/S$. For each Hilbert polynomial the relative Hilbert scheme of $\mathcal{A}/S$ is projective over S (Grothendieck), so it has finitely many irreducible components; altogether there are countably many, $T_1, T_2, \dots$, each projective over S and carrying a universal closed subscheme of $\mathcal{A} \times_S T_\nu$, flat over T_ν. For a finite tuple $\nu = (\nu_1, \dots, \nu_m)$, let T be an irreducible component of $T_{\nu_1} \times_S \cdots \times_S T_{\nu_m}$ and $\phi\colon \widetilde{T} \to S$ the composite of a resolution of singularities $\widetilde{T} \to T$ with the projection to S: $\widetilde{T}$ is smooth and connected, and ϕ is proper. Pulled back to $\widetilde{T}$, the universal subschemes give closed subschemes $\mathcal{Z}_1, \dots, \mathcal{Z}_m$ of $\mathcal{A} \times_S \widetilde{T}$, flat over $\widetilde{T}$, whose fibres over $t \in \widetilde{T}$ are subschemes $Z_{i,t}$ of $\mathcal{A}_{\phi(t)}$. The total space $\mathcal{A} \times_S \widetilde{T}$ is smooth, and the restriction of $\mathrm{cl}(\mathcal{Z}_i)$ to the fibre over t is $\mathrm{cl}(Z_{i,t})$, by the compatibility of cycle classes with specialization in flat families [Ful98, Ch. 10, Ch. 19]. The restriction of a global class to the fibres is a flat section of $\phi^* R^{2r} f_* \mathbb{Q}$, so $t \mapsto \mathrm{cl}(Z_{i,t})$ is a flat section over the connected $\widetilde{T}$, and so is $t \mapsto \mathfrak{w}_{\phi(t)}$. For $q \in \mathbb{Q}^m$, the flat sections $\sum_i q_i \mathrm{cl}(Z_{i,t})$ and $\mathfrak{w}_{\phi(t)}$ therefore agree either at every point of $\widetilde{T}$ or at none. Every algebraic class with rational coefficients in a fibre $\mathcal{A}_s$ is $\sum_i q_i \mathrm{cl}(Z_i)$ for closed subschemes $Z_i \subseteq \mathcal{A}_s$, each of which is a point of some T_ν over s; so the algebraicity locus is the union of the images $\phi(\widetilde{T})$ over the countably many triples (ν, T, q) for which the two sections agree. Each image is closed and algebraic, ϕ being proper. For the second statement, let $W \neq \emptyset$ be open in the locus. W is an open subset of a complex manifold, so by Baire's theorem one of the closed sets $Z_k \cap W$ has non-empty interior in W. A closed algebraic subset of the irreducible S (smooth and connected) that is not all of S has smaller dimension and empty interior, so $Z_k = S$. □

For a curve S the closed subsets are finite sets or all of S, so the algebraicity locus of a flat class over a curve is either all of S or countable.

Exercise 12.5 (Baire in $\mathbb{C}^2$). (a) Show that a closed algebraic subset $Z \subsetneq \mathbb{C}^2$ has empty interior. (Hint: a polynomial vanishing on a ball vanishes identically.) (b) Deduce that $\mathbb{C}^2$ is not a countable union of proper closed algebraic subsets, and find a point outside $\bigcup_{q \in \mathbb{Q}(\mathrm{i})} L_q \cup \bigcup_{q \in \mathbb{Q}(\mathrm{i})} \{z = q\}$. (c) Give a countable union of proper closed algebraic subsets of $\mathbb{C}^2$ that contains every point with both coordinates in $\overline{\mathbb{Q}}$.

Proposition 12.9 (step (vii): globalization). *Assume the conclusion of step (vi): for every $t \in \mathcal{H}$, every class in $\mathrm{HW}(A_{C_t})$ is algebraic. Assume Theorem 6.6. Then every class in HW_s is algebraic, for every $s \in S'$.*

The argument. Let S_0'' be a connected component of the cover S'' of Lemma 12.3, $\pi\colon S_0'' \to S'$ the projection, and $\mathcal{A}'' = \mathcal{A} \times_{S'} S_0''$, a projective abelian scheme over the smooth connected quasi-projective S_0''. On S_0'' the local system $\pi^* \mathbb{HW}$ is trivial; let $\mathfrak{w}_1, \mathfrak{w}_2$ be flat sections forming a basis. Fix i. By Proposition 12.8, the set of $s \in S_0''$ where $\mathfrak{w}_{i,s}$ is algebraic is a countable union of closed algebraic subsets. It contains $\pi^{-1}(U')$: for $s \in \pi^{-1}(U')$, $\pi(s) = \mathcal{P}(t)$ for some $t \in \mathcal{H}'$ (Proposition 12.2(b)), the fibre $\mathcal{A}_s'' = \mathcal{A}_{\mathcal{P}(t)}$ is K-isogenous to A_{C_t} (Proposition 12.2(c)), whose Weil classes are algebraic by step (vi), and a K-isogeny carries Weil classes to Weil classes and algebraic classes to algebraic classes (Proposition 12.11). The set $\pi^{-1}(U')$ is non-empty, since π is onto, and open. By Proposition 12.8, $\mathfrak{w}_{i,s}$ is algebraic for every $s \in S_0''$. Since $\mathfrak{w}_{1,s}, \mathfrak{w}_{2,s}$ span $\mathrm{HW}_{\pi(s)}$ and π is onto, the Weil classes are algebraic at every point of S'. □

The argument needs algebraicity only on a Zariski-dense open set of the image, not at every member of $\mathcal{H}$, although step (vi) gives every member. And it does not use any genericity of the points of S': special points, including those with complex multiplication, are reached by the closedness of the algebraicity locus, not by deforming to them.

Remark 12.10 (an analytic open set suffices). The second statement of Proposition 12.8 needs a set with non-empty interior in the *analytic* topology, and the image of $\mathcal{P}$ contains one without Borel or Chevalley: at a point where $\mathcal{P}$ has rank $9 = \dim S'$ it is a submersion, hence

open near that point (Exercise 12.2(c)). So the algebraicity of $\mathcal{P}$ in Proposition 12.2(a) and the Zariski-open set of (b) are more than step (vii) needs; what it does need is the algebraicity of S' and of the abelian scheme $\mathcal{A}$ in (c), for the Hilbert schemes. The observation removes no hypothesis of the theorem, since the rank 9 is still needed.

What just happened. Borel makes $S' = \Gamma'\backslash X^+$ an algebraic variety and the period map algebraic, and the rank 9 of the G_{32} period map gives a Zariski-open set U' in its image, on which the Weil classes are algebraic by step (vi). Over S' the Weil classes form a rank-2 local system with monodromy in μ_6, trivial on a finite cover. The algebraicity locus of a flat section is a countable union of closed algebraic subsets, by Hilbert schemes and specialization of cycles; containing an open set, it is everything, by Baire and irreducibility.

12.5. Isogenies and Landherr: every sixfold of class 2

PROPOSITION 12.11 (K-isogeny invariance). *Let $\varphi\colon A \to A'$ be an isogeny of abelian sixfolds with K-actions ι, ι' such that $\varphi \circ \iota(a) = \iota'(a) \circ \varphi$ in $\mathrm{Hom}(A, A') \otimes \mathbb{Q}$ for all $a \in K$. Then $\bigwedge^6 \varphi^*$ maps $\mathrm{HW}(A', \iota')$ isomorphically onto $\mathrm{HW}(A, \iota)$, and $\mathrm{HW}(A, \iota)$ is algebraic if and only if $\mathrm{HW}(A', \iota')$ is.*

PROOF. $\varphi^*\colon H^1(A', \mathbb{Q}) \to H^1(A, \mathbb{Q})$ is an isomorphism of Hodge structures, and it is K-linear by the hypothesis. A K-linear isomorphism F of six-dimensional K-spaces induces an isomorphism $\bigwedge^6_K F$ of their top exterior powers over K, and the inclusion $\bigwedge^6_K \subset \bigwedge^6_{\mathbb{Q}}$ [Del82, Lemma 4.3(b)] is natural in K-linear maps, so $\bigwedge^6 \varphi^* = \varphi^*|_{H^6}$ maps $\mathrm{HW}(A')$ onto $\mathrm{HW}(A)$. Pull-back by the morphism φ maps algebraic classes to algebraic classes. The kernel of φ is killed by $m = \deg \varphi$, so there is $\psi\colon A' \to A$ with $\psi \circ \varphi = [m]$ [Mil08, I, Rem. 8.6]; then also $\varphi \circ \psi = [m]$ (Exercise 12.6), so $t = m^{-6}\psi^*\varphi^* t$ for $t \in H^6(A', \mathbb{Q})$, and ψ^* preserves algebraic classes too. Hence a class in $\mathrm{HW}(A')$ is algebraic if and only if its image in $\mathrm{HW}(A)$ is. □

EXERCISE 12.6 (the quasi-inverse is K-linear too). In Proposition 12.11, (a) show that ψ is K-equivariant in $\mathrm{Hom}(A', A) \otimes \mathbb{Q}$; (b) verify $t = m^{-6}\psi^*\varphi^* t$ on $H^6(A', \mathbb{Q})$ from $\psi^*\varphi^* = m$ on $H^1(A', \mathbb{Q})$; (c) suppose instead that φ is K-antilinear, $\varphi \circ \iota(a) = \iota'(\bar{a}) \circ \varphi$; show that $\bigwedge^6 \varphi^*$ still maps $\mathrm{HW}(A', \iota')$ onto $\mathrm{HW}(A, \iota)$, because $\mathrm{HW}(A, \iota) = \mathrm{HW}(A, \bar{\iota})$.

The next example is the arithmetic of step (viii) in miniature: Landherr's theorem as a statement about rational isometries.

EXAMPLE 12.12 (Landherr in dimension two, with numbers). In dimension 2 Landherr's theorem says that two Hermitian forms of signature $(1,1)$ over K are isometric over K if and only if their determinants agree modulo norms [Del82, Prop. 4.1]. Take $h_1 = \langle 2, -1\rangle$, of determinant -2, and write $h(x,y) = x^{\mathsf{T}} H \bar{y}$, so that the basis given by the columns of P has Gram matrix $P^{\mathsf{T}} H \bar{P}$.

(i) $h' = \langle 5, -10\rangle$ has determinant $-50 = -2 \cdot 5^2$, the class -2 although 5 itself is not a norm. The vector $x = (2, \theta)$ has $h_1(x,x) = 2 \cdot 4 - 3 = 5$, and $y = (\theta, -4)$ has $h_1(x,y) = 2 \cdot 2 \cdot \bar{\theta} - \theta \cdot (-4) = 0$ and $h_1(y,y) = 2 \cdot 3 - 16 = -10$. So

$$P = \begin{pmatrix} 2 & \theta \\ \theta & -4 \end{pmatrix}, \qquad P^{\mathsf{T}} h_1 \bar{P} = \langle 5, -10\rangle, \qquad \det P = -5, \quad \mathrm{Nm}(\det P) = 25 = \frac{-50}{-2}.$$

(ii) $h'' = \langle 7, -2\rangle$ has determinant -14, again $\equiv -2$ since $7 = \mathrm{Nm}(2 + \theta)$. The first attempt, $x = (2,1)$ with $h_1(x,x) = 7$, forces $y = (c, 4c)$ with $-14\,\mathrm{Nm}(c) = -2$, so $c = (2 - \theta)/7$: a rational isometry with denominators, $P = \left(\begin{smallmatrix} 2 & c \\ 1 & 4c \end{smallmatrix}\right)$, $\det P = 2 - \theta$. There are also integral ones, for instance $P = \left(\begin{smallmatrix} 2+\theta & 1 \\ 2+\theta & 2 \end{smallmatrix}\right)$ with $\det P = 2 + \theta$. Landherr's theorem only promises the rational kind; in step (viii) the lattice is irrelevant, because an isogeny absorbs it.

(iii) The hyperbolic plane $\langle 1, -1\rangle$ has determinant $-1 \not\equiv -2$ and the isotropic vector $(1,1)$. The form $\langle 2, -1\rangle$ has none: $2\,\mathrm{Nm}(a) = \mathrm{Nm}(b)$ with $(a,b) \neq 0$ would make 2 a norm. So no isometry exists, as Landherr predicts.

Adding two hyperbolic planes gives the same picture in dimension 6: $\langle 2,-1\rangle \perp H \perp H$ is h_2 of Example 2.8.

Exercise 12.7 (two Landherr comparisons). (a) Verify the three matrix identities of Example 12.12. (b) Show that $\mathrm{Nm}(\det P) = \det h'/\det h_1$ for any isometry P, and deduce again that $\langle 2,-1\rangle \ncong \langle 1,-1\rangle$. (c) Is $\langle 2,-1\rangle$ isometric to $\langle -1,2\rangle$? To $\langle 10,-5\rangle$? To $\langle 3,-6\rangle$? To $\langle 3,-3\rangle$?

Exercise 12.8 (signs and scaling in rank 6). (a) Show that $\det(-h) = \det h$ for a form of rank 6, so the sign convention for E does not affect the class. (b) For a Hermitian form of rank 3, show that scaling by 2 changes the determinant class. Why does this not matter for the theorem?

Theorem 12.13 (step (viii): every sixfold of class 2). *Assume the conclusions of step (vi) and Theorem 6.6, as in Proposition 12.9. Let A be a complex abelian sixfold, $\iota\colon K \to \mathrm{End}(A)\otimes\mathbb{Q}$, and λ a polarization satisfying (a)–(c) of the Main Theorem: the Rosati involution of λ restricts to complex conjugation on $\iota(K)$, both eigenspaces of $\iota(\theta)$ on $H^{1,0}(A)$ have dimension 3, and $\det h_\lambda \equiv -2 \bmod \mathrm{Nm}\,K^\times$. Then every class in $\mathrm{HW}(A,\iota)$ is algebraic.*

The argument. *The point x_A.* As in Section 2.7: h_λ and h are Hermitian of dimension 6, signature $(3,3)$ and determinant $\equiv -2$ (the scaling of E in Section 12.2 changes the determinant by a norm, Exercise 2.1), so by Proposition 2.6(a) there is a K-linear isometry $f\colon (H^1(A,\mathbb{Q}),h_\lambda)\to(V,h)$, which carries E_λ to E. The complex structure J_A of A commutes with K, preserves E_λ and satisfies $E_\lambda(x,J_Ax)>0$, because E_λ is the Riemann form of a polarization with the same sign convention as E. So $x_A = fJ_Af^{-1}$ satisfies the conditions of Definition 2.14: $x_A\in X^+$. No similitude with negative multiplier is needed, and nothing about X^- enters.

The isogeny. Let $s=[x_A]\in S'$. By Proposition 12.2(c), $H^1(\mathcal{A}_s,\mathbb{Q}) = (V,x_A)$, so f is a K-linear isomorphism of rational Hodge structures $H^1(A,\mathbb{Q})\to H^1(\mathcal{A}_s,\mathbb{Q})$. Choose an integer $M\ge 1$ with $Mf(H^1(A,\mathbb{Z}))\subseteq V_\mathbb{Z}$. Then Mf is an injective map of lattices respecting the Hodge structures, and a morphism of integral Hodge structures $H^1(A,\mathbb{Z})\to H^1(\mathcal{A}_s,\mathbb{Z})$ is the pull-back by a unique homomorphism $\varphi\colon \mathcal{A}_s\to A$ of complex tori, and homomorphisms of complex tori between abelian varieties are homomorphisms of abelian varieties [Mil08, I, Thm. 2.9]; it has finite kernel because $Mf\otimes\mathbb{Q}$ is bijective, so φ is an isogeny. Since f is K-linear, φ commutes with the actions of K in $\mathrm{Hom}(\mathcal{A}_s,A)\otimes\mathbb{Q}$. This is the description of abelian varieties as polarizable complex tori, used in one direction.

Conclusion. By Proposition 12.9, HW_s is algebraic, and by Proposition 12.11 so is $\mathrm{HW}(A,\iota)$. □

The theorem uses hypothesis (c) for one polarization only, and Landherr only for rational Hermitian spaces. Neither the type of λ nor the order of K acting integrally on A enters: they change the lattice $H^1(A,\mathbb{Z})$ inside the rational Hodge structure, and the integer M absorbs the change. The proof of Proposition 12.2(c) used the same construction for the fibres over $\mathcal{P}(t)$.

Remark 12.14 (other components). Theorem 12.13 lands every (A,ι,λ) in the one connected variety $S'=\Gamma'\backslash X^+$, so no statement about the other connected components of the Shimura variety of $\mathrm{GU}(V,h)$ [Mil05, App. B, case (A)] is needed.

Remark on the proof 12.1: standard facts in steps (vii) and (viii). The arguments of this chapter rest on standard facts that are cited, not proved: the theorems of Baily–Borel and Borel [Mil05, Thms. 3.12, 3.14], [Bor72]; the fine moduli space $\mathcal{A}_{6,\delta,N}$ with a projective universal abelian scheme for $N\ge 3$ [MFK94, Ch. 7]; relative Hilbert schemes [Nit05, Thm. 5.1, with $E=\mathcal{O}_X$] and the specialization of cycle classes in flat families [Ful98, Ch. 10, Ch. 19]; Chevalley's theorem; and the Riemann existence theorem for finite étale covers. What a reader should check is that the hypotheses are met as used:

smoothness and quasi-projectivity of $\mathcal{H}'$ and S_0'', torsion-freeness and arithmeticity of Γ', and projectivity of $\mathcal{A}$, which is what the arguments above spell out.

Remark 12.15 (Where this is used). Proposition 12.9 and Theorem 12.13 are the last two links of the chain of Chapter 1; Chapter 13 lists them with their status. Lemma 12.3 and Proposition 12.8 are general and do not depend on G_{32}: any family of Weil sixfolds of class 2 with algebraic Weil classes and a dominant period map would feed into them in the same way. The rank of Theorem 6.6 enters only through the open set U' of Proposition 12.2(b).

CHAPTER 13

The whole proof: what it rests on

Proof steps: all

Where we are. This chapter puts the eight steps of Chapter 1 back together as a chain of implications and records what the proof rests on: the published theorems it quotes, the standard facts it recalls, and the computations it uses, with what each computation proves. It then names the steps that are least secure and the order in which a reader might check them. It adds no mathematics, and it expands Table 1.2. How the proof and this companion were produced, and their status, is stated in the preface.

13.1. The proof as a chain of implications

Write $\mathcal{H}$ for the Hurwitz family of G_{32}-covers $C \to \mathbb{P}^1$ of Example 5.19, finite étale over the 13-dimensional space $\mathcal{U}_{13}$ of branch configurations (Definition 5.6), $\mathcal{B}_C$ for the six-dimensional K-space and A_C for the Weil sixfold with $H^1(A_C, \mathbb{Q}) = \mathcal{B}_C$ (Definition 5.17). The Main Theorem is reached by eight implications. Each line names what is assumed, what is concluded, and where the companion explains it.

(i) *Family.* For every $C \in \mathcal{H}$, A_C is a Weil sixfold with $\det h \equiv -2$ (Example 2.13 and Proposition 5.18); the period map $\mathcal{H} \to \mathrm{Sh}_3^{(2)}$ has rank 9 (Theorem 6.6).

(ii) *Relation.* $e_\rho = \sum_{j=1}^{11} c_j v_j$ with $c_j = \pm 1$, for the pieces of Table 7.1 (Proposition 7.8).

(iii) *Ramified pieces.* For $j \in \{1, 2, 3, 4, 6, 9, 10, 11\}$ the Weil line Λ_{ψ_j} of $X_j \to Y_j$ lies in $\mathrm{Alg}(J(X_j)) \otimes \mathbb{C}$ (the cyclic-cover theorem, Theorem 8.1; Table 9.1).

(iv) *Étale pieces.* The same for $j \in \{5, 7, 8\}$ (Theorem 10.10 and Proposition 10.12).

(v) *Pull-back.* (iii) and (iv) give eleven lines $\Lambda_j = \bigwedge^{\mathrm{top}} W_j \subset \mathrm{Alg}(J(C)) \otimes \mathbb{C}$ (Proposition 7.10).

(vi) *Assembly.* (ii) and (v) give $\mathrm{HW}(A_C)$ algebraic for every $C \in \mathcal{H}$ (Theorem 11.8 and Proposition 11.15).

(vii) *Globalization.* (vi) and the rank 9 of (i) give algebraicity of the Weil classes at every point of $S' = \Gamma' \backslash X^+$ (Proposition 12.9).

(viii) *Every sixfold.* (vii), Landherr and K-isogeny invariance give the Main Theorem (Theorem 12.13).

Steps (ii)–(vi) use no genericity and hold on every member of $\mathcal{H}$; only (vii) uses the dominance half of (i). Figure 13.1 draws the chain with the published theorem or the kind of computation each link uses.

13.2. What the proof takes from outside

Table 13.1 lists, for every step and every group of pieces, the published theorems and standard facts it uses and the computations it rests on, with what each computation proves. Everything else is argued in the chapters named in the last column.

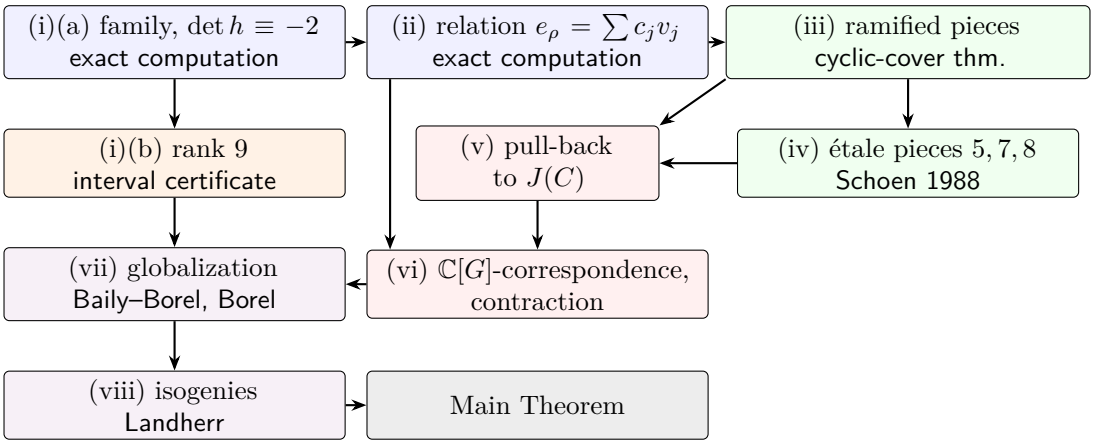


FIGURE 13.1. The chain of Section 13.1; arrows mean "is used by". The second line of a box names the published theorem or the kind of computation that the step uses; the steps without a second line rest on arguments written out in the companion and on standard facts. The arrow from (iii) to (iv) only records the order in which the companion treats them.

TABLE 13.1. What each step and each group of pieces rests on.

item	published theorems and standard facts	computation, and what it proves	companion
(i)(a) family, $\det h \equiv -2$	Landherr's classification [Del82, Prop. 4.1]; the identification of the braid action on parabolic cohomology with the monodromy of $\mathcal{B}_C$ [DW06]	exact, in GAP: the tuple (product 1, membership in G_{32}, generation, classes; Chapter C); the invariant Hermitian form from braid monodromy: the forms invariant under the action of 67 braids are the multiples of one form of determinant -128 (Example 2.13), which gives $\det h \equiv -2$ on every member of $\mathcal{H}$	Chapters 2 and 5
(i)(b) rank 9	Griffiths' formula for the derivative of the period map, smoothing of admissible covers, Picard–Lefschetz, and the other standard facts (S1)–(S7) of Section 6.1	exact, in GAP: the Dehn-twist relation on M_ρ and the fixed spaces on the non-abelian component; ball arithmetic: an enclosure of a 9×9 minor of the limit matrix that excludes 0, with the two 18×18 pivot minors before it; given the formulas for the entries, this proves that the period map has rank 9	Chapter 6
(ii) relation	Frobenius reciprocity, Chevalley–Weil, Brauer's theorem as motivation	exact, in GAP: the characters, subgroups and subcovers (the 334 pieces), and the integral relation; this proves the relation and the data of Table 7.1	Chapters 3 and 7
pieces $1, 2, 3, 4, 6, 9, 10, 11$	Theorem 8.1, with Shioda's theorems on Fermat varieties [Shi79], Poincaré's formula, Lieberman's theorem [Lie68; Kle68] and Zariski's main theorem	full pairing is part of the data of (ii); $d \le g(X)$ is automatic (Remark 8.2)	Chapters 8 and 9

item	published theorems and standard facts	computation, and what it proves	companion
pieces $5, 7, 8$	Schoen 1988 with $r = 0$ [Sch88, p. 13]; Patel–Zhang [PZ27] as an alternative	étaleness, m and $g(Y)$ are part of the data of (ii)	Chapter 10
(v) pull-back	the norm map of Jacobians	none	Chapter 7
(vi) assembly	Wedderburn's theorem; abelian varieties as polarizable complex tori [Mil08, I, Thm. 2.9]; Hodge–Riemann relations on curves	uses the multiplicity data of (ii)	Chapter 11
(vii) globalization	Baily–Borel and Borel [Mil05, Thms. 3.12, 3.14]; relative Hilbert schemes [CS14, proof of Prop. 40]; fine moduli at level $N \geq 3$ [MFK94, Ch. 7]	none	Chapter 12
(viii) every sixfold	Landherr's classification [Del82, Prop. 4.1]; isogenies of abelian varieties	none	Chapters 2 and 12

The computer-assisted parts of the proof are exact computations in GAP (the tuple: product 1, membership in G_{32}, generation, classes; the invariant Hermitian form from braid monodromy, of determinant -128; the characters, subgroups and subcovers and the integral relation; the Dehn-twist relation on M_ρ; and the fixed spaces on the non-abelian component) and one ball-arithmetic certificate (the 9×9 minor, with the two 18×18 pivot minors before it). The hypotheses of (iii) and (iv) rest on the computation of the subcovers in (ii). The chapters explain what each computation proves and illustrate it on small cases; they do not reproduce it.

13.3. The least secure steps

In decreasing order of the weight we give them:

1. The cyclic-cover theorem at the scale of the ramified pieces. The proof of Theorem 8.1 is uniform in m, r and g (Chapters 8 and 9), but its injectivity step (Theorem 9.4) is the step the argument leans on most, and the worked examples and exercises of that step have $r \leq 4$ and base genus at most 2, with piece 9 only as a degree count (Exercise 9.2), while the pieces have r up to 672 and base genus up to 6793. An error in step 6, or in one of the recalled inputs (proper base change for $j_!$, the Thom isomorphism with compact supports, Riemann's existence theorem for the algebraicity of the lift in step 5, Zariski's main theorem), would remove the Weil lines of the eight ramified pieces as they are obtained here. The statement of Schoen's theorem contains these pieces as well (Remark 9.13).

2. Dominance. The rank-nine theorem rests on seven standard facts recalled in Chapter 6 and on two lemmas proved there; the certificate is an enclosure of a determinant whose entries come from derived formulas, so it certifies the arithmetic, not the formulas (Section 6.6).

3. Step (vi). The assembly is short and uses only linear algebra, Wedderburn's theorem and the contraction of Proposition 11.11, and the $\mathrm{SL}(2, 3)$ example (Example 11.10) shows that its bookkeeping can be realized exactly.

4. One computation of the group data. The pieces, the relation and the étale data all rest on the character table and the subgroup classes of G_{32} as computed by GAP (Section 7.5).

5. Recalled standard facts. They are listed where they are used (Chapters 6, 8 and 9, Remark on the proof 12.1). None is in doubt as a statement; whether each is applied within its hypotheses is what a reader has to check.

13.4. What to check first

A reader who wants to check the proof might proceed in this order.

(a) The cyclic-cover theorem, steps 3–8, line by line, with particular attention to the injectivity of step 6 at the scale of the ramified pieces (Theorem 9.4).
(b) Schoen 1988 against the étale pieces: that his Theorem 2.0 and Corollary 3.1 apply to pieces $5, 7, 8$ as stated (Chapter 10).
(c) The smoothing of admissible covers, and the identification of the smooth fibres of the plumbing family with $\mathcal{H}$ or its complex conjugate (Chapter 6).
(d) Dominance: the isotypic Griffiths formula for a non-abelian group; $\dim F^1_\rho(C_0) = 3$ via Grauert and Rosenlicht; the independence of the Picard–Lefschetz constant of the node; and, above all, the formulas for the entries of the limit matrix (Section 6.6).
(e) Step (vi) line by line, including the isotropy lemma Lemma 11.12 and the conventions of Remark 11.2.
(f) Steps (vii)–(viii): that the recalled facts are used within their hypotheses (Remark on the proof 12.1), including the treatment of the centre $Z(G)$ in Section 12.2, and Remark 12.10.
(g) The G_{32} data by a computation that does not go through GAP's character table, for instance the coset actions of the three étale subgroups from explicit generators.
(h) The identification of the braid action on parabolic cohomology with the monodromy of $\mathcal{B}_C$ (Chapter 5).
(i) The recalled standard inputs of the cyclic-cover theorem (Chapters 8 and 9).
(j) The literature: whether the cyclic-cover theorem in this generality appears in the literature beyond Schoen's papers, and the relation to Severn's preprint of 14 October 2026, which claims the Main Theorem by a different route [Sev26b] (Section 1.5).

13.5. What a gap would and would not mean

Every item above is a statement about the chain of arguments, not about the Weil classes. If one of them failed, the conclusion would be that this proof does not establish the Main Theorem as written; it would not suggest that any Weil class on a non-split sixfold is non-algebraic, and the Hodge conjecture [Del06, §1] predicts that all of them are. The worked examples of the chapters test special cases, small groups and small covers; they can expose an error in a formula or a number, but not in an argument that is correct in every small case.

The failures would also be local: a failure of dominance would leave steps (ii)–(vi) intact (Chapter 6).

Exercise 13.1. Suppose the proof had to do without the Weil lines of pieces 4 and 10, while those of the other nine pieces were available. (a) Which of the implications (i)–(viii) of Section 13.1 would lose an input, and which would survive unchanged? (b) Using the counts of Section 7.5, explain why no other choice of pieces from the 291 that are étale or satisfy the prime condition could replace them in a relation with $k = 1$. (c) What would step (vi) produce from a relation $2e_\rho = \sum_j c_j v_j$, and why does the contraction of Proposition 11.11 not obviously finish the argument (Remark on the proof 11.1)?

Exercise 13.2. Suppose the period map of $\mathcal{H}$ had rank 8 instead of 9. (a) Which conclusions of Section 13.1 survive, and on which subset of $\mathrm{Sh}_3^{(2)}$ would the Weil classes then be known to be algebraic? (b) Why does the Baire argument of step (vii) fail, even though the image is infinite? (c) Does step (viii) still apply to the sixfolds that are K-isogenous to members of the family?

APPENDIX A

Glossary of symbols

The symbols of the companion, in the order in which the proof needs them, with the place where each is defined or first explained and the proof step that uses it. Letters that are reused with different meanings in different chapters are listed once per meaning.

symbol	meaning	where	step
The statement (Chapters 1, 2)			
K, ζ, ω, θ	$K=\mathbb{Q}(\sqrt{-3})$; $\zeta=e^{2\pi \mathrm{i}/6}$, $\omega=\zeta^2$, $\theta=\sqrt{-3}$	Section 1.4	all
$\mathrm{HW}(A,\iota)$	Weil classes $\bigwedge^6_K H^1(A,\mathbb{Q})$ of a Weil sixfold	Definition 1.1	all
h_λ, E_λ	Hermitian form of (A,ι,λ) on $H^1(A,\mathbb{Q})$, $E_\lambda=\mathrm{Tr}_{K/\mathbb{Q}}(\theta h_\lambda)$	Definition 2.1 and Remark 2.2	(i), (viii)
$\det h$, $\mathrm{disc}\, h$	determinant class in $\mathbb{Q}^\times/\mathrm{Nm}\, K^\times$; $\mathrm{disc}\, h=(-1)^n \det h$	Section 2.1	(i), (viii)
$\mathrm{Sh}_3^{(2)}$	moduli of Weil sixfolds of class 2 (indexed by $\mathrm{disc}\, h$)	Section 2.1	(i), (vii)
h_2	model form $\langle 2,1,1,-1,-1,-1\rangle$ of the class 2	Example 1.4	(viii)
X^+	the domain $\mathrm{U}(3,3)/\mathrm{U}(3)\times\mathrm{U}(3)$, $\dim_{\mathbb{C}}=9$	Definition 2.14	(i), (vii)
Θ	class of a polarization in H^2 (the letter θ is reserved for $\sqrt{-3}$)	Section 1.2	(vi)
Groups and representations (Chapters 3, 4)			
$M_{\rho'}$	multiplicity space $\mathrm{Hom}_G(\rho', H^1(C,\mathbb{C}))$	Proposition 3.5	(ii), (vi)
$\kappa_{\rho'}$, κ_0	central character of ρ'; that of ρ	Definition 3.7	(ii)
$\rho_1,\dots,\rho_{14}$	the irreducibles of G_{32} with central character κ_0, $\rho_1=\rho$, ordered by degree, then by $\dim M_{\rho'}$	Example 5.13 and Table 7.2	(ii), (vi)
$v_{H,\psi}$	multiplicity vector $\rho'\mapsto\langle \mathrm{Res}_H\,\rho',\psi\rangle$	Theorem 3.10	(ii)
$k_{\min}$	least $k\geq 1$ with ke_ρ in the $\mathbb{Z}$-span of a family of vectors	Definition 3.17	(ii)
E^ρ_{ab}	matrix units of $\mathbb{C}[G]$	Definition 3.21	(vi)
V_ρ	K^4 as an 8-dimensional $\mathbb{Q}[G]$-module; $V_\rho\otimes\mathbb{C}=\rho\oplus\bar\rho$, $\mathrm{End}_{\mathbb{Q}[G]}V_\rho=K$	Proposition 3.27	(i), (vi)
G_4	$\mathrm{SL}(2,3)$ as a reflection group over $\mathbb{Z}[\omega]$	Examples 4.3 and 3.14	toy
$G=G_{32}$, ρ	the reflection group of order 155520; its reflection representation	Proposition 4.10	(i)
$\mathcal{R}_\omega$, $\mathcal{R}_{\omega^2}$, $\mathcal{I}_{90}$	conjugacy classes of the branch data: six reflections of each determinant and one involution of the class of size 90	Section 4.5	(i)
Covers and the family (Chapters 5, 6)			
$\boldsymbol{\gamma}$, $\mathcal{H}$	the branching tuple; its Hurwitz component, a finite étale cover of the 13-dimensional space of 13-point subsets of $\mathbb{C}$ ($13-3=10$ moduli modulo $\mathrm{PGL}_2(\mathbb{C})$)	Example 5.19 and Chapter C	(i)
H^1_{par}	parabolic cohomology of a local system on $\mathbb{P}^1$ minus points	Definition 5.14	(i)

continued on the next page

symbol	meaning	where	step
$\mathcal{B}_C$, A_C	$\mathrm{Hom}_{\mathbb{Q}[G]}(V_\rho, H^1(C,\mathbb{Q}))$; the Weil sixfold, up to isogeny, with $H^1(A_C,\mathbb{Q}) = \mathcal{B}_C$	Definition 5.17 and Proposition 5.18	all
E_u	the polarization on $\mathcal{B}_C$ used from Chapter 5 on	Proposition 5.18	(i)
Ω_C	canonical sheaf of a smooth curve; dualizing sheaf of a nodal curve	Section 5.4	(i)
F^1_ρ	$\mathrm{Hom}_G(\rho, H^0(C,\Omega_C))$, three-dimensional	Section 6.1	(i)(b)
$\mathcal{P}$	period map $\mathcal{H} \to \Gamma\backslash X^+$; on a finite cover, $\mathcal{H}' \to S'$	Sections 6.2 and 12.2	(i)(b), (vii)
Γ (Chapter 6)	the monodromy group of $\mathcal{B}_C$ over $\mathcal{H}$, acting on X^+	Section 6.2	(i)(b)
(S1)–(S7)	the standard facts on curves and period maps used for the dominance	Section 6.1	(i)(b)
N_e, c_{PL}	monodromy logarithm at a node; Picard–Lefschetz constant	Section 6.5	(i)(b)
$M^* = [R \mid T \mid F]$	limit codifferential matrix: node, point-moving and first-order columns	Section 6.1 and Theorem 6.6	(i)(b)
Pieces (Chapter 7)			
$W_{H,\psi}$, W_j	piece: the ψ-eigenspace of $X = C/\ker\psi \to Y = C/H$, pulled back to $H^1(C,\mathbb{C})$	Definition 7.1	(ii)–(v)
e_ρ, c_j	indicator vector of ρ; coefficients ±1 of the relation	Proposition 7.8	(ii)
m, r, d_j	degree, number of branch points, $\dim W_j = 2g(Y_j) - 2 + r$	Table 7.1	(ii)
fully paired, balanced	exponent conditions of a piece	Definition 7.6	(iii)
$\Lambda_{H,\psi}$, Λ_j	the line $\bigwedge^{\mathrm{top}} W_{H,\psi}$ in the cohomology of $J(C)$; the line of piece j	Proposition 7.10	(v), (vi)
The cyclic-cover theorem (Chapters 8, 9)			
steps 1–9	the steps of the proof of the cyclic-cover theorem, not the steps (i)–(viii) of the main proof	Section 8.1	(iii)
W_ψ	the ζ_m-eigenspace of the deck transformation σ on $H^1(X,\mathbb{C})$, of dimension d	Section 8.1	(iii)
$\mathcal{G}$, $\mathcal{N}$	$\mu_m^d \rtimes S_d$ acting on X^d; the kernel of its product map to μ_m	Section 8.3	(iii)
$\mathcal{H}_i$	the hypersurface of divisors through the branch point b_i in $\mathrm{Sym}^d Y$ (not the Hurwitz component)	Section 8.2	(iii)
d', m', α'	$\gcd(m,a_i)$, m/d', reduced exponent vector	Section 8.1	(iii)
Λ_ψ	Weil line $\bigwedge^d W_\psi \subset H^d(J(X),\mathbb{C})$	Theorem 8.1	(iii)
$Z = X^d/\mathcal{N}$	quotient of the self-product (not the centre of G)	Section 8.3	(iii)
S, AJ_d	$\mathrm{Sym}^d Y$; Abel–Jacobi map $S \to \mathrm{Pic}^d(Y)$	Proposition 8.4	(iii)
$\mathbb{L}_\psi$, $j_!$	rank-one local system on $U \subset S$; extension by zero	Proposition 8.9	(iii)
$\mathcal{L}_0$, U_0, U_R	$K_Y(b_1+\cdots+b_r)$; open parts of $\lvert\mathcal{L}_0\rvert$ and of the base of its cone	Propositions 8.12 and 8.15	(iii)
F, G_F, $V(\alpha)$	Fermat hypersurface $X^{r-2}_{m'}$; its diagonal group; character eigenline	Section 8.1	(iii)

continued on the next page

symbol	meaning	where	step
$\mathfrak{C}^{r-2}_{m'}$	characters whose line is spanned by algebraic classes	Theorem 8.1	(iii)
Q, ϖ, $\hat{q}$, ξ	smooth model of the cone; its maps to F and Z; hyperplane class	Proposition 9.1	(iii)
Γ, ϵ_ψ, ℓ_ψ	correspondence $Q \to X^d$; projector; ψ-line in $H^d(X^d)$	Lemma 9.6	(iii)
L, $L^\vee$	Lefschetz operator of a polarization; its adjoint, algebraic by Lieberman's theorem	Proposition 9.7	(iii), (iv)
Étale pieces (Chapter 10)			
$\mathrm{Prym}(X/Y)$, $\mathrm{Prym}^{\mathrm{prim}}$	generalized Prym $(\ker \mathrm{Nm})^0$; its primitive part	Definition 10.2	(iv)
U', U_{Weil}	Schoen's Weil structures of an étale cover	Definition 10.4	(iv)
U'', Ξ	the pull-back of U' to $H^d(J(X), \mathbb{Q})$; the sum map $X^d \to J(X)$	Definition 10.4 and Section 10.5	(iv)
Λ_χ, z_χ	Weil line of a character χ; Schoen's cycle combination	Section 10.1 and Proposition 10.8	(iv)
Assembly (Chapter 11)			
P, Alg	$J(C)$; span of classes of algebraic cycles	Section 11.1	(v), (vi)
$U_\pm$, $D_\pm$	sums of the pieces with $c_j = \pm 1$; their dimensions $D_+ = 28374$, $D_- = 28368$	Section 11.1	(vi)
$\mathfrak{u}_+(\rho')$, $\mathfrak{t}(\rho')$, $\mathcal{T}$	the ρ'-parts of U_+ and of the target; the target	Section 11.4	(vi)
$n_\pm(\rho')$	$\sum_{c_j=\pm 1} v_j(\rho')$	Section 11.4	(vi)
Ψ	matrix over $\mathbb{C}[G]$ realizing $U_+ \cong (u \otimes M_\rho) \oplus U_-$	Definition 11.7	(vi)
Λ_u	the line $\bigwedge^6(u \otimes M_\rho)$, $u \in \rho$	Section 11.1	(vi)
B_u	quotient of P with $H^1 = \mathcal{B}_C$ evaluated at $u \in V_\rho$	Proposition 11.15	(vi)
Globalization (Chapter 12)			
Γ, Γ', S'	$\mathrm{U}(V_{\mathbb{Z}}, h)$; its torsion-free subgroup $\mathrm{U}(V_{\mathbb{Z}}, h) \cap \Gamma_{\mathrm{Sp}}(N)$, $N \geq 3$; $\Gamma' \backslash X^+$	Definition 12.1	(vii)
$\mathcal{A}_{6,\delta,N}$, $\mathcal{A}$	fine moduli of polarized sixfolds with level N; the family over S'	Proposition 12.2	(vii)
S'', $\mathfrak{w}$	finite cover trivializing the Weil classes; a flat Weil class	Lemma 12.3	(vii)
x_A	point of X^+ of a sixfold (A, ι, λ)	Theorem 12.13	(viii)

APPENDIX B

Sources and where to read them

The sources the companion relies on or refers to, grouped by the part of the proof they serve, with what is used from each and the chapters (including their exercises) where it is cited. Unrefereed preprints are marked as such in the bibliography.

source	used for (chapter)
Inputs of the proof	
[Sch88]	Thm. 2.0, Cor. 3.1, Lemma 2.8, the étale case p. 13: the theorem used for the étale pieces, and the construction behind the cyclic-cover theorem (1, 7, 8, 9, 10, 13)
[Shi79]	linear subspaces on Fermat varieties, Cor. to Thm. II(i), p. 179; condition (P_m) (1, 8, 9, 13)
[Lie68; Kle68]	algebraic inverse of hard Lefschetz on abelian varieties (1, 9, 10, 13)
[Lan35; Del82]	classification of Hermitian forms, Prop. 4.1, Cor. 4.2; Hermitian form of a polarization, Lemma 4.6; Weil classes are Hodge classes, Prop. 4.4 (1, 2, 5, 6, 10, 11, 12, 13)
[PZ27]	alternative reference for the étale pieces (1, 7, 9, 10, 13)
[Joh17]	ball arithmetic in Arb (1, 6)
Hodge classes, Weil classes and Hermitian forms	
[Del06]	statement of the Hodge conjecture, §1; Lefschetz $(1,1)$, §2 (1, 13)
[Dem12]	hard Lefschetz theorem, Ch. VI, (8.17) (1, 8, 9)
[Wei79; Gor99; MZ98; MZ99; Mil99]	Weil classes, their Hodge type and non-Lefschetz property, Hodge and Lefschetz groups (1, 2)
[Mil22; Voi26; Mar25]	Weil type, split Weil classes, naming of the discriminant (1, 2, 12)
[Mil20; Ser73; Gro21]	norms from $\mathbb{Q}(\sqrt{-3})$, Ex. 3.44 and p. 54; Meyer's theorem, Ch. IV; Landherr's theorem over any quadratic extension, Thm. 3.1 (2, 3)
[Mil08]	Rosati involution, Riemann forms, cohomology of complex tori, isogenies (1, 2, 7, 8, 10, 11, 12, 13)
[Mil05]	compactness criterion, Thm. 3.3; arithmetic quotients and Baily–Borel, §3; unitary Shimura data, App. B (2, 12, 13)
[Moo10; Mat22]	eigenspaces of cyclic covers of the line; intersection numbers of twisted cycles (2, 5, 6)
Prior art and context	
[Sch89]	mechanism of the cyclic-cover theorem; Thm. 3.4, Rem. 3.5 (1, 8, 9)
[Sch98]	fourfolds and split sixfolds over $\mathbb{Q}(\sqrt{-3})$ (1, 2)
[Koi04]	split sixfolds over $\mathbb{Q}(\mathrm{i})$ (1)
[GV03]	cyclic subcovers of a non-abelian cover (1, 7)
[Mos26]	known cases (1)
[Ope26a; Ope26b; Ope26c]	related claims; not used in the proof (1, 8, 9, 11)
[Sev26b]	a preprint claiming the same theorem by a different route; not used in the proof (1, 13)
[Sev26c; Sev26a]	later claims for every discriminant; not used in the proof (1)
Finite groups and reflection groups	
[Ser77]	representation theory, Brauer's theorem (3, 4)
[Eti+11]	open introduction to representation theory (3)
[GM03]	Shephard–Todd, Chevalley, degrees, Table 1 (4)

continued on the next page

source	used for (chapter)
[ST54; Che55; Spr74; Ben76]	classification, invariants, regular elements, Schur indices (4)
Covers and Hurwitz spaces	
[CW34]	Chevalley–Weil formula (5, 7)
[FK92]	Eichler trace formula (5)
[Völ96; FV91; SGA1]	Riemann existence, Hurwitz spaces (5, 9)
[DW06]	braid action on parabolic cohomology (5, 13)
Period maps and degenerations	
[Sch73; Voi03]	variations of Hodge structure, period maps, transversality, nilpotent orbits; local systems, Picard–Lefschetz (5, 6, 12)
[DLMF]	hypergeometric and elliptic integrals in the Legendre example (6)
[Wol13]	plumbing and the codifferential near a node, Lemma 4 (6)
[CFG15]	Griffiths' formula as a multiplication map (6)
[HM82; Gra60; Lam81; FM12; Ser88]	admissible covers, base change, Picard–Lefschetz, Dehn twists, Rosenlicht's dualizing sheaf (6)
Symmetric products, Fermat varieties and sheaf theory	
[ACGH85; Mac62]	symmetric products, Abel–Jacobi, Poincaré's formula, Betti numbers of $\mathrm{Sym}^n Y$ (8, 9, 10)
[SK79; SSL10; Sil21]	blow-ups and Lefschetz facts for Fermat varieties; lines on Fermat surfaces; known cases of the Hodge conjecture for Fermat varieties (8, 9)
[Mil02; Mil13]	Lieberman's theorem as recorded there; action of correspondences (9)
[Ive86; BT82; Hat02; Stacks; Ful98]	compact supports, proper base change, Thom isomorphism, Zariski's main theorem, cycle classes (5, 7, 8, 9, 10, 11, 12)
Shimura varieties and loci of algebraic classes	
[Bor72; MFK94; Har77]	extension theorem, moduli at level $N \geq 3$, constructible images, Bertini's theorem (8, 12, 13)
[CS14; Voi13]	the Hilbert-scheme argument for loci of algebraic classes (1, 12, 13)
[Nit05]	existence and projectivity of relative Hilbert schemes (12)
Computation	
[LLL82]	lattice reduction, used to find the relation of Proposition 7.8 (7)

APPENDIX C

The tuple

This appendix prints the tuple $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_{13})$ of Section 4.5, which defines the family $\mathcal{H}$ of Example 5.19. Its entries are matrices in $\rho(G_{32}) \subset \mathrm{GL}_4(K)$, acting on column vectors, with $\omega = e^{2\pi \mathrm{i}/3}$; γ_i is the local monodromy at the i-th of the real branch points $t_1 < \cdots < t_{13}$, normalized as in Remark 5.3 (the order used in Section 6.4).

$$\gamma_1 = \tfrac{1}{36}\begin{pmatrix} 12{-}12\omega & 6{+}12\omega & 18 & -24{-}12\omega \\ 6{-}6\omega & 30{-}3\omega & 9\omega & 6{-}6\omega \\ 18{+}18\omega & -9\omega & 18{-}9\omega & 18{+}18\omega \\ -24{-}12\omega & 6{+}12\omega & 18 & 12{-}12\omega \end{pmatrix} \qquad \gamma_2 = \tfrac{1}{36}\begin{pmatrix} 24{+}12\omega & 12{+}6\omega & -18\omega & -12{+}12\omega \\ -6{-}12\omega & 33{+}3\omega & 9{+}9\omega & -6{-}12\omega \\ 18 & -9{-}9\omega & 27{+}9\omega & 18 \\ -12{+}12\omega & 12{+}6\omega & -18\omega & 24{+}12\omega \end{pmatrix}$$

$$\gamma_3 = \tfrac{1}{36}\begin{pmatrix} 24{+}12\omega & 6{-}6\omega & 6{+}30\omega & 0 \\ 6{-}6\omega & 33{+}3\omega & -3{-}15\omega & 0 \\ -30{-}6\omega & 15{+}3\omega & 15{+}21\omega & 0 \\ 0 & 0 & 0 & 36 \end{pmatrix} \qquad \gamma_4 = \tfrac{1}{36}\begin{pmatrix} 24{+}12\omega & 24{+}12\omega & 24{+}12\omega & 0 \\ -12{-}24\omega & 24{+}12\omega & -12{+}12\omega & 0 \\ -12{-}24\omega & -12{+}12\omega & 24{+}12\omega & 0 \\ 0 & 0 & 0 & 36 \end{pmatrix}$$

$$\gamma_5 = \tfrac{1}{36}\begin{pmatrix} 12{-}12\omega & 30{+}24\omega & -6{-}12\omega & 0 \\ 30{+}6\omega & -6{-}21\omega & 12{+}15\omega & 0 \\ -6{+}6\omega & 12{-}3\omega & 30{-}3\omega & 0 \\ 0 & 0 & 0 & 36 \end{pmatrix} \qquad \gamma_6 = \tfrac{1}{36}\begin{pmatrix} 12{-}12\omega & 18\omega & -12{-}6\omega & -12{-}24\omega \\ -18\omega & 18{-}9\omega & -9\omega & 18 \\ -12{-}6\omega & 9\omega & 30{-}3\omega & -6{-}12\omega \\ -12{+}12\omega & 18{+}18\omega & -6{+}6\omega & 12{-}12\omega \end{pmatrix}$$

$$\gamma_7 = \tfrac{1}{36}\begin{pmatrix} 24{+}12\omega & 18{+}18\omega & -6{+}6\omega & -24{-}12\omega \\ -18{-}18\omega & 27{+}9\omega & -9{-}9\omega & -18\omega \\ -6{+}6\omega & 9{+}9\omega & 33{+}3\omega & -12{-}6\omega \\ 12{+}24\omega & 18 & 6{+}12\omega & 24{+}12\omega \end{pmatrix} \qquad \gamma_8 = \tfrac{1}{36}\begin{pmatrix} 24{+}12\omega & -12{+}12\omega & -12{+}12\omega & 0 \\ -12{+}12\omega & 24{+}12\omega & -12{+}12\omega & 0 \\ -12{+}12\omega & -12{+}12\omega & 24{+}12\omega & 0 \\ 0 & 0 & 0 & 36 \end{pmatrix}$$

$$\gamma_9 = \tfrac{1}{36}\begin{pmatrix} 12{-}12\omega & -24{-}12\omega & -24{-}12\omega & 0 \\ -24{-}12\omega & 12{-}12\omega & -24{-}12\omega & 0 \\ -24{-}12\omega & -24{-}12\omega & 12{-}12\omega & 0 \\ 0 & 0 & 0 & 36 \end{pmatrix} \qquad \gamma_{10} = \tfrac{1}{36}\begin{pmatrix} 12{-}12\omega & 30{+}24\omega & -6{-}12\omega & 0 \\ 30{+}6\omega & -6{-}21\omega & 12{+}15\omega & 0 \\ -6{+}6\omega & 12{-}3\omega & 30{-}3\omega & 0 \\ 0 & 0 & 0 & 36 \end{pmatrix}$$

$$\gamma_{11} = \tfrac{1}{36}\begin{pmatrix} 36 & 0 & 0 & 0 \\ 0 & 12{-}12\omega & 24{+}12\omega & -24{-}12\omega \\ 0 & 24{+}12\omega & 12{-}12\omega & 24{+}12\omega \\ 0 & -24{-}12\omega & 24{+}12\omega & 12{-}12\omega \end{pmatrix} \qquad \gamma_{12} = \tfrac{1}{36}\begin{pmatrix} 36 & 0 & 0 & 0 \\ 0 & 24{+}12\omega & 12{-}12\omega & -12{+}12\omega \\ 0 & 12{-}12\omega & 24{+}12\omega & 12{-}12\omega \\ 0 & -12{+}12\omega & 12{-}12\omega & 24{+}12\omega \end{pmatrix}$$

$$\gamma_{13} = \tfrac{1}{36}\begin{pmatrix} 0 & 36{+}18\omega & 18\omega & 0 \\ 18{-}18\omega & -9 & 9{+}18\omega & 0 \\ -18{-}18\omega & -9{-}18\omega & -27 & 0 \\ 0 & 0 & 0 & 36 \end{pmatrix}$$

An exact computation over K in GAP verifies:

- $\gamma_1\gamma_2\cdots\gamma_{13} = 1$ (product from left to right);
- every γ_i lies in $G_{32} = \langle r_1, r_2, r_3, r_4\rangle$ (Section 4.1), and $\langle \gamma_1, \ldots, \gamma_{13}\rangle = G_{32}$;
- $\gamma_1, \ldots, \gamma_{12}$ are reflections of order 3, with $\det \gamma_i = \omega$ for $i \in \{2, 3, 4, 7, 8, 12\}$ (class $\mathcal{R}_\omega$) and $\det \gamma_i = \omega^2$ for $i \in \{1, 5, 6, 9, 10, 11\}$ (class $\mathcal{R}_{\omega^2}$);
- γ_{13} has order 2, determinant 1 and $\mathrm{rk}(\gamma_{13} - 1) = 2$, and lies in the class $\mathcal{I}_{90}$ (Table 4.2);
- $\gamma_5 = \gamma_{10}$ (this is not a misprint);
- $\gamma_1\gamma_2 = \gamma_6\gamma_7 = \gamma_8\gamma_9 = \gamma_{11}\gamma_{12} = 1$ and $\gamma_3\gamma_4$ has order 6, facts about the boundary point used in Chapter 6.

APPENDIX D

Solutions to the exercises

Chapter 1

Solution to Exercise 1.1. Everything is a binomial coefficient, because the cohomology of a complex torus is the exterior algebra of H^1 [Mil08, I, Thm. 2.3].

(a) $\dim_{\mathbb{Q}} H^1 = 12$, so $\dim H^6 = \binom{12}{6} = 924$. With $H^1 \otimes \mathbb{C} = H^{1,0} \oplus H^{0,1}$, both of dimension 6, $H^{3,3} = \bigwedge^3 H^{1,0} \otimes \bigwedge^3 H^{0,1}$ has dimension $\binom{6}{3}^2 = 20^2 = 400$. (b) A 6-dimensional K-vector space W has $\dim_K \bigwedge^6_K W = \binom{6}{6} = 1$, and a one-dimensional K-space is two-dimensional over $\mathbb{Q}$. (c) $\dim H^{2n} = \binom{4n}{2n}$, which is $6, 70, 924, 12870$ for $n = 1, 2, 3, 4$; the Weil classes take the fractions $1/3$, $1/35$, $1/462$, $1/6435$ of it. For $n = 1$ they lie in H^2 and are divisor classes by the Lefschetz $(1,1)$ theorem; for $n = 2$ they sit in the 70-dimensional H^4. □

Solution to Exercise 1.2. Compute the pull-back of $\mathrm{d}z$ and $\mathrm{d}\bar{z}$ under the two actions of ω on one factor.

(a) Under multiplication by ω, $z \mapsto \omega z$, one has $\omega^* \mathrm{d}z = \omega\, \mathrm{d}z$ and $\omega^* \mathrm{d}\bar{z} = \bar{\omega}\, \mathrm{d}\bar{z}$; so $\mathrm{d}z$ lies in V_ι (where a acts by a) and $\mathrm{d}\bar{z}$ in $V_{\bar{\iota}}$. Under the conjugate action, $z \mapsto \bar{\omega} z$, the roles swap: $\mathrm{d}\bar{z} \in V_\iota$. The factors with the standard action are $1, 2, 6$ and those with the conjugate action $3, 4, 5$, which gives the span claimed. (b) $\mathrm{d}z_1, \mathrm{d}z_2, \mathrm{d}z_6$ have type $(1,0)$ and $\mathrm{d}\bar{z}_3, \mathrm{d}\bar{z}_4, \mathrm{d}\bar{z}_5$ type $(0,1)$, and the type of a wedge product is the sum of the types: $(3,3)$. (c) Now V_ι is spanned by four $\mathrm{d}z_k$ and two $\mathrm{d}\bar{z}_l$, so $\ell = \bigwedge^6 V_\iota$ has type $(4,2)$ and $\bar{\ell}$ type $(2,4)$. A rational class w in $\bigwedge^6_K H^1$ has $w = a\ell + \bar{a}\bar{\ell}$ with $a \in \mathbb{C}$ (it is fixed by complex conjugation); it is of type $(3,3)$ only if $a = 0$. The space $\bigwedge^6_K H^1$ is still 2-dimensional over $\mathbb{Q}$ by the argument of Exercise 1.1(b), since the dimension count does not see the multiplicities. □

Solution to Exercise 1.3. For (a) clear denominators and compare powers of 2; the rest follows.

(a) Suppose $x = a/c$, $y = b/c$ with integers a, b, c, $c > 0$, $\gcd(a,b,c) = 1$, and $a^2 + 3b^2 = 2c^2$. If c is odd, the right side is $\equiv 2 \pmod 4$, while $a^2 + 3b^2 \equiv a^2 - b^2 \pmod 4$ takes only the values $0, 1, 3$. If c is even, $2c^2 \equiv 0 \pmod 8$; then $a^2 - b^2 \equiv 0 \pmod 4$ forces a, b of the same parity, both odd gives $a^2 + 3b^2 \equiv 1 + 3 = 4 \pmod 8$, and both even contradicts $\gcd(a,b,c) = 1$. So there is no solution. (b) Every norm $x^2 + 3y^2$ is positive, so -2 is not a norm, and (a) says 2 is not. If -2 and -1 had the same class, $-2/-1 = 2$ would be a norm. (c) $x^2 + x + 1$ takes the value 1 at $x = 0$ and $3 \equiv 1$ at $x = 1$ modulo 2, so it has no root and is irreducible over $\mathbb{F}_2$; hence $\mathbb{Z}[\omega]/(2) \cong \mathbb{F}_2[x]/(x^2 + x + 1) = \mathbb{F}_4$ is a field and 2 is prime in $\mathbb{Z}[\omega]$. If $2 = \mathrm{Nm}(\alpha)$ with $\alpha \in K$, write $\alpha = \beta/n$ with $\beta \in \mathbb{Z}[\omega]$, $n \in \mathbb{Z}$; then $2n^2 = \beta\bar{\beta}$. Since 2 is prime in $\mathbb{Z}[\omega]$ and $\bar{\beta}$ is divisible by 2 exactly as often as β, the power of 2 dividing $\beta\bar{\beta}$ is even, while that dividing $2n^2$ is odd. This is (a) again, in the language of primes. In general, $p \equiv 2 \pmod 3$ is inert, and this is the reason behind the norm test of Lemma 2.4. (d) $3 = 0^2 + 3 \cdot 1^2$, $7 = 2^2 + 3 \cdot 1^2$, $13 = 1^2 + 3 \cdot 2^2$ are norms; $5 \equiv 2 \pmod 3$ divides 5 to an odd power, so 5 is not. □

Solution to Exercise 1.4. All five forms have three positive and three negative entries, so signature $(3,3)$; by Landherr only the determinant class matters (Proposition 2.6(a)).

The determinants are -2, $2 \cdot (-2)(-2)(-1) = -8$, -3, -5 and -14. Since $4 = \mathrm{Nm}(2)$, $3 = \mathrm{Nm}(\sqrt{-3})$ and $7 = \mathrm{Nm}(2 + \sqrt{-3})$, these are $\equiv -2, -2, -1, -5, -2$. So $\langle 1,1,1,-1,-1,-2\rangle$, $\langle 2,1,1,-2,-2,-1\rangle$ and $\langle 7,2,1,-1,-1,-1\rangle$ are isometric to h_2; $\langle 3,1,1,-1,-1,-1\rangle$ is split;

$\langle 5,1,1,-1,-1,-1\rangle$ is neither, since $5/2$ and 5 are not norms (5 has odd valuation and $5 \equiv 2 \pmod 3$). □

Solution to Exercise 1.5. Write $\delta = \det h_{\lambda_1} \in \mathbb{Q}^\times$ for the Gram determinant of h_{λ_1} in a K-basis e_1, e_2, e_3 of $H^1(B_1, \mathbb{Q})$.

(a) On the second factor the K-structure is conjugated: $a \in K$ acts by $\iota_1(\bar a)$. The form $h'(x,y) = -\overline{h_{\lambda_1}(x,y)}$ is linear in x for this structure, Hermitian, and $\mathrm{Tr}(\sqrt{-3}\,h') = \mathrm{Tr}(\overline{\sqrt{-3}\,h_{\lambda_1}}) = \mathrm{Tr}(\sqrt{-3}\,h_{\lambda_1}) = E_{\lambda_1}$, so h' is the form of $(B_1, \bar\iota_1, \lambda_1)$ by uniqueness. Its Gram matrix in e_1, e_2, e_3 is $-\bar H$, of determinant $(-1)^3\bar\delta = -\delta$. Scaling the polarization by c scales the form by $c^{\pm 1}$ and the 3×3 determinant by $c^{\pm 3}$. The product polarization makes the two factors orthogonal, so $\det h_{\lambda_1 \times c\lambda_1} = \delta \cdot (-c^{\pm 3}\delta) = -c^{\pm 3}\delta^2 \equiv -c$, because $\delta^2 = \mathrm{Nm}(\delta)$ and $c^{\pm 2} = \mathrm{Nm}(c^{\pm 1})$. Signature: $(2,1)$ for h_{λ_1} gives $(1,2)$ for $-\bar h_{\lambda_1}$, in total $(3,3)$ (the H^1/H_1 convention may swap all signatures at once, Remark 2.2). (b) $-c \equiv -1$ iff c is a norm, iff every prime $\equiv 2 \pmod 3$ divides c to an even power: $c \in \{1,3,4,7,9,12\}$. $-c \equiv -2$ iff $c/2$ is a norm: $c \in \{2,6,8\}$. The others, $5, 10, 11$, give the classes $-5, -10, -11$. (c) The space $\mathrm{HW}(A,\iota) = \bigwedge^6_K H^1(A,\mathbb{Q})$ is defined without λ, so the theorem's conclusion is a property of (A,ι). The class is a property of (A,ι,λ), and the hypothesis asks only that one λ have class -2. Here $c = 2$ does; that $c = 1$ gives a split triple only means that (A,ι) also belongs to the split case. Different compatible polarizations of one (A,ι) can have different classes because they need not differ by a scalar: here the two factors are scaled independently. □

Solution to Exercise 1.6. Dimension of a fibre = dimension of the source minus the rank.

(a) On the 10 moduli the generic fibre has dimension $10 - 9 = 1$. On $\mathcal{H}$ it has dimension $13 - 9 = 4$: the three dimensions of a $\mathrm{PGL}_2(\mathbb{C})$-orbit of configurations, which do not change the curve, and the one dimension above. (b) Step (vii) needs a non-empty open subset of the Shimura variety on which the Weil classes are algebraic, because a countable union of proper closed subvarieties contains no open set (Baire, Section 12.1). If the period map has rank 9, its image contains such an open set. If its rank is smaller, for instance if the image is a curve, the image lies in a proper closed subvariety, and the countable-union argument gives algebraicity only on the Zariski closure of the image, however many members the family has. This is what happens for any family whose curves have fewer than 9 moduli, such as a cyclic family of μ_6-covers of the line branched over 8 points, whose image has dimension at most $8-3 = 5 < 9$. (c) $\sum_i (1 - 1/e_i) = 12 \cdot \frac{2}{3} + \frac{1}{2} = \frac{17}{2}$, so $2g - 2 = 155520 \cdot (-2 + \frac{17}{2}) = 155520 \cdot \frac{13}{2} = 1010880$ and $g = 505441$ (Example 5.13 computes it from the group). □

Solution to Exercise 1.7. Exchange the order of summation in (b).

(a) $D_+ = d_2+d_4+d_5+d_6+d_8+d_9+d_{11} = 2208+3726+4680+7128+9360+312+960 = 28374$ and $D_- = d_1 + d_3 + d_7 + d_{10} = 2520 + 3168 + 8424 + 14256 = 28368$, so $D_+ - D_- = 6$. (b) $\sum_j c_j d_j = \sum_j c_j \sum_{\rho'} v_j(\rho') \dim M_{\rho'} = \sum_{\rho'} (\sum_j c_j v_j(\rho')) \dim M_{\rho'} = \sum_{\rho'} e_\rho(\rho') \dim M_{\rho'} = \dim M_\rho = 6$. It is a necessary condition, not a proof of the relation; the relation itself is checked entrywise (Proposition 7.8). (c) The pieces 4 and 10, whose $(6,6)$ pairs at $m = 12$ fail the prime condition of Section 7.5, since 6 is divisible by both primes of 12. The ramified pieces that satisfy the prime condition, together with the étale pieces, span $2e_\rho$ but not e_ρ ($k_{\min} = 2$), and whether step (vi) can be run with $2e_\rho$ is not known (Section 7.5). □

Chapter 2

Solution to Exercise 2.1. Use the uniqueness in Definition 2.1: a Hermitian form is determined by its trace form $\mathrm{Tr}(\theta h)$.

(a) $E_{c\lambda} = cE_\lambda = \mathrm{Tr}(\theta\, ch_\lambda)$, and ch_λ is Hermitian, so $h_{c\lambda} = ch_\lambda$ by uniqueness. A Gram matrix is multiplied by c, its 6×6 determinant by c^6, and $c^6 = (c^3)(\overline{c^3}) = \mathrm{Nm}(c^3)$ since $c^3 \in \mathbb{Q}$ is real. (b) $\mathrm{Tr}(u\theta \cdot u^{-1}h_\lambda) = \mathrm{Tr}(\theta h_\lambda) = E_\lambda$, so the form attached to θ' is $u^{-1}h_\lambda$. (c) For odd m, $\det(2h) = 2^m \det h$ and 2^m has odd 2-adic valuation, so it is not a norm (Proposition 2.6(d)): the class changes. For the forms in this companion this never matters, because they have even rank: 6 for sixfolds, 2 for the surface of Example 2.9. □

Solution to Exercise 2.2. The determinant is $-c_1\cdots c_6/3^6$ and $3^6 = \mathrm{Nm}(27)$.

(a) $\det\psi_c \equiv -c_1\cdots c_6$, which is $\equiv -2$ iff $c_1\cdots c_6 \equiv 2$, i.e. iff $c_1\cdots c_6$ is twice a norm. With entries in $\{1,2\}$ the product is 2^k with k the number of entries equal to 2, and $2^k \equiv 2$ iff k is odd. Writing a sorted triple by its number of 2s ($(0,1,2,3$ for $(1,1,1),(2,1,1),(2,2,1),(2,2,2)$), the pairs (k_1,k_2) with k_1+k_2 odd are $(0,1),(0,3),(1,0),(1,2),(2,1),(2,3),(3,0),(3,2)$: eight vectors, for instance $c=(2,1,1,1,1,1)$, $(1,1,1,2,2,2)$ and $(2,2,2,2,2,1)$. (b) On the first three factors K acts through σ (multiplication), so they make up $\mathrm{Lie}\,A_\sigma$, and there ψ_c has the positive entries $c_i/3$; on the last three it has $-c_i/3<0$. This is what the computation in the proof of Proposition 2.15(a) gives on H_1 with the sign of Remark 2.2(2): the form is positive on the part of $\mathrm{Lie}\,A$ where K acts through σ. □

Solution to Exercise 2.3. Compute with the Gram matrix $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$.

(a) $\det = 0\cdot 0 - 1\cdot 1 = -1$. For $x_\pm = e \pm \frac12 f$: $h(x_\pm,x_\pm) = \pm\frac12 \pm \frac12 = \pm 1$, and $h(x_+,x_-) = -\frac12+\frac12 = 0$. So in the basis x_+,x_- the form is $\langle 1,-1\rangle$. (b) The orthogonal sum has Gram matrix block diagonal with three such blocks: signature $(3,3)$, $\det = (-1)^3 = -1$, $\mathrm{disc} = (-1)^3\det = 1$, and $\mathrm{span}_K(e_1,e_2,e_3)$ (the first vectors of the three pairs) is a three-dimensional totally isotropic subspace. (c) By [Del82, proof of Cor. 4.2], a basis of a three-dimensional totally isotropic subspace extends to a basis in which the Gram matrix consists of three hyperbolic blocks, so a split space of dimension 6 is an orthogonal sum of three hyperbolic planes and has $\det \equiv -1$ by (b). If $\det h \equiv -2$ and h were split, then $-2\equiv -1$, i.e. $2\in \mathrm{Nm}\,K^\times$, which is false (Exercise 1.3). Only the easy direction of Landherr is used. □

Solution to Exercise 2.4. For (b) imitate, modulo 5, the descent after Proposition 2.6.

(a) $3 = 0^2+3\cdot 1^2 = \mathrm{Nm}(\theta)$, $7 = 2^2+3\cdot 1^2 = \mathrm{Nm}(2+\theta)$, $13 = 1^2+3\cdot 2^2 = \mathrm{Nm}(1+2\theta)$. (b) If 5 were a norm, $5c^2 = a^2+3b^2$ with integers a,b,c, $c\neq 0$, not all divisible by 5. Modulo 5, $a^2 \equiv -3b^2 = 2b^2$. The squares modulo 5 are $0,1,4$, and 2 is not one of them, so $b\equiv 0$ would be forced (otherwise $2\equiv (a/b)^2$), and then $a\equiv 0$. Then $25 \mid a^2+3b^2 = 5c^2$, so $5\mid c$: all three are divisible by 5, a contradiction. (c) $6 = 2\cdot 3$ represents the class 2; $24 = 2^3\cdot 3$ the class 2 (2 to an odd power); $108 = 2^2\cdot 3^3$ the class 1, so 108 is a norm ($108 = 9^2+3\cdot 3^2 = \mathrm{Nm}(9+3\theta)$, and also $108 = 3\cdot 6^2 = \mathrm{Nm}(6\theta)$); $128 = 2^7$ the class 2. (d) A prime $p\equiv 1 \pmod 3$ has no prime factor $\equiv 2 \pmod 3$, so it is a norm by Lemma 2.4. For $p=31$: $31 - 3\cdot 3^2 = 4 = 2^2$, so $31 = \mathrm{Nm}(2+3\theta)$. □

Solution to Exercise 2.5. Write $x=(x_1,\dots,x_6)$ and $h_2(x,y) = 2x_1\bar y_1 + x_2\bar y_2 + x_3\bar y_3 - x_4\bar y_4 - x_5\bar y_5 - x_6\bar y_6$.

(a) $h_2(w_1,w_1) = 1-1 = 0$, $h_2(w_2,w_2) = 1-1=0$, and $h_2(w_1,w_2) = 0$ because w_1 and w_2 have disjoint supports. (b) A three-dimensional totally isotropic subspace would make h_2 split, hence $\det h_2 \equiv -1$ by Exercise 2.3(c); but $\det h_2 = -2 \not\equiv -1$. So the maximal totally isotropic subspaces have dimension exactly 2 (the Witt index is 2). (c) $x = e_2 - e_4$ has $h_2(x,x) = 1-1=0$ and is not in $\mathrm{span}_K(w_1,w_2)$ (its third and fifth coordinates vanish but it is not a multiple of w_1). But $h_2(x,w_1) = h_2(e_2-e_4,e_2+e_4) = h_2(e_2,e_2) - h_2(e_4,e_4) = 1+1 = 2\neq 0$, so x,w_1,w_2 do not span a totally isotropic subspace, as (b) predicts. □

Solution to Exercise 2.6. Use $\langle r\rangle \in [0,1)$ for the fractional part and Chevalley–Weil, $\dim V_{\zeta^k} = -2 + \#\{i : ka_i \not\equiv 0 \bmod 6\}$.

(a) Over t_i there are $\gcd(a_i,6) = 3,1,2,2$ points, so $2g-2 = -12 + (3+5+4+4) = 4$ and $g=3$. For $k=1,\dots,5$, the numbers of i with $ka_i\not\equiv 0$ are $4,3,2,3,4$ (for $k=3$: $3a_i\equiv 0$ for $a_i = 4,4$, and $3\cdot 3 = 9\not\equiv 0$, $3\cdot 1\not\equiv 0$, so 2), giving dimensions $2,1,0,1,2$, total $6 = 2g$. Then $d_1 = -1 + \langle -\frac36\rangle + \langle -\frac16\rangle + 2\langle -\frac46\rangle = -1+\frac12+\frac56+2\cdot\frac13 = 1$ and $d_5 = -1+\langle -\frac{15}{6}\rangle + \langle -\frac56\rangle + 2\langle -\frac{20}{6}\rangle = -1+\frac12+\frac16+2\cdot\frac23 = 1$. (b) $-\frac32 = -2\cdot\frac34$ and $\frac34 = \mathrm{Nm}(\theta/2)$, so $-\frac32$ is -2 times a norm. The class is not -1 because $2\notin \mathrm{Nm}\,K^\times$. (c) Gram–Schmidt: with $e_2' = e_2 - \frac{\psi(e_2,e_1)}{\psi(e_1,e_1)}e_1$ one gets $\psi \cong \langle -\frac32, \det\psi/(-\frac32)\rangle = \langle -\frac32, 1\rangle$. A vector $(x,y)\neq 0$ in

this basis is isotropic iff $\frac{3}{2}\operatorname{Nm}(x) = \operatorname{Nm}(y)$, which needs $\frac{3}{2}$ to be a norm (if $x = 0$ then $y = 0$); but $\frac{3}{2} = 2 \cdot \frac{3}{4}$ is not. So there is no nonzero isotropic vector. □

Solution to Exercise 2.7. Factor and look at the exponent of 2 (and the sign).

(a) $-128 = -2^7$: negative, 2 to an odd power, no other prime; so $-128 = -2 \cdot 2^6 = -2 \cdot \operatorname{Nm}(8)$, class -2. For $r \in \mathbb{Q}^\times$, $r^6 = \operatorname{Nm}(r^3)$ is a norm, so $-128r^6$ has the same class. (b) For $V = K^2$ and $\Gamma = \{1\}$, every sesquilinear form is invariant, $\dim_K S_\Gamma = 4$, and $\langle 1, -1\rangle$, $\langle 1, -2\rangle$ are invariant Hermitian forms with determinants $-1 \not\equiv -2$. If the computed space had been larger than a line, the polarization could have been any invariant Hermitian form in it, and the determinant of one computed form would say nothing about the determinant of the polarization; Lemma 2.10(a) is exactly what excludes this. □

Solution to Exercise 2.8. Repeat the computation in the proof of Proposition 2.15(a) without assuming the signature.

Let $V_\pm = \{Jx = \pm \mathrm{i}x\}$. The proof of Proposition 2.15(a) used the signature only in its last sentence: for any J commuting with K, preserving E and with $E(x, Jx) > 0$, it shows that h is positive definite on V_+, negative definite on V_-, and $V_+ \perp_h V_-$. Hence $(\dim V_+, \dim V_-)$ is the signature of h, by Sylvester's law of inertia applied to the decomposition $V_\mathbb{R} = V_+ \oplus V_-$. The multiplicities of K on $(V_\mathbb{R}, J)$ are $(\dim V_+, \dim V_-)$, because on $V_\pm$ the element θ acts as $\mathrm{i}\sqrt{3} = \pm\sqrt{3}J$, i.e. through σ resp. $\bar\sigma$ for the complex structure J. So multiplicities (n, n) force signature (n, n), and conversely signature (p, q) forces multiplicities (p, q) for every such J. □

Solution to Exercise 2.9. Count real parameters of the linear equations.

(a) Write $X = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ in blocks of sizes p, q. The equation $X^* I_{p,q} + I_{p,q} X = 0$ says $A^* = -A$, $D^* = -D$, $C = B^*$. Skew-Hermitian $p \times p$ matrices form a real space of dimension p^2, and B is arbitrary, $2pq$ real parameters. Total $p^2 + q^2 + 2pq = (p+q)^2$. So $\dim_\mathbb{R} \mathrm{U}(n, n) = 4n^2$, $\dim_\mathbb{R} \mathrm{U}(n) \times \mathrm{U}(n) = 2n^2$, and the quotient has real dimension $2n^2$, complex dimension n^2; for $n = 3$, 9. (b) The chart is the open unit ball in $\mathrm{Mat}_3(\mathbb{C})$, of complex dimension 9. (c) A positive complex line is not orthogonal to e_1 (otherwise it would lie in the negative line $\mathbb{C}e_2$), so it is spanned by a vector $e_1 + we_2$, and $h(e_1 + we_2, e_1 + we_2) = 1 - \delta\,|w|^2$; so the domain is the disc $|w| < \delta^{-1/2}$. Rescaling the second basis vector by $\delta^{-1/2}$ (over $\mathbb{R}$) turns the form into $\langle 1, -1\rangle$ and w into $z = \sqrt{\delta}\, w$, and the chart of Proposition 2.15(c) becomes the unit disc $\{|z| < 1\}$, for every δ. The class of δ changes only the arithmetic group acting on it. □

Chapter 3

Solution to Exercise 3.1. Restricting means reading off the values on 1 and (12); inducing needs the conjugates of each element that land in C_2.

Restriction: on $C_2 = \{1, (12)\}$ the characters $\mathbf{1}, \mathrm{sgn}, \rho_2$ take the values $(1, 1)$, $(1, -1)$, $(2, 0)$. Hence $\langle \mathrm{Res}\,\mathbf{1}, \mathrm{sgn}\rangle = \frac{1}{2}(1-1) = 0$, $\langle \mathrm{Res}\,\mathrm{sgn}, \mathrm{sgn}\rangle = \frac{1}{2}(1+1) = 1$, $\langle \mathrm{Res}\,\rho_2, \mathrm{sgn}\rangle = \frac{1}{2}(2+0) = 1$, giving $v = (0, 1, 1)$. Induction: $\mathrm{Ind}_{C_2}^{S_3} \mathrm{sgn}$ has degree 3. At $g = (12)$ the elements x with $x^{-1}gx \in C_2$ are those with $x^{-1}gx = (12)$, i.e. the centralizer $\{1, (12)\}$, so the value is $\frac{1}{2} \cdot 2 \cdot \mathrm{sgn}((12)) = -1$. No conjugate of (123) lies in C_2, so the value there is 0. Thus $\mathrm{Ind}\,\mathrm{sgn} = (3, -1, 0)$, and with class sizes $1, 3, 2$: $\langle \mathrm{Ind}, \mathbf{1}\rangle = \frac{1}{6}(3-3) = 0$, $\langle \mathrm{Ind}, \mathrm{sgn}\rangle = \frac{1}{6}(3+3) = 1$, $\langle \mathrm{Ind}, \rho_2\rangle = \frac{1}{6}(6) = 1$. Both computations give $(0, 1, 1)$, i.e. $\mathrm{Ind}_{C_2}^{S_3} \mathrm{sgn} = \mathrm{sgn} + \rho_2$. □

Solution to Exercise 3.2. Use the explicit model $\bigoplus_i g_i \otimes \mathbb{C}$ of the induced representation and move z past g_i.

In $\mathrm{Ind}_H^G \psi = \bigoplus_i g_i \otimes \mathbb{C}$ we have $z(g_i \otimes w) = zg_i \otimes w = g_i z \otimes w = g_i \otimes \psi(z)w$, because z is central and $z \in H$. So z acts as the scalar $\psi(z)$ on the whole induced representation, hence on every subrepresentation. On an irreducible constituent ρ' it also acts as $\kappa_{\rho'}(z)$, so $\kappa_{\rho'}(z) = \psi(z)$ for all $z \in Z$. If $\kappa_{\rho'}|_Z \neq \psi|_Z$, then ρ' is not a constituent, and $v_{H,\psi}(\rho') = \langle \rho', \mathrm{Ind}_H^G \psi\rangle = 0$ by Theorem 3.10. □

Solution to Exercise 3.3. Use the class sizes $1, 1, 4, 4, 6, 4, 4$ and $|\omega| = 1$, $1 + \omega + \omega^2 = 0$.

The degrees are $1,1,1,2,2,2,3$ and $1+1+1+4+4+4+9=24$. Next, $\langle\chi_5,\chi_5\rangle = \frac{1}{24}(4+4+4+4+0+4+4)=1$. Since χ_4 is real, $\langle\chi_5,\chi_4\rangle = \frac{1}{24}(4+4+4\omega+4\omega+0+4\omega^2+4\omega^2) = \frac{1}{24}(8+8(\omega+\omega^2))=0$. The central character is $\chi(-1)/\chi(1)$: it is $-2/2=-1$ for χ_4,χ_5,χ_6, and $1/1=1$ resp. $3/3=1$ for $\chi_1,\chi_2,\chi_3,\chi_7$. □

Solution to Exercise 3.4. A subgroup of index 2 is normal with quotient of order 2, so it contains all squares.

(a) Let $[G:H]=2$. Then H is normal and $G/H\cong C_2$, so $g^2\in H$ for all g. The element -1 is the only element of order 2 (one class of size 1). An element g of order 3 satisfies $g=g^4=(g^2)^2\in H$; an element of order 4 has $g^2=-1$, so $-1\in H$; an element g of order 6 has $g^3=-1$, so $g=g^4\cdot g^{-3}=(g^2)^2\cdot(-1)\in H$. So H contains 1, -1, the 8 elements of order 3 and the 8 of order 6: $18>12$ elements, a contradiction. (b) A subgroup of index 2 would give a surjection $G\to C_2$, i.e. a one-dimensional character of order 2. The one-dimensional characters χ_1,χ_2,χ_3 form a group of order 3 under multiplication, with no element of order 2. Conclusion: an irreducible monomial $\rho'=\operatorname{Ind}_H^G\psi$ has $[G:H]=\dim\rho'$; for $\dim\rho'=2$ this needs a subgroup of index 2, which does not exist. So χ_4,χ_5,χ_6 are not monomial. □

Solution to Exercise 3.5. Each induced character from a subgroup of order k has degree $4/k$ and is a sum of distinct one-dimensional characters.

Write the characters as χ_{ab}, $a,b\in\{0,1\}$, with $\chi_{ab}(x,y)=(-1)^{ax+by}$. The cyclic subgroups are 1 and $A=\langle(1,0)\rangle$, $B=\langle(0,1)\rangle$, $C=\langle(1,1)\rangle$. $\operatorname{Ind}_1^G\mathbf{1}=\sum_{a,b}\chi_{ab}$ (coefficient sum 4). For $X\in\{A,B,C\}$ and ψ one of the two characters of X, $\operatorname{Ind}_X^G\psi$ is the sum of the two χ_{ab} restricting to ψ on X (Frobenius reciprocity), coefficient sum 2. That is $1+3\cdot2=7$ characters, all with even coefficient sum, so every integral combination has even coefficient sum, while χ_{00} has coefficient sum 1. Writing χ_X for the non-trivial character trivial on X, $\operatorname{Ind}_X^G\mathbf{1}=\chi_{00}+\chi_X$, and $\sum_X\operatorname{Ind}_X^G\mathbf{1}-\operatorname{Ind}_1^G\mathbf{1}=3\chi_{00}+\chi_A+\chi_B+\chi_C-(\chi_{00}+\chi_A+\chi_B+\chi_C)=2\chi_{00}$, so

$$\chi_{00}=\tfrac{1}{2}\big(\operatorname{Ind}_A^G\mathbf{1}+\operatorname{Ind}_B^G\mathbf{1}+\operatorname{Ind}_C^G\mathbf{1}-\operatorname{Ind}_1^G\mathbf{1}\big)\ ,$$

as a computation in GAP also finds. There is no contradiction with Brauer's theorem: G is itself elementary (a 2-group), so $\chi_{00}=\operatorname{Ind}_G^G\chi_{00}$ is allowed there, but G is not cyclic, so this term is not allowed in Artin's theorem. □

Solution to Exercise 3.6. For the first part use injectivity of the Wedderburn map; for the second use $sc^ks=c^{-k}$.

The map of Theorem 3.20 is an injective algebra homomorphism, and by Proposition 3.22 it sends E_{ab} to the elementary matrix e_{ab} in the ρ-factor. Since $e_{ab}e_{cd}=\delta_{bc}e_{ad}$ in $\mathrm{Mat}_d(\mathbb{C})$, both sides of $E_{ab}E_{cd}=\delta_{bc}E_{ad}$ have the same image, hence are equal. For S_3, $sc^ks=c^{-k}$ gives $(sc^i)(sc^j)=c^{j-i}$. With $E_{12}=\frac{1}{3}\sum_i\omega^i sc^i$ and $E_{21}=\frac{1}{3}\sum_j\omega^{-j}sc^j$ $(i,j\in\{0,1,2\})$,

$$E_{12}E_{21}=\tfrac{1}{9}\sum_{i,j}\omega^{i-j}c^{j-i}=\tfrac{1}{9}\sum_k\Big(\sum_i\omega^{-k}\Big)c^k=\tfrac{1}{3}(1+\omega^2c+\omega c^2)=E_{11}\ .$$

□

Solution to Exercise 3.7. For (a) the squaring map tells you which value to sum; for (b) multiply the rows of χ_2 and χ_5.

(a) $\nu(\chi_4)=\frac{1}{24}(\chi_4(1)+\chi_4(1)+4\chi_4(3b)+4\chi_4(3b)+6\chi_4(-1)+4\chi_4(3a)+4\chi_4(3a))=\frac{1}{24}(2+2-4-4-12-4-4)=-1$. (b) $\chi_2(g)$ is a scalar, so $g\mapsto\chi_2(g)\rho(g)$ is multiplicative, with character $\chi_2\chi_5$. Its values on $1,-1,6a,3a,4,3b,6b$ are 2, -2, $\omega^2\omega=1$, $\omega^2(-\omega)=-1$, 0, $\omega(-\omega^2)=-1$, $\omega\cdot\omega^2=1$, which is χ_4. The reflections r_1,r_2 have trace $1+\omega=-\omega^2$ and determinant ω, so they lie in class $3b$, where $\chi_2=\omega$. Hence χ_4 is realized by $\omega r_1,\omega r_2\in\mathrm{GL}_2(\mathbb{Z}[\omega])$, i.e. over K, while $\nu(\chi_4)=-1$ means it is not realisable over $\mathbb{R}$ [Eti+11, §4.1], so not over $\mathbb{Q}$. □

Solution to Exercise 3.8. Fermat's little theorem gives the polynomial.

x^3-x is nonzero and vanishes at $0,1,2\in\mathbb{F}_3$. In the proof of Proposition 3.26(a), the case $N=1$ uses that a nonzero polynomial has finitely many roots while $\mathbb{Q}$ is infinite, and the induction step applies the case $N=1$ to $f(q,x_N)$ on all of $\mathbb{Q}$; both fail over $\mathbb{F}_3$. The

same proof works with $\mathbb{Z}$ in place of $\mathbb{Q}$, since only infiniteness is used, so $\mathbb{Z}^N$ is Zariski dense in $\mathbb{C}^N$, and in (b) the span of $\Phi(\mathbb{Z}^N)$ already contains $\Phi(\mathbb{C}^N)$. □

Solution to Exercise 3.9. Multiplication by ω has an irreducible characteristic polynomial over $\mathbb{Q}$.

(a) In the basis $(1,\omega)$, multiplication by ω sends $1 \mapsto \omega$ and $\omega \mapsto \omega^2 = -1-\omega$, so c acts by $M = \left(\begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix}\right)$. For $X = \left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right)$, $XM = \left(\begin{smallmatrix} q & -p-q \\ s & -r-s \end{smallmatrix}\right)$ and $MX = \left(\begin{smallmatrix} -r & -s \\ p-r & q-s \end{smallmatrix}\right)$, so $XM = MX$ means $r = -q$ and $s = p+q$: $X = \left(\begin{smallmatrix} p & q \\ -q & p+q \end{smallmatrix}\right) = p \cdot 1 - q\, M$. The commutant is $\mathbb{Q}[M] \cong \mathbb{Q}[x]/(x^2+x+1) = K$, and X is the matrix of multiplication by $p - q\omega$. (b) For χ_4 realized over K on $V = K^2$, Proposition 3.27(a) still gives $V \otimes_{\mathbb{Q}} \mathbb{C} \cong \rho_0 \oplus \bar{\rho}_0$, but now the character is real, so $\bar{\rho}_0 \cong \rho_0$ and $V \otimes \mathbb{C} \cong \rho_0^{\oplus 2}$. By Schur's lemma $\operatorname{End}_{\mathbb{C}[G]}(\rho_0^{\oplus 2}) = \operatorname{Mat}_2(\mathbb{C})$, of dimension 4, so $\dim_{\mathbb{Q}} \operatorname{End}_{\mathbb{Q}[G]}(V) = 4$, as a computation in GAP confirms. □

Chapter 4

Solution to Exercise 4.1. Use $a^*a > 0$ and $a^*v = 0$ for $v \perp a$; for (b), solve $(r_1 - \omega)x = 0$.

(a) $sa = a + (\lambda-1)a\,\frac{a^*a}{a^*a} = \lambda a$, and for $v \perp a$, $sv = v + (\lambda-1)a\,\frac{a^*v}{a^*a} = v$. So s has the eigenvalue 1 on the hyperplane $a^\perp$ and λ on $\mathbb{C}a$, whence $\det s = \lambda$ and $\operatorname{Tr} s = n - 1 + \lambda$. With $P = aa^*/(a^*a)$ (so $P^* = P = P^2$), $s^*s = (1 + (\bar{\lambda}-1)P)(1 + (\lambda-1)P) = 1 + (\lambda + \bar{\lambda} - 2 + (\bar{\lambda}-1)(\lambda-1))P = 1 + (|\lambda|^2 - 1)P$, which is 1 when $|\lambda| = 1$. (b) $r_1 e_1 = e_1$, so the fixed line is $\mathbb{C}\binom{1}{0}$. For the eigenvalue ω, $(r_1 - \omega)\binom{x}{y} = \binom{(1-\omega)x+y}{0}$, so the eigenline is $\mathbb{C}\binom{1}{\omega-1}$. $\operatorname{Tr} r_1 = 1+\omega = -\omega^2$ and $\det r_1 = \omega$. A matrix $1 + (\omega-1)aa^*/(a^*a)$ has its fixed line orthogonal to its ω-eigenline for the standard Hermitian form, by (a); here $\binom{1}{0}^*\binom{1}{\omega-1} = 1 \neq 0$. Equivalently, r_1 is not unitary for the standard form; it is unitary for the averaged form $\sum_{g \in G_4} g^*g$. □

Solution to Exercise 4.2. Put both fractions over $(1-\lambda)(1-\bar{\lambda}) = |1-\lambda|^2$.

$\frac{1}{1-\lambda} + \frac{1}{1-\bar{\lambda}} = \frac{2-\lambda-\bar{\lambda}}{1-\lambda-\bar{\lambda}+\lambda\bar{\lambda}}$, and the denominator equals the numerator when $\lambda\bar{\lambda} = 1$. For $\lambda = -1$ each term is $\frac{1}{2}$. For G_4: $N = 8$, $|G| = 24$, so $\frac{N}{2|G|} = \frac{8}{48} = \frac{1}{6}$, and $\frac{(4-1)+(6-1)}{2\cdot 4\cdot 6} = \frac{8}{48} = \frac{1}{6}$. □

Solution to Exercise 4.3. Expand numerator and denominator separately to first order in $u = 1 - t$.

(a) $4 \cdot 6 = 24 = |G_4|$ and $(4-1) + (6-1) = 8$, the number of reflections found in Example 4.3. (b) $1 + t^{12} = 1 + (1-u)^{12} = 2 - 12u + O(u^2)$, $1 - t^6 = 6u(1 - \frac{5}{2}u + O(u^2))$, $1 - t^8 = 8u(1 - \frac{7}{2}u + O(u^2))$, so

$$\frac{1+t^{12}}{(1-t^6)(1-t^8)} = \frac{2-12u}{48u^2}(1+6u) + O(1) = \frac{1}{24u^2} + \frac{0}{u} + O(1)\ .$$

On the other side of Molien's formula, $g = 1$ contributes $\frac{1}{24}u^{-2}$; an element $g \neq 1$ of SL_2 has no eigenvalue 1 (its eigenvalues are μ, μ^{-1}, and $\mu = 1$ forces $g = 1$ since g is diagonalisable), so $\det(1-tg)^{-1}$ is regular at $t = 1$. Hence the u^{-1} coefficient is 0 on both sides, consistent with $N = 0$ in the proof of Proposition 4.6. □

Solution to Exercise 4.4. $|G| = \prod d_i$, $N = \sum(d_i - 1)$, $|Z| = \gcd(d_i)$; a hyperplane with reflections of order 3 carries exactly two of them.

(a) G_{25}: $6\cdot 9\cdot 12 = 648$, $5+8+11 = 24$, $\gcd = 3$. G_{26}: $6\cdot 12\cdot 18 = 1296$, $5+11+17 = 33$, $\gcd = 6$. G_{32}: $12\cdot 18\cdot 24\cdot 30 = 155520$, $11+17+23+29 = 80$, $\gcd = 6$. (b) The reflections in one hyperplane, with 1, form a cyclic group (they all act on the complementary line by roots of unity and fix the hyperplane); if all reflections have order 3 it has order 3 and contains exactly two reflections. So G_{25} has $24/2 = 12$ and G_{32} has $80/2 = 40$ reflecting hyperplanes. (c) $33 - 24 = 9$ reflections of order 2. A hyperplane cannot carry reflections of orders 2 and 3

at once, since then its cyclic group would have order 6 and contain reflections of order 6, and G_{26} has none. So the hyperplanes number $12 + 9 = 21$, as computed. □

Solution to Exercise 4.5. A diagonalisable matrix with a single eigenvalue ξ is $\xi \cdot 1$.

(a) $\gcd(12, 18, 24, 30) = 6$, so by the easy half of Proposition 4.8 every central element has order dividing 6 and $|Z(G_{32})|$ divides 6. (b) c has finite order, so it is diagonalisable, and its order is the least common multiple of the orders of its eigenvalues; $e^{2\pi i/30}$ has order 30, and the others have order dividing 30, so c has order 30. The eigenvalues of c^5 are $e^{2\pi i \cdot 5k/30} = e^{2\pi i k/6}$ with $k \equiv 1 \pmod 6$ in all four cases, so $c^5 = \zeta \cdot 1$. Hence $\mu_6 \cdot 1 \subseteq Z(G_{32})$, and with (a), $Z(G_{32}) = \mu_6$. (c) $c^{10} = \zeta^2 = \omega$ and $c^{15} = \zeta^3 = -1$. For G_4, $c = r_1 r_2 = \left(\begin{smallmatrix} 0 & 1 \\ -\omega^2 & \omega \end{smallmatrix}\right)$ has trace ω and determinant ω^2, hence eigenvalues the roots of $x^2 - \omega x + \omega^2$, which are $-1 = e^{-2\pi i \cdot 3/6}$ and $\zeta = e^{-2\pi i \cdot 5/6}$ (the exponents $3, 5$ are $d_i - 1$ for $d = 4, 6$). So c has order 6, and c^3 has both eigenvalues -1: $c^3 = -1$ generates $Z(G_4)$, of order $2 = \gcd(4, 6)$. □

Solution to Exercise 4.6. Compute dim of the κ-eigenspace of Z on $\mathbb{C}[G]$ twice.

Z acts on $\mathbb{C}[G]$ by left multiplication freely, permuting the basis G in $|G| / |Z|$ orbits of size $|Z|$; on the span of one orbit Z acts by its regular representation, which contains each character of Z once. So the κ-eigenspace of Z has dimension $|G| / |Z|$. On the other hand $\mathbb{C}[G] = \bigoplus_\chi (\chi\text{-isotypic part})$ with the χ-part of dimension $\chi(1)^2$ (Theorem 3.3), and Z acts on it by the scalar κ_χ. Comparing, $\sum_{\kappa_\chi = \kappa} \chi(1)^2 = |G| / |Z|$. For G_{32}: $2 \cdot 16 + 5 \cdot 400 + 2 \cdot 1296 + 3 \cdot 3600 + 4096 + 6400 = 32 + 2000 + 2592 + 10800 + 4096 + 6400 = 25920 = 155520/6$. For G_4 and the sign: χ_4, χ_5, χ_6 give $4 + 4 + 4 = 12 = 24/2$. □

Solution to Exercise 4.7. A representation with central character κ_0 has the value $\zeta\chi(1)$ at $\zeta \cdot 1$.

(a) At $z = \zeta \cdot 1$: $\bar\rho(z) = \zeta^{-1}$ and $\det(z)^2 = (\zeta^4)^2 = \zeta^8 = \zeta^2$, so z acts on $\bar\rho \otimes \det^2$ by $\zeta^{-1}\zeta^2 = \zeta = \kappa_0(z)$; since $\zeta \cdot 1$ generates the centre, the central character is κ_0. (b) $\chi(\zeta \cdot 1) = \zeta\chi(1)$ is not real, so χ is not real-valued, while a representation over $\mathbb{R}$ (a fortiori over $\mathbb{Q}$) has real trace. And $\mathbb{Q}(\chi) \ni \zeta\chi(1)/\chi(1) = \zeta$, so $\mathbb{Q}(\chi) \supseteq \mathbb{Q}(\zeta) = K$. (c) Character values in K say only that the simple factor of $K[G]$ belonging to χ has centre K; it is $\mathrm{Mat}_n(D)$ for a division algebra D with centre K, and χ is realisable over K exactly when $D = K$, i.e. when the Schur index of χ over K is 1 (Section 3.6, [Ser77, Ch. 12]). The splitting-field theorem [Ben76], [GM03, Thm. 4.2] says that the character field K of the reflection representation splits G_{32}, i.e. all these Schur indices are 1. For G_4, χ_4 is rational with Schur index 2 over $\mathbb{Q}$ (the factor of $\mathbb{Q}[G_4]$ is the quaternion algebra), but over K it is realized by $\omega r_1, \omega r_2$, so its index over K is 1, as the theorem predicts for $K_{G_4} = K$. □

Solution to Exercise 4.8. Determinants are multiplicative; the central -1 multiplies every eigenvalue by -1.

(a) $\omega^6(\omega^2)^6 \cdot 1 = \omega^{18} = 1$. With seven factors ω and five ω^2 the product is $\omega^{7+10} = \omega^{17} = \omega^2 \neq 1$, so no such tuple has product 1. (b) For the type of $\boldsymbol{\gamma}$: $-8 + 12 \cdot 1 + 1 \cdot 2 = 6$. Seven reflections of each kind: $-8 + 14 \cdot 1 = 6$. (c) $155520/3888 = 40$ and $155520/1728 = 90$, by the orbit-stabilizer theorem. (d) If $\gamma \in \mathcal{I}_{90}$, then $-\gamma$ has eigenvalues $-1, -1, 1, 1$: it is an involution other than ± 1, and $\mathcal{I}_{90}$ is the only such class, so $-\mathcal{I}_{90} = \mathcal{I}_{90}$. If $s \in \mathcal{R}_\omega$, then $-s$ has eigenvalues $-1, -1, -1, -\omega$, with $-\omega = e^{5\pi i/3}$ of order 6; so $-s$ has order 6 and no eigenvalue 1, hence $\mathrm{rk}(-s - 1) = 4$ and it is not a reflection. □

Chapter 5

Solution to Exercise 5.1. Riemann–Hurwitz for (a); for (b), find the stabilizers and use the Lefschetz count with no fixed points.

(a) $-2 + 12 \cdot \frac{2}{3} + \frac{1}{2} = -2 + 8 + \frac{1}{2} = \frac{13}{2}$, so $2g(C) - 2 = 155520 \cdot \frac{13}{2} = 1010880$ and $g(C) = 505441$. It is an integer because $|G|$ is even; in general $|G| / e_i$ is the number of points over t_i, so $|G|(1 - 1/e_i)$ is always an integer.

(b) By Theorem 5.2 the stabilizers of points of C are the conjugates of the $\langle\gamma_i\rangle$. Their nontrivial elements are reflections of order 3 (γ_i and γ_i^2, with $\mathrm{rk}(g-1)=1$) or the involution ($\mathrm{rk}(g-1)=2$), while a nontrivial scalar $\zeta^j I_4$ has $\mathrm{rk}(g-1)=4$. So no nontrivial element of Z fixes a point: $C\to C/Z$ is unramified of degree 6, and $2g(C)-2=6\,(2g(C/Z)-2)$ gives $g(C/Z)=1+1010880/12=84241$. For a character κ of Z, the Lefschetz formula as in the proof of Theorem 5.10(a) gives $\chi_{H^1}(z)=2$ for $z\neq 1$ (no fixed points) and $2g(C)$ for $z=1$, so

$$\langle\chi_{H^1}|_Z,\kappa\rangle=\frac{1}{6}\Big(2g(C)+2\sum_{z\neq 1}\overline{\kappa(z)}\Big)=\frac{2g(C)-2}{6}=2g(C/Z)-2=168480$$

for $\kappa\neq 1$, using $\sum_{z\neq 1}\overline{\kappa(z)}=-1$. The central character κ_0 is nontrivial (Z consists of scalar matrices and acts on ρ by these scalars), so the κ_0-part has dimension 168480.

(c) The κ_0-part is $\bigoplus \rho'\otimes M_{\rho'}$ over the ρ' with central character κ_0, of dimension $\sum \dim\rho'\dim M_{\rho'}=4\cdot 6+4\cdot 42+20\cdot(102+102+138+174+174)+36\cdot(162+270)+60\cdot(342+378+450)+64\cdot 384+80\cdot 552=24+168+13800+15552+70200+24576+44160=168480$. □

Solution to Exercise 5.2. Compute the product of the two changed entries, and note that they are conjugates of the old ones.

(a) Only positions $j,j+1$ change, and $(\gamma_j\gamma_{j+1}\gamma_j^{-1})\gamma_j=\gamma_j\gamma_{j+1}$, so the product is unchanged. The new entries lie in the old subgroup, and conversely γ_j is a new entry and $\gamma_{j+1}=\gamma_j^{-1}(\gamma_j\gamma_{j+1}\gamma_j^{-1})\gamma_j$, so the subgroups agree. The new entry $\gamma_j\gamma_{j+1}\gamma_j^{-1}$ is conjugate to γ_{j+1}, and γ_j moves to position $j+1$, so the multiset of classes is unchanged.

(b) $\sigma_j^{-1}\colon(\dots,\gamma_j,\gamma_{j+1},\dots)\mapsto(\dots,\gamma_{j+1},\gamma_{j+1}^{-1}\gamma_j\gamma_{j+1},\dots)$. Applied after σ_j: the pair $(\gamma_j\gamma_{j+1}\gamma_j^{-1},\gamma_j)$ goes to $(\gamma_j,\gamma_j^{-1}\gamma_j\gamma_{j+1}\gamma_j^{-1}\gamma_j)=(\gamma_j,\gamma_{j+1})$.

(c) If $\gamma_j=\gamma_{j+1}=x$, then σ_j gives $(xxx^{-1},x)=(x,x)$. If $\gamma_{j+1}=x^{-1}$, then σ_j gives $(xx^{-1}x^{-1},x)=(x^{-1},x)$, and σ_j again gives $(x^{-1}xx,x^{-1})=(x,x^{-1})$. □

Solution to Exercise 5.3. Parity decides the last entry; a subgroup of S_3 generated by transpositions is either of order 2 or all of S_3.

A product of five transpositions is an odd permutation, and the odd permutations of S_3 are exactly the three transpositions. So for each of the $3^5=243$ choices of $x_1,\dots,x_5$, the element $x_6=(x_1\cdots x_5)^{-1}$ is a transposition, and the tuple has product 1. Two distinct transpositions generate S_3, so the tuple fails to generate only if all six entries are the same transposition, which gives 3 tuples (with product $x^6=1$). Hence 240 generating tuples. An element of S_3 commuting with two distinct transpositions is the identity, so conjugation acts freely, and there are $240/6=40$ classes. □

Solution to Exercise 5.4. Do $j=0$ by symmetry and then compare consecutive j.

Let $L(j)=\frac{1}{e}\sum_{k=1}^{e-1}\varepsilon^{-jk}/(\varepsilon^k-1)$. For $j=0$, pair k with $e-k$: $\frac{1}{\varepsilon^k-1}+\frac{1}{\varepsilon^{-k}-1}=\frac{1}{\varepsilon^k-1}-\frac{\varepsilon^k}{\varepsilon^k-1}=-1$, so the sum over $k=1,\dots,e-1$ is $-\frac{e-1}{2}$ and $L(0)=-\frac{e-1}{2e}$. Next, $\frac{\varepsilon^{-k}-1}{\varepsilon^k-1}=-\varepsilon^{-k}$, so

$$L(j+1)-L(j)=\frac{1}{e}\sum_{k=1}^{e-1}\varepsilon^{-jk}\frac{\varepsilon^{-k}-1}{\varepsilon^k-1}=-\frac{1}{e}\sum_{k=1}^{e-1}\varepsilon^{-(j+1)k}=-\frac{1}{e}(0-1)=\frac{1}{e}$$

for $1\le j+1\le e-1$, since the full sum over $k=0,\dots,e-1$ vanishes. Hence $L(j)=\frac{j}{e}-\frac{e-1}{2e}$ for $0\le j<e$. □

Solution to Exercise 5.5. Use the class sizes $1,3,2$ for (a) and the closed formulas of Theorem 5.10 for (b), (c).

(a) $\chi_{1,0}(1)=g=4$; for a transposition, six fixed points with $\epsilon=-1$ give $1+6\cdot(-2)^{-1}=-2$; a 3-cycle has no fixed points and $\chi_{1,0}=1$. Then $\langle\chi_{1,0},\mathrm{sgn}\rangle=\frac{1}{6}(4+3\cdot 2+2\cdot 1)=2$, $\langle\chi_{1,0},\rho_2\rangle=\frac{1}{6}(8+0-2)=1$, $\langle\chi_{1,0},\mathbf{1}\rangle=\frac{1}{6}(4-6+2)=0$: $H^{1,0}\cong\mathrm{sgn}^{\oplus 2}\oplus\rho_2$.

(b) With $2k$ transpositions ($k \geq 2$ for the tuple to generate): $2g-2 = 6(-2+k)$, $g = 3k-5$. A transposition acts on sgn by -1 ($N_0 = 0$, $N_1 = 1$) and on ρ_2 with eigenvalues $1, -1$. So $\dim M_{\mathrm{sgn}} = -2+2k$ with $h^{1,0} = -1+2k\cdot\frac{1}{2} = k-1$, and $\dim M_{\rho_2} = -4+2k$ with $h^{1,0} = -2+k$; both are balanced. Check: $(2k-2)+2(2k-4) = 6k-10 = 2g$. For $k=3$ this is Example 5.11.

(c) The tuple $(a,a,b,b,(123),(132))$ has product 1 and generates. Now $2g-2 = 6(-2+4\cdot\frac{1}{2}+2\cdot\frac{2}{3}) = 8$, $g=5$. A 3-cycle acts trivially on sgn and with eigenvalues ω, ω^2 on ρ_2 (rank 2). So $\dim M_{\mathrm{sgn}} = -2+4 = 2$, $h^{1,0} = -1+4\cdot\frac{1}{2} = 1$; $\dim M_{\rho_2} = -4+4+2\cdot 2 = 4$, $h^{1,0} = -2+4\cdot\frac{1}{2}+2(\frac{1}{3}+\frac{2}{3}) = 2$. Check: $2+2\cdot 4 = 10 = 2g$. □

Solution to Exercise 5.6. Apply Theorem 5.10(b) to the new exponents, and in general replace k by $e_i - k$.

(a) For exponents $(3,5,2,2)$, $h^{1,0}(M_{\chi^n}) = -1+\sum_i \langle na_i/6\rangle$ gives, for $n = 1,\dots,5$: $-1+\frac{3+5+2+2}{6} = 1$; $-1+\frac{0+4+4+4}{6} = 1$; $-1+\frac{3+3+0+0}{6} = 0$; $-1+\frac{0+2+2+2}{6} = 0$; $-1+\frac{3+1+4+4}{6} = 1$. The dimensions are unchanged, $2,1,0,1,2$, so the Hodge numbers $(h^{1,0}, h^{0,1})$ are $(1,1),(1,0),(0,0),(0,1),(1,1)$, against $(1,1),(0,1),(0,0),(1,0),(1,1)$ for $(3,1,4,4)$: exchanged.

(b) The eigenvalues of $\rho'(\gamma_i^{-1})$ are the inverses of those of $\rho'(\gamma_i)$, so $N_{i,k}$ becomes N_{i,e_i-k} for $k \neq 0$ and $N_{i,0}$ is unchanged; (a) is unchanged. In (b), $\sum_k N_{i,k}k/e_i$ becomes $\sum_{k\neq 0} N_{i,k}(1-k/e_i) = (d-N_{i,0}) - \sum_k N_{i,k}k/e_i$. Summing over i, the new $h^{1,0}$ is $\delta - d + \sum_i (d-N_{i,0}) - (h^{1,0}-\delta+d) = \dim M_{\rho'} - h^{1,0}$, by (a). □

Solution to Exercise 5.7. Write out Definition 5.14 for $V = \mathbb{C}$.

(a) Let $s = \#\{i : c_i \neq 1\}$; $s \geq 2$, since the c_i multiply to 1 and are not all 1. Here $\mathrm{im}(g_i - 1)$ is $\mathbb{C}$ if $c_i \neq 1$ and 0 otherwise, so the u_i with $c_i \neq 1$ are free, subject to $\sum_i c_1\cdots c_{i-1}u_i = 0$, one nonzero linear condition: $\dim Z^1_{\mathrm{par}} = s-1$. Coboundaries $v \mapsto ((c_i-1)v)$ form a line because some $c_i \neq 1$. So $\dim H^1_{\mathrm{par}} = s-2$.

(b) For the eigenspace V_n of a cyclic cover with data $(m; a_1,\dots,a_r)$ the local monodromies are $c_i = \zeta_m^{\pm na_i}$ (the sign depends on the direction convention), and $c_i \neq 1$ exactly when $na_i \not\equiv 0$; so $s-2 = r_n - 2 = \dim V_n$, as in Section 5.4.

(c) Without the parabolic condition all r components u_i are free, subject to the one condition, so $\dim Z^1 = r-1$ and $\dim H^1(U,\mathbb{L}) = r-2$ (also $= -\chi(U)$, as $H^0 = H^2 = 0$). The parabolic condition removes the $r-s$ directions u_i at the punctures with $c_i = 1$: the classes that do not extend across those punctures (for $\mathbb{L}$ trivial near t_i, the classes with a nonzero "residue" there, like $\mathrm{d}x/(x-t_i)$). □

Solution to Exercise 5.8. Complex conjugation exchanges M_ρ and $M_{\bar\rho}$; evaluation at u is injective.

(a) $\dim_{\mathbb{Q}} \mathcal{B}_C = \dim_{\mathbb{C}}(\mathcal{B}_C \otimes \mathbb{C}) = \dim M_\rho + \dim M_{\bar\rho}$. Complex conjugation c on $H^1(C,\mathbb{C}) = H^1(C,\mathbb{Q})\otimes\mathbb{C}$ commutes with G, so $f \mapsto c\circ f\circ c$ maps $\mathrm{Hom}_G(\rho, H^1(C,\mathbb{C}))$ antilinearly and bijectively onto $\mathrm{Hom}_G(\bar\rho, H^1(C,\mathbb{C}))$. Hence $\dim_{\mathbb{Q}}\mathcal{B}_C = 2\dim M_\rho$ and $\dim_K \mathcal{B}_C = \dim M_\rho$.

(b) ev_u is injective (Proposition 3.27(e)), so $\dim_{\mathbb{Q}}\mathcal{B}_C = \dim_{\mathbb{Q}} \mathrm{ev}_u(\mathcal{B}_C)$. After $\otimes\mathbb{C}$, $f = f' + f''$ with $f' \in M_\rho$, $f'' \in M_{\bar\rho}$ and $f(u) = f'(u') + f''(u'')$; under $H^1(C,\mathbb{C})_\rho \cong \rho\otimes M_\rho$ the first term is $u'\otimes f'$. Since u is rational and nonzero, $u'' = \overline{u'}$ and both are nonzero, so the image is $u'\otimes M_\rho \oplus u''\otimes M_{\bar\rho}$, of dimension $2\dim M_\rho$. For the branching type of the family, $\dim M_\rho = 6$ (Example 5.13), so $\dim_{\mathbb{Q}}\mathcal{B}_C = 12 = 2\cdot 6$, and A_C is a sixfold. □

Chapter 6

Solution to Exercise 6.1. Count parameters and use Riemann–Roch on $\mathbb{P}^1$.

(a) Ordered 13-tuples of distinct points form an open subset of $(\mathbb{P}^1)^{13}$, of dimension 13. The group $\mathrm{PGL}_2(\mathbb{C})$ has dimension 3 and acts freely as soon as there are at least 3 points (a Möbius map fixing three points is the identity), so the quotient has dimension $13-3 = 10$. The line bundle $\Omega^{\otimes 2}_{\mathbb{P}^1}(t_1+\cdots+t_{13})$ has degree $2\cdot(-2)+13 = 9 \geq 0$, so $h^0 = 9+1 = 10$ (on

$\mathbb{P}^1$, $h^0(\mathcal{O}(d)) = d+1$ for $d \geq 0$). The two numbers agree because the second space is the cotangent space of the first.

(b) If $\mathcal{P}$ is dominant, then at a generic point $d\mathcal{P}$ is surjective onto the nine-dimensional TX^+ of (6.1), so its kernel in the 13-dimensional tangent space of $\mathcal{H}$ has dimension $13-9=4$, and by the constant rank theorem the fibre through a generic point is a smooth fourfold near that point. The kernel contains the three Möbius directions, along which the curve does not change; modulo them it has dimension $10-9=1$. □

Solution to Exercise 6.2. Row-reduce, or use that the columns of a non-zero minor are independent.

If a $k \times k$ minor, with rows I and columns J, is non-zero, then the k columns J restricted to the rows I are linearly independent, hence so are the full columns J: the rank is at least k. Conversely, if the rank is at least k, choose k independent columns; the $p \times k$ matrix they form has rank k, so it has k independent rows (row rank equals column rank), and the $k \times k$ submatrix on those rows is invertible, so its determinant is non-zero. For a 9×13 matrix, a non-zero 9×9 minor gives rank ≥ 9, and the rank is at most the number of rows, 9. □

Solution to Exercise 6.3. Diagonal examples for (a); for (b), the generic rank is the maximum.

(a) $M(s) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & s \end{smallmatrix}\right)$ has rank 1 at $s=0$ and 2 elsewhere; $M(s) = \left(\begin{smallmatrix} 1 & s \\ 0 & 0 \end{smallmatrix}\right)$ has rank 1 everywhere. If $\operatorname{rk} M(0) = 2$, then $\det M(0) \neq 0$, and by continuity $\det M(s) \neq 0$ for s near 0, so $\operatorname{rk} M(s) = 2$ there: rank cannot drop when one moves away from a point, only when one moves into it.

(b) By Proposition 6.3 applied to the Jacobian matrix of f in local coordinates (on overlapping charts the ranks agree, and S is connected), the rank of df equals its maximum r on a dense open set, the generic rank. Every single $\operatorname{rk} df_s$ is at most this maximum. □

Solution to Exercise 6.4. Expand F and G to first order, then differentiate the logarithm separately.

The $n=1$ term of G is $\binom{2}{1}^2 \frac{\lambda}{16}(H_2 - H_1) = 4 \cdot \frac{\lambda}{16} \cdot \frac{1}{2} = \frac{\lambda}{8}$, and $F(\lambda) = 1 + \frac{1}{4}\lambda + O(\lambda^2)$ (the $n=1$ term of $\sum \binom{2n}{n}^2 (\lambda/16)^n$). So $G/F = \frac{\lambda}{8} + O(\lambda^2)$, and from $\tau = \mathrm{i}F(1-\lambda)/F(\lambda)$ and (6.2),

$$\tau(\lambda) = \frac{\mathrm{i}}{\pi}\Big(\log 16 - \log\lambda - 4\frac{G(\lambda)}{F(\lambda)}\Big) = -\frac{\mathrm{i}}{\pi}\log\lambda + \frac{4\mathrm{i}\log 2}{\pi} - \frac{\mathrm{i}}{2\pi}\lambda + O(\lambda^2).$$

Applying $\lambda\,\mathrm{d}/\mathrm{d}\lambda$ gives $-\frac{\mathrm{i}}{\pi} - \frac{\mathrm{i}}{2\pi}\lambda + O(\lambda^2)$, which is (6.3). The $O(\lambda^2)$ is legitimate because G/F is holomorphic at 0, so its expansion can be differentiated term by term. Under $\lambda = \mu^2$ one has $\mu\,\mathrm{d}/\mathrm{d}\mu = 2\lambda\,\mathrm{d}/\mathrm{d}\lambda$, so $\mu\,\mathrm{d}\tau/\mathrm{d}\mu = -\frac{2\mathrm{i}}{\pi} - \frac{\mathrm{i}}{\pi}\mu^2 + O(\mu^4)$: the constant term doubles (the monodromy around $\mu=0$ is the square of that around $\lambda=0$, so its logarithm doubles), and the first-order term moves to order μ^2. This is why, at a node whose plumbing parameter is a root of the natural coordinate, first-order contributions appear at a fractional order. □

Solution to Exercise 6.5. Contract q with v to get a meromorphic 1-form and apply the residue theorem.

Write $q = f(x)\,\mathrm{d}x^2$; then $v \lrcorner q = v(x)f(x)\,\mathrm{d}x$ is a meromorphic 1-form on $\mathbb{P}^1$. At ∞, with $w = 1/x$, holomorphy of q means $f(x) = O(x^{-4})$, and $v\,\mathrm{d}x = -(a + b/w + c/w^2)\,\mathrm{d}w/w^2$, so $vf\,\mathrm{d}x = O(w^4)\cdot O(w^{-4})\,\mathrm{d}w$ is holomorphic. By the residue theorem the residues at the p_j add up to 0. Near p_j write $q = \sum_{k\geq -2} q_k X^k\,\mathrm{d}X^2$ with $X = x - p_j$, so $R_j = q_{-2}$ and $T_j = q_{-1}$, and $v = v(p_j) + v'(p_j)X + \ldots$; the residue of $vf\,\mathrm{d}x$ is $v(p_j)q_{-1} + v'(p_j)q_{-2} = (a + bp_j + cp_j^2)T_j + (b + 2cp_j)R_j$. Summing over j gives the identity. Applied to each row $q = \phi_a\eta_b$ with $v \in \{1, x, x^2\}$, it gives three linear relations among the columns T_j, R_e of $\tilde{M}$ at $t=0$ (the same three hold at every point, with the moving positions). The thirteen columns of M^* (8 of type T, 4 of type R, one F) therefore span at most $13-3=10$ dimensions, matching the ten parameters of the family. □

Solution to Exercise 6.6. Use the skew-adjointness twice, then the structure of split spaces.

(a) For $x, y \in V$, $h(Nx, Ny) = -h(x, N^2 y) = 0$, so $\operatorname{im} N$ is totally isotropic. An isotropic subspace W lies in $W^\perp$, which has dimension $6 - \dim W$ by non-degeneracy, so $\dim W \leq 3$. Since N is K-linear and defined over $\mathbb{Q}$, $\operatorname{im} N$ is a K-subspace of the rational space V.

(b) A six-dimensional Hermitian space with a totally isotropic K-subspace of dimension 3 is split by definition (Definition 2.1), hence an orthogonal sum of three hyperbolic planes [Del82, proof of Cor. 4.2], and by Exercise 2.3 its determinant is $(-1)^3 = -1$ modulo norms.

(c) The monodromy logarithms N_e around the boundary components through a point commute, are nilpotent with $N_e^2 = 0$ (Picard–Lefschetz, all vanishing cycles at one node disjoint) and $N_e N_f = 0$ (disjoint vanishing cycles at different nodes, so their sum N also satisfies $N^2 = 0$), are skew for h, commute with K and are defined over $\mathbb{Q}$. If $\dim_K \operatorname{im} N = 3$, (b) would give $\det h \equiv -1$, contradicting $\det h \equiv -2$ (Example 2.13); here $-1 \not\equiv -2$ because 2 is not a norm from K. So $\dim_K \operatorname{im} N \leq 2$. □

Solution to Exercise 6.7. Multiply the midpoints, then bound the error term by term.

For $x = 3 + u$, $y = 2 + v$ with $|u| \leq 0.1$, $|v| \leq 0.05$: $xy = 6 + 3v + 2u + uv$, and $|3v + 2u + uv| \leq 3 \cdot 0.05 + 2 \cdot 0.1 + 0.1 \cdot 0.05 = 0.355$. So the ball is $[6 \pm 0.355] = [5.645, 6.355]$. The exact range is $[2.9 \cdot 1.95,\ 3.1 \cdot 2.05] = [5.655, 6.355]$, so the upper end is sharp and the lower end is overestimated by 0.01: the bound treats u and v as independent worst cases, but uv is positive at both extreme corners, which the symmetric radius cannot express. Arb returns $[6 \pm 0.356]$: slightly larger, because 0.1 and 0.05 are not binary numbers and their conversions are rounded outwards. A ball can never be too small because every step bounds the error of exact operations on all points of its inputs, and rounding is always done so as to enlarge the radius. It can be much too large because these worst cases compound: in a long computation, or one with cancellation (Example 6.8 at 53 bits, where the radius is 10^{16} times the true value), the radius grows although the true error does not; the cure is more precision, not a different conclusion. □

Chapter 7

Solution to Exercise 7.1. A vector is in $\rho'^{(H,\psi)}$ as soon as the generator acts on it by $\psi(g)$; for (b) use the eigenvalues $1, \omega$ of r_1.

(a) Since $H = \langle g \rangle$, a vector x satisfies $\rho'(h)x = \psi(h)x$ for all $h \in H$ if and only if $\rho'(g)x = \psi(g)x$. So $\rho'^{(H,\psi)}$ is the $\psi(g)$-eigenspace of $\rho'(g)$, and $\rho'(g)$ is diagonalisable (finite order), so its dimension is the multiplicity of $\psi(g)$ as an eigenvalue. By Theorem 3.10 this dimension is $\langle \operatorname{Res}_H \rho', \psi \rangle = v_{H,\psi}(\rho')$.

(b) $g = -r_1$ has order 6 because r_1 has order 3 and -1 is central of order 2, and $g^3 = -r_1^3 = -1$, so $Z \leq H$. A character of H is fixed by $\psi(g)$, a sixth root of unity, and $\psi(-1) = \psi(g)^3 = -1$ leaves $\psi(g) \in \{-1, -\omega, -\omega^2\}$. The eigenvalues of $r_1 = \left(\begin{smallmatrix} 1 & 1 \\ 0 & \omega \end{smallmatrix}\right)$ are $1, \omega$, so those of $\rho(g)$ are $-1, -\omega$. By (a), $v_{H,\psi}(\rho) = 1$ for $\psi(g) = -1$ and for $\psi(g) = -\omega$, and $v_{H,\psi}(\rho) = 0$ for $\psi(g) = -\omega^2$. The character with $\psi(g) = -1$ has order $m = 2$: it is trivial on $g^2 = r_1^2$, of order 3. These are the three rows (C_6, ψ_2), (C_6, ψ_6), (C_6, ψ_6') of Example 7.7, whose last entries $1, 1, 0$ agree. □

Solution to Exercise 7.2. Count each value against its negative, and look separately at $m/2$.

(a) Fully paired: $(2, 4), (2, 4), (3, 3)$, where $3 = 6/2$ occurs twice. (b) Not fully paired: 1 occurs once and $-1 = 5$ not at all. (c) Not fully paired: 4 and $8 = -4$ pair, but $6 = 12/2$ occurs once, an odd number of times. (d) Fully paired: 324 pairs $(4, 8)$, and 6 occurs 6 times, giving three pairs $(6, 6)$. If t is a unit, $a \mapsto ta$ is a bijection of $\mathbb{Z}/m \smallsetminus \{0\}$ that commutes with $a \mapsto -a$, so it carries the pairs $(b, -b)$ to pairs $(tb, -tb)$; for m even, t is odd and $t \cdot m/2 \equiv m/2$, so the number of entries equal to $m/2$ does not change. Replacing ψ by ψ^t multiplies every exponent by t, since $\psi^t(h) = \zeta_m^{ta}$ when $\psi(h) = \zeta_m^a$. □

Solution to Exercise 7.3. Use $\langle x\rangle + \langle -x\rangle = 1$ for $x \notin \mathbb{Z}$.

(a) Arrange the exponents as pairs $(b, -b)$. For a unit t, $tb \not\equiv 0$, so $\langle tb/m\rangle + \langle -tb/m\rangle = 1$; a pair $(m/2, m/2)$ also contributes $\frac{1}{2} + \frac{1}{2} = 1$. With $k = r/2$ pairs the sum is $r/2$.

(b) $1 + 3 + 4 + 4 = 12 \equiv 0 \bmod 6$, and $\sum\langle a_i/6\rangle = \frac{1+3+4+4}{6} = 2 = r/2$. For $t = 5$ the exponents become $(5, 3, 2, 2)$ with $\sum\langle a_i/6\rangle = \frac{12}{6} = 2$. Not fully paired by the previous exercise.

(c) $(\mathbb{Z}/6)^\times = \{1, 5\}$, and for $t = 5$, $\sum_i\langle -a_i/6\rangle = \sum_i(1 - \langle a_i/6\rangle) = r - \sum_i\langle a_i/6\rangle$ because no a_i is 0. So the condition for $t = 5$ is the condition for $t = 1$. □

Solution to Exercise 7.4. Riemann–Hurwitz without ramification, and the multiplicity formula of Proposition 7.4 with $s = 0$.

With no branch points, $2g(X) - 2 = m(2g(Y) - 2)$, i.e. $g(X) = m(g(Y) - 1) + 1$. In the proof of Proposition 7.4 the multiplicity of a non-trivial character χ^t (faithful or not) is $2g(Y) - 2 + \#\{i : \chi^t|_{S_i} \neq 1\} = 2g(Y) - 2$, because there are no stabilizers. The Hodge numbers $g(Y) - 1 + \sum_i\langle \pm ta_i/m\rangle$ have empty sums, so both equal $g(Y) - 1$: étale covers are balanced. Pieces 5, 7, 8: $24 \cdot 2340 + 1 = 56161$, $24 \cdot 4212 + 1 = 101089$, $12 \cdot 4680 + 1 = 56161$, as in Table 7.1, and $\dim W = 2g(Y) - 2 = 4680, 8424, 9360$. □

Solution to Exercise 7.5. Use $\dim M = (8, 2, 2)$ for (ρ_a, ρ_b, ρ).

(Z, sgn): $2 \cdot 8 + 2 \cdot 2 + 2 \cdot 2 = 24 = 2 \cdot 13 - 2$. (C_4, ψ_4): $8 + 2 + 2 = 12 = 2 \cdot 7 - 2$. (C_6, ψ_2): $0 + 2 + 2 = 4 = 2 \cdot 3 - 2$. (C_6, ψ_6): $8 + 0 + 2 = 10 = 2 \cdot 3 - 2 + 6$. (C_6, ψ_6'): $8 + 2 + 0 = 10$. The centre acts freely, so $C \to C/Z$ is étale of degree 2 and $48 = 2(2g(Y) - 2)$ gives $g(Y) = 13$. A relation with (C_6, ψ_6): $(1, 0, 1) - (1, 1, 1) + (0, 1, 1) = (0, 0, 1)$, i.e. $e_\rho = v_{C_6,\psi_6} - v_{C_4,\psi_4} + v_{C_6,\psi_2}$, with $10 - 12 + 4 = 2 = \dim M_\rho$. It uses two étale pieces, over bases of genus 7 and 3. □

Solution to Exercise 7.6. For (c), count the points of X_9 over the points of Y described in Section 7.4.

(a) $2 \cdot 77 - 2 + 160 = 312$ and $42 + 270 = 312$. (b) $2g(Y) - 2 = -80 + 12 \cdot 9 \cdot 2 + 16 = 152$, so $g(Y) = 77$; $2g(X_9) - 2 = 6 \cdot 152 + 78 \cdot 4 + 4 \cdot 3 + 78 \cdot 4 = 912 + 312 + 12 + 312 = 1548$, so $g(X_9) = 775$.

(c) $\ker\psi_9$ has index 6 in H_9, so $X_9 \to \mathbb{P}^1$ has degree $6 \cdot 40 = 240$. Over a point t_i of a reflection, Y has 13 points of index 1, each a branch point of $X_9 \to Y$ with stabilizer of order 3 (so $6/3 = 2$ points of X_9 above it), and 9 points of index 3, unbranched in $X_9 \to Y$ (6 points each): $13 \cdot 2 + 9 \cdot 6 = 80 = 240/3$ points, all of index 3. Over the involution, Y has 8 points of index 1, four with exponent 3 (3 points each) and four unbranched (6 each), and 16 points of index 2, unbranched (6 each): $12 + 24 + 96 = 132$ points. Riemann–Hurwitz: $2g(X_9) - 2 = 240 \cdot (-2) + 12 \cdot (240 - 80) + (240 - 132) = -480 + 1920 + 108 = 1548$, as in (b). □

Solution to Exercise 7.7. Split each row into the entries with $c_j = +1$ and those with $c_j = -1$; for (c), use that $\psi|_Z = \kappa_0$.

(a) Row k, plus side minus minus side: ρ_1: $1 - 0 = 1$ (only v_{11}); ρ_2: $1 - 1$ (v_9 against v_{10}); ρ_3: $3 - 3$; ρ_4: $2 - 2$; ρ_5: $4 - 4$; ρ_6: $4 - 4$; ρ_7: $5 - 5$; ρ_8: $5 - 5$; ρ_9: $7 - 7$; ρ_{10}: $9 - 9$; ρ_{11}: $9 - 9$; ρ_{12}: $11 - 11$; ρ_{13}: $10 - 10$; ρ_{14}: $14 - 14$. So $\sum_j c_j v_j = e_\rho$. (b) For instance $d_9 = 42 + 270 = 312$ and $d_{11} = 6 + 162 + 342 + 450 = 960$; all eleven values agree with the column d: $2520, 2208, 3168, 3726, 4680, 7128, 8424, 9360, 312, 14256, 960$. (c) For instance $\sum_k v_9(\rho_k)\dim\rho_k = 4 + 36 = 40 = 155520/3888$ and $\sum_k v_{11}(\rho_k)\dim\rho_k = 4 + 36 + 60 + 60 = 160 = 155520/972$; the other nine agree as well. Only the ρ_k contribute because $Z \leq H$ acts on $\rho'^{(H,\psi)}$ both through $\psi|_Z = \kappa_0$ and through the central character of ρ', so $\langle \mathrm{Res}_H\,\rho', \psi\rangle = 0$ unless that central character is κ_0. Check (b) compares the vectors with the coset data of Table 7.1, and (c) compares them with the subgroup orders. □

Solution to Exercise 7.8. Apply one linear functional to both sides.

(a) Let $f(v) = \sum_{\rho'} v(\rho')\dim M_{\rho'}$. By Proposition 7.4, $f(v_j) = \dim W_j$, and $f(e_\rho) = \dim M_\rho$; f is linear. (b) $2208 + 3726 + 4680 + 7128 + 9360 + 312 + 960 = 28374$ and $2520 + 3168 + 8424 + 14256 = 28368$; the difference is 6. (c) The column d comes from

$2g(Y) - 2 + r$, i.e. from the coset actions, not from the vectors. So (b) detects a wrong sign c_j (unless compensated elsewhere), a wrong $g(Y_j)$ or r_j, and an error in a vector that changes $f(v_j)$. It misses errors that f cannot see: entries moved between irreducibles with the same $\dim M$ (the two degree-20 irreducibles with $\dim M = 102$, for instance), compensating errors in two vectors, and anything common to both computations, such as an error in the character table or in the choice of κ_0. □

Solution to Exercise 7.9. There is no prime $p \mid m$ with $p \nmid a$ exactly when every prime factor of m divides a.

(a) A prime $p \mid m$ with $p \nmid a$ fails to exist iff every prime divisor of m divides a, i.e. iff $\mathrm{rad}(m) \mid a$. (b) $m = 12$, rad $= 6$: $a = 6$. $m = 18$, rad $= 6$: $a \in \{6, 12\}$. $m = 24$, rad $= 6$: $a \in \{6, 12, 18\}$. $m = 30$, rad $= 30$: none. (c) For squarefree m, $\mathrm{rad}(m) = m$ and no nonzero a is a multiple of m, so every simple tuple satisfies the prime condition. For $m = 12$ the only failing value is $a = 6 = -6$, so the only pair that fails it is $(6, 6)$, the pair that occurs in pieces 4 and 10. □

Solution to Exercise 7.10. Lemma 7.3 and the étale multiplicity $2g - 2$; for (b), $\pi_*\pi^*D = (\deg \pi)D$ for divisors.

(a) $g(X) = 2(2 - 1) + 1 = 3$, so $\dim H^1(X) = 6$. The $+1$-eigenspace of the involution is $\pi^*H^1(Y)$, of dimension 4 (Lemma 7.3); the -1-eigenspace has dimension $2g(Y) - 2 = 2$ (previous exercise), and $4 + 2 = 6$. (b) For a divisor D on Y, $\pi_*\pi^*D = (\deg \pi)D$, so $\mathrm{Nm} \circ \pi^* = [\deg \pi]$ on $J(Y)$, where now $\mathrm{Nm}\colon J(X) \to J(Y)$. Pulling back, $(\pi^*)^* \circ \mathrm{Nm}^* = [\deg \pi]^*$, which is multiplication by $\deg \pi$ on $H^1(J(Y))$. Hence Nm^* is injective on H^1, in accordance with Proposition 7.10. □

Chapter 8

Solution to Exercise 8.1. Riemann–Roch with Serre duality; for (b), the Weierstrass function x has a double pole at O and no other pole.

(a) $h^0(\mathcal{L}) - h^0(K_Y \otimes \mathcal{L}^{-1}) = d + 1 - g$, and $\deg(K_Y \otimes \mathcal{L}^{-1}) = 2g - 2 - d < 0$ for $d \geq 2g - 1$, so $h^0(\mathcal{L}) = d - g + 1$ and $\dim |\mathcal{L}| = d - g$. For $d = 2g - 2$ the same formula gives $h^0(\mathcal{L}) = g - 1 + h^0(K_Y \otimes \mathcal{L}^{-1})$, where $K_Y \otimes \mathcal{L}^{-1}$ has degree 0 and has a section only if it is trivial. So $\dim |\mathcal{L}| = g - 2$ for $\mathcal{L} \ncong K_Y$ and $\dim |K_Y| = g - 1$: the fibre over K_Y is one dimension larger, and AJ_{2g-2} is not a projective bundle.

(b) For $g = 1$, $2g - 1 = 1$, so by (a) and Proposition 8.4 every AJ_d with $d \geq 1$ is a $\mathbb{P}^{d-1}$-bundle over $\mathrm{Pic}^d(E) \cong E$. For $d = 2$, $H^0(\mathcal{O}(2O))$ is spanned by 1 and the Weierstrass function x (its only pole is a double pole at O), so $|2O| = \{\mathrm{div}(x - c) + 2O : c \in \mathbb{C}\} \cup \{2O\}$, i.e. the fibres $x^{-1}(c)$, $c \in \mathbb{P}^1$, of the double cover $x\colon E \to \mathbb{P}^1$; this is the fibre $\mathbb{P}^1$ of AJ_2 over $\mathcal{O}(2O)$. □

Solution to Exercise 8.2. Decompose each factor by characters of μ_m and use the elements $(\sigma, \sigma^{-1}; \mathrm{id}) \in \mathcal{N}$ and the swap.

(a) $H^*(X) = H^0 \oplus H^1 \oplus H^2$ with $H^1 = \bigoplus_k W_{\psi^k}$, $W_{\psi^0} = \pi^*H^1(Y)$, and H^0, H^2 invariant. The element $(\sigma, \sigma^{-1}; \mathrm{id})$ acts on $H^*(X)_{\kappa_1} \otimes H^*(X)_{\kappa_2}$ by $\kappa_1(\sigma)/\kappa_2(\sigma)$, so the $\mathcal{N}$-invariant components have $\kappa_1 = \kappa_2$; together with the swap they generate $\mathcal{N}$ (its elements are $(c, -c; \mathrm{id})$ and $(c, -c; \mathrm{swap})$). For $\kappa_1 = \kappa_2 = \psi^k$, $k \neq 0$, the component is $W_{\psi^k} \otimes W_{\psi^k}$, on which $(\sigma, 1; \mathrm{id})$ acts by ζ_m^k, so it belongs to the character ψ^k of $\mathcal{G}/\mathcal{N}$. The swap acts on $H^1 \otimes H^1$ by $v \otimes w \mapsto (-1)^{1 \cdot 1} w \otimes v$ (the Koszul sign: the swap of $X \times X$ maps $\mathrm{pr}_1^*v \cup \mathrm{pr}_2^*w$ to $\mathrm{pr}_2^*v \cup \mathrm{pr}_1^*w = -\mathrm{pr}_1^*w \cup \mathrm{pr}_2^*v$), so the invariants are the antisymmetric tensors $\bigwedge^2 W_{\psi^k}$. For the trivial character the invariants are $(H^*(Y)^{\otimes 2})^{S_2} = H^*(\mathrm{Sym}^2 Y)$. Hence $H^*(X^2/\mathcal{N}) = H^*(\mathrm{Sym}^2 Y) \oplus \bigoplus_{k \neq 0} \bigwedge^2 W_{\psi^k}$.

(b) For $(1, 3, 1, 3)$ mod 4 over $\mathbb{P}^1$: $\dim W_{\psi^k} = -2 + 4 = 2$ for $k = 1, 2, 3$ (all $ka_i \not\equiv 0 \bmod 4$), so $d = 2$ and each $\bigwedge^2 W_{\psi^k}$ is a line in degree 2; with $H^*(\mathbb{P}^2)$ the Betti numbers of Z are $1, 0, 4, 0, 1$.

(c) The inverse of q^* on invariants is $|\mathcal{N}|^{-1} q_*$, which needs $|\mathcal{N}|$ to be invertible in the coefficients. With $\mathbb{Z}$ or $\mathbb{F}_p$, $p \mid |\mathcal{N}|$, the quotient can have torsion phenomena not seen by invariants. □

Solution to Exercise 8.3. Count, for each i, the $k \in \mathbb{Z}/m$ with $ka_i \equiv 0$: there are $\gcd(a_i, m)$ of them.

For $k \neq 0$, $d_k = 2g - 2 + \#\{i : ka_i \not\equiv 0\}$, so

$$\begin{aligned}\sum_{k=0}^{m-1} d_k &= 2g + (m-1)(2g-2) + \sum_i \#\{k \neq 0 : ka_i \not\equiv 0\} \\ &= 2g + (m-1)(2g-2) + \sum_i (m - \gcd(a_i, m)),\end{aligned}$$

because $\#\{k \in \mathbb{Z}/m : ka_i \equiv 0\} = \gcd(a_i, m)$, and $k = 0$ is among them. The right side is $m(2g-2) + 2 + \sum_i (m - \gcd(a_i, m)) = 2g(X)$ by Riemann–Hurwitz. For $(3, 1, 4, 4)$ over $\mathbb{P}^1$: $(d_0, \dots, d_5) = (0, 2, 1, 0, 1, 2)$, sum $6 = 2 \cdot 3$. For $(2, 4, 2, 4)$ over E: $d_0 = 2$, $d_1 = d_5 = 4$, $d_2 = d_4 = 4$ ($2a_i \in \{4, 2\}$), $d_3 = 0$ ($3a_i \equiv 0$), sum $18 = 2 \cdot 9$; Riemann–Hurwitz gives $2g(X) - 2 = 0 + 4 \cdot (6 - 2) = 16$. □

Solution to Exercise 8.4. Repeat the proof of Proposition 8.9 with $d = 1$; for (b) use that U is connected and non-compact; for (c) multiply Euler characteristics.

(a) μ_m acts freely on $\pi^{-1}(U)$, so $\pi_* \mathbb{C}_X|_U$ is a local system of rank m with a rank-one ψ-part $\mathbb{L}_\psi$. Over b_i the fibre is $\mu_m/\langle \sigma^{a_i} \rangle$, and $\mathbb{C}[\mu_m/\langle \sigma^{a_i} \rangle]$ contains only characters trivial on σ^{a_i}; $\psi(\sigma^{a_i}) = \zeta_m^{a_i} \neq 1$, so the ψ-stalk is 0 and $(\pi_* \mathbb{C})_\psi = j_! \mathbb{L}_\psi$. A small loop around b_i lifts to the path from x to $\sigma^{a_i} x$, which is the definition of the local monodromy; the meridian acts on the ψ-part by $\zeta_m^{a_i}$.

(b) A compactly supported section of a local system on the connected non-compact surface U vanishes outside a compact set, hence on a nonempty open set, hence everywhere (a locally constant section that vanishes at one point of a connected space vanishes identically). So $H^0_c = 0$. By Poincaré duality $H^2_c(U, \mathbb{L}_\psi) \cong H^0(U, \mathbb{L}_\psi^\vee)^\vee$, and a global section of $\mathbb{L}_\psi^\vee$ is invariant under the monodromy $\zeta_m^{-a_1} \neq 1$, so it is 0.

(c) Compute χ_c with a finite cell structure as in Example 8.10: every cochain group of a rank-one system has the same dimension as for $\mathbb{C}$, so $\chi_c(U, \mathbb{L}_\psi) = \chi_c(U) = \chi(U) = 2 - 2g - r$ ($\chi_c = \chi$ on an even-dimensional manifold, by Poincaré duality). By (b) only H^1_c survives, so $\dim H^1_c(U, \mathbb{L}_\psi) = 2g - 2 + r$, and $H^1(X)_\psi = H^1(Y, j_! \mathbb{L}_\psi) = H^1_c(U, \mathbb{L}_\psi)$ by step 2. □

Solution to Exercise 8.5. Duality for (a); for (b) write the differential with the Koszul sign $(-1)^{\deg}$ of the first factor.

(a) By the Čech computation in Example 8.10, $H^0(\mathbb{C}^*, \mathbb{L}_{\bar\zeta}) = H^1(\mathbb{C}^*, \mathbb{L}_{\bar\zeta}) = 0$ since $\bar\zeta \neq 1$ (the dual system has inverse monodromy). Poincaré duality $H^k_c(\mathbb{C}^*, \mathbb{L}_\zeta) \cong H^{2-k}(\mathbb{C}^*, \mathbb{L}_{\bar\zeta})^\vee$ gives $H^*_c(\mathbb{C}^*, \mathbb{L}_\zeta) = 0$, as in Example 8.10.

(b) The cells are $e^1 \otimes e^1$ (degree 2), $e^2 \otimes e^1$ and $e^1 \otimes e^2$ (degree 3), $e^2 \otimes e^2$ (degree 4). With $\delta e^1 = (1 - \lambda) e^2$ on each factor and $\delta(a \otimes b) = \delta a \otimes b + (-1)^{\deg a} a \otimes \delta b$, and $\lambda_1 = 1$, $\lambda_2 = \zeta$:

$$\delta(e^1 \otimes e^1) = -(1 - \zeta)\, e^1 \otimes e^2, \qquad \delta(e^2 \otimes e^1) = (1 - \zeta)\, e^2 \otimes e^2, \qquad \delta(e^1 \otimes e^2) = 0.$$

So $\delta^2 = 0$, $\ker \delta$ in degree 2 is 0, in degree 3 it is $\mathbb{C}\, e^1 \otimes e^2 = \operatorname{im} \delta$, and in degree 4 the image is everything. The complex is acyclic, although the first factor alone has cohomology: the twisted factor kills everything.

(c) $\mathbb{C}^l$ is simply connected, so every local system on it is trivial (it corresponds to a representation of the trivial group, Section 5.5); a nontrivial twist on $(\mathbb{C}^*)^k \times \mathbb{C}^l$ must have some $\lambda_i \neq 1$. □

Solution to Exercise 8.6. $H^0(K_Y)$ and R are both 2-dimensional; for (b) note that any two coordinates are independent on R.

(a) $h^0(K_Y) = 2$ and $R = \{w \in \mathbb{C}^3 : \sum w_i = 0\}$ has dimension 2, so $|\mathcal{L}_0| = \mathbb{P}(H^0(K_Y) \oplus R) = \mathbb{P}^3$, containing the skew lines $\mathbb{P}(H^0(K_Y))$ (the divisors $D + D''$, $D'' \in |K_Y|$, the vertex) and $\mathbb{P}(R)$ (for a chosen splitting). A point $[\eta : w]$ off both lines lies on the unique line through $[\eta : 0]$ and $[0 : w]$, so $\mathbb{P}^3$ is their join. Projection from $\mathbb{P}(H^0(K_Y))$ sends $[\eta : w]$ to $[w]$; the fibre over $[w]$ is $\{[\eta : w]\} \cong H^0(K_Y)$, and globally $\mathcal{V} = H^0(K_Y) \otimes \mathcal{O}_{\mathbb{P}^1}(1) \cong \mathcal{O}(1)^{\oplus 2}$.

(b) $\mathbb{P}(R) \cong \mathbb{P}^1$ with the three points $\{w_i = 0\}$, e.g. $[0 : 1 : -1]$, $[1 : 0 : -1]$, $[1 : -1 : 0]$, which are distinct; U_R is their complement. $F = \{u_1^3 + u_2^3 + u_3^3 = 0\} \subset \mathbb{P}^2$ is the Fermat cubic curve, and $u \mapsto (u_i^3)$ is the quotient by $G_F = \mu_3^3/\mu_3$, of order 9. Check by Riemann–Hurwitz: over each of the three points the fibre has 3 points ($u_i = 0$, $u_j^3 = -u_k^3$) with stabilizers of order 3, so $2g(F) - 2 = 9 \cdot (-2) + 3 \cdot (9 - 3) = 0$ and $g(F) = 1$, as it must be for a smooth plane cubic.

(c) $d = 2g - 2 + r = 5 = 2 \cdot 2 + (3 - 2)$: the Thom shift is 4 and the Fermat line sits in $H^1(F)$. For $\alpha = \pm(1, 1, 1)$, $|\alpha| = 1$ or 2, so $V(\alpha)$ has type $(1, 0)$ or $(0, 1)$ by Shioda's rule $q = |\alpha| - 1$ (Section 8.1; [Shi79, Thm. I(ii)]). Theorem 8.1(a) does not apply: it assumes r even, and here $V(\alpha)$ sits in odd degree, where there are no algebraic classes, and $d = 5$ is odd; fully paired data always have r even. □

Solution to Exercise 8.7. For (a) sum the full alternating series and subtract the two missing terms.

(a) An intersection of $j \le r - 2$ of the hyperplanes is a $\mathbb{P}^{r-2-j}$, with $\chi = r - 1 - j$, and $j \ge r-1$ of them have empty intersection (any $r-1$ of the w_i are independent on R). Inclusion–exclusion gives $\chi(U_R) = \sum_{j=0}^{r-2}(-1)^j\binom{r}{j}(r - 1 - j)$. The full sum $\sum_{j=0}^{r}(-1)^j\binom{r}{j}(r - 1 - j)$ vanishes for $r \ge 2$ (both $\sum(-1)^j\binom{r}{j}$ and $\sum(-1)^j j\binom{r}{j}$ vanish), and the missing terms are $j = r - 1$, which is 0, and $j = r$, which is $(-1)^r \cdot (-1)$. So $\chi(U_R) = (-1)^r$. Since $\chi_c(U_R, \mathbb{L}') = \chi(U_R)$ for a rank-one system, this agrees with (8.3): one line in degree $r - 2 \equiv r \pmod 2$.

(b) $F \cap \{u_i = 0\}$ is the Fermat cubic curve ($\chi = 0$), $F \cap \{u_i = u_j = 0\}$ is X_3^0, three points, and three coordinates cannot vanish on F without the fourth vanishing. So $\chi(F^\circ) = 9 - 4 \cdot 0 + 6 \cdot 3 = 27 = 3^3 = |G_F| \cdot \chi(U_R)$ with $|G_F| = 3^{4-1}$ and $\chi(U_R) = (-1)^4$. In general $\chi(F^\circ) = |G_F|\,\chi(U_R) = (-1)^r m'^{r-1}$, since $F^\circ \to U_R$ is a G_F-torsor.

(c) $\chi(\mathbb{C}^*) = 0$, and Euler characteristics multiply, so the product has $\chi = 0$. But the trivial system on the same space also has $\chi = 0$ and nonzero H_c^* (Lemma 8.11), so vanishing of χ says nothing about vanishing in each degree; Proposition 8.12 needs the full Lemma 8.11. □

Chapter 9

Solution to Exercise 9.1. Count cube roots of the residue vector, modulo the diagonal; for (c), compare connectedness of $Q^{\circ\circ}$ with irreducibility of Q.

(a) A point of U_0 is $[\eta : w]$ with all $w_i \ne 0$. A preimage is $(u, [\eta' : \nu])$ with $u \in F$ and $\varphi(u, [\eta' : \nu]) = [\eta' + \nu(u_i^3)] = [\eta : w]$, so $[u_1^3 : \cdots : u_4^3] = [w]$; once u is fixed, $[\eta' : \nu]$ is unique (normalize ν so that $\nu(u_i^3) = w$, then $\eta' = \eta$). Fix cube roots ρ_i of w_i; the solutions are $u = [c_1\rho_1 : \cdots : c_4\rho_4]$, $c_i \in \mu_3$, and (c_i) and (ζc_i) give the same point of $\mathbb{P}^3$. So there are $3^4/3 = 27$ preimages, and $G_F = \mu_3^4/\mu_3$, acting by $u_i \mapsto c_i u_i$, permutes them simply transitively. (b) If $w_1 = 0$ and $w_2, w_3, w_4 \ne 0$, then $u_1 = 0$ and u_2, u_3, u_4 are cube roots up to a common factor: $3^3/3 = 9$ points. The first factor $\mu_3 \subset G_F$ fixes each of them (it multiplies $u_1 = 0$), so the orbit of G_F has $27/3 = 9$ elements. (c) In the proof, irreducibility of the graph closure $\overline{\Gamma}$ is what makes $\overline{\Gamma} \to Q$ birational; $\overline{\Gamma}$ is irreducible because it is the closure of a connected (smooth, hence irreducible) set $Q^{\circ\circ}$. Since $Q^{\circ\circ}$ is open and dense in Q, its connectedness is the same as the irreducibility of Q, i.e. of F. For $r = 2$, $F = X_{m'}^0$ is m' points, Q is a disjoint union of m' copies of $\mathbb{P}^g$, and the argument must be run on each component separately; Theorem 8.1 assumes $r \ge 3$, and every piece has $r \ge 64$. □

Solution to Exercise 9.2. The α-part of the boundary cohomology is $V(\alpha)\otimes H^*(\mathbb{P}^{g-1})$; compare parities, or compare with the real dimension of $\mathbb{P}^{g-1}$.

(a) $H^*(X_6^{158}\times\mathbb{P}^{76})_\alpha = H^{158}(F)_\alpha\otimes H^*(\mathbb{P}^{76}) = V(\alpha)\otimes H^*(\mathbb{P}^{76})$ lives in degrees $158+2j$, $0\le j\le 76$, i.e. the even degrees $158, 160, \dots, 310$. Here $d-1=311$ is odd, so the α-part of H^{311} of the boundary vanishes. (b) $r=3$, $g=2$, $d=5$: the boundary is $F\times\mathbb{P}^1$ with F the Fermat cubic curve, whose H^1 is 2-dimensional (genus 1) and contains $V(\alpha)$. The α-part of $H^*(F\times\mathbb{P}^1)$ is $V(\alpha)\otimes H^*(\mathbb{P}^1)$, in degrees 1 and 3; $d-1=4$ is not among them. The odd cohomology of F is nonzero, but it sits in the wrong degree. (c) $d-1-2j=r-2$ means $2g+r-3-2j=r-2$, i.e. $2(g-j)=1$, impossible for integers. By dimension: the only α-part of $H^*(F)$ is in degree $r-2$, so the α-part of $H^{d-1}(F\times\mathbb{P}^{g-1})$ is $V(\alpha)\otimes H^{d-1-(r-2)}(\mathbb{P}^{g-1}) = V(\alpha)\otimes H^{2g-1}(\mathbb{P}^{g-1})$, and $2g-1$ exceeds the real dimension $2g-2$ of $\mathbb{P}^{g-1}$. □

Solution to Exercise 9.3. Pair both sides with $H^1(J)$ and use $\int_J f_*(a)\cup w = \int_X a\cup f^*w$.

(a) For $d=1$, f^* is the identity $H^1(J)=H^1(X)$, and for $v,w\in H^1(J)$, $\int_J f_*f^*v\cup w = \int_X v\cup w$. On the other side, $a_1\Theta = a_1a_2b_2$, $b_1\Theta = b_1a_2b_2$, $a_2\Theta = a_2a_1b_1$, $b_2\Theta = b_2a_1b_1$. Pairing with the matching w and reordering to $a_1b_1a_2b_2$: $\int a_1a_2b_2b_1 = +1$ (two transpositions), $\int b_1a_2b_2a_1 = -1$ (three), $\int a_2a_1b_1b_2 = +1$ (two), $\int b_2a_1b_1a_2 = -1$ (three); all other pairings vanish. On the curve, $\int_X a_1b_1 = 1$, $\int_X b_1a_1 = -1$, $\int_X a_2b_2 = 1$, $\int_X b_2a_2 = -1$. The two tables agree, and the pairing $H^3(J)\times H^1(J)\to\mathbb{C}$ is perfect, so $f_*f^*v = v\cup\Theta$. (b) f_* lowers degrees by $2(\dim X^d - \dim J) = 2(d-g_X)$. For $d>g_X$, f_*1 would lie in $H^{-2(d-g_X)}(J)=0$, so $f_*f^*v = v\cup f_*1 = 0$ for every v and the transfer recovers nothing; the formula with Θ^{g_X-d} is meaningless. Steps 1–7 never use $d\le g_X$; only the birationality of $\mathrm{Sym}^d X\to W_d$ and Poincaré's formula in step 8 do. (c) Piece 10: $d=14256\le g(X)=84169$ (Table 7.1). □

Solution to Exercise 9.4. Two lines in $\mathbb{P}^3$ meet iff their four linear equations have a common nonzero solution.

(a) A standard line is fixed by a pairing (three choices) and ϵ_1,ϵ_2 with $\epsilon^m=-1$ (m choices each): $3m^2$ lines, and each lies on X_m^2 since $u_i^m+u_j^m = u_j^m(\epsilon^m+1)=0$ on each pair. Fix $L=\{u_1=\epsilon_1u_2,\ u_3=\epsilon_2u_4\}$. Same pairing, $L'=\{u_1=\delta_1u_2,\ u_3=\delta_2u_4\}\ne L$: if $\delta_1=\epsilon_1$, $\delta_2\ne\epsilon_2$, then $u_3=u_4=0$ and $L\cap L'$ is the point $[\epsilon_1:1:0:0]$; similarly with the roles exchanged; if both differ, $u_1=\dots=u_4=0$, so they are disjoint. That gives $2(m-1)$ lines. Pairing $\{13,24\}$, $L'=\{u_1=\gamma_1u_3,\ u_2=\gamma_2u_4\}$: on $L\cap L'$, $u_1=\epsilon_1\gamma_2u_4=\gamma_1\epsilon_2u_4$, so they meet iff $\gamma_2=\gamma_1\epsilon_2/\epsilon_1$, and this γ_2 satisfies $\gamma_2^m=-1$; so exactly m lines of this pairing meet L, and likewise m of the pairing $\{14,23\}$. Total $2(m-1)+2m=4m-2$. (b) $L\cong\mathbb{P}^1$ has genus 0, and adjunction on the surface gives $-2=2g(L)-2=L^2+K\cdot L=L^2+(m-4)$, so $L^2=2-m$. □

Solution to Exercise 9.5. Write $\alpha(g)$ through the two ratios that move L; for (b), find the stabilizer of L.

(a) For $g=[\zeta_1:\zeta_2:\zeta_3:\zeta_4]\in G=\mu_3^4/\mu_3$, $gL=\{u_1=x\epsilon u_2,\ u_3=y\epsilon u_4\}$ with $x=\zeta_1/\zeta_2$, $y=\zeta_3/\zeta_4$, and $\alpha(g)=\zeta_1\zeta_2^2\zeta_3\zeta_4^2=xy\,\zeta_2^3\zeta_4^3=xy$. The map $g\mapsto(x,y)$ is onto μ_3^2 with fibres of size $27/9=3$. By Exercise 9.4, $L\cdot gL=-1$ if $x=y=1$, 1 if exactly one of x,y is 1, and 0 otherwise. Hence

$$\sum_g \alpha(g)(L\cdot gL) = 3\Big(-1+\sum_{y\ne1}y+\sum_{x\ne1}x\Big) = 3(-1-1-1) = -9,$$

and dividing by $|G|=27$ gives $-1/3$, as in Example 9.11. (b) For $\alpha=(1,3,4,4)$ mod 6 and the pairing $\{12,34\}$, the stabilizer of L is $H=\{\zeta_1=\zeta_2,\ \zeta_3=\zeta_4\}$ and $\alpha|_H=\zeta_1^{1+3}\zeta_3^{4+4}$, nontrivial at $\zeta_1=\zeta_6$, $\zeta_3=1$. For $\{13,24\}$, $\alpha|_H=\zeta_1^{1+4}\zeta_2^{3+4}=\zeta_1^5\zeta_2$, and for $\{14,23\}$, $\alpha|_H=\zeta_1^{1+4}\zeta_2^{3+4}$ again (with $\zeta_4=\zeta_1$, $\zeta_3=\zeta_2$): nontrivial. Splitting the sum $e_\alpha[L]=|G|^{-1}\sum_g\alpha(g)^{-1}g^*[L]$ into cosets of H produces the factor $\sum_{h\in H}\alpha(h)^{-1}=0$, since $h^*[L]=[L]$. (c) No: an entry 1 needs a partner 5. It is a Hodge character: $|\alpha|=(1+1+16)/6=3$ and $|5\alpha|=(5+5+8)/6=3$, the value $n/2+1=3$ for $n=4$. In Shioda's notation it is the element $(2,0,0,4,0;3)$ (two

entries 1, four entries 4), one of the indecomposable elements listed for $m = 6$ in [Shi79, p. 181, table], so step 9 does not apply to it. □

Solution to Exercise 9.6. Divide by $d' = 2$; the components of the induced torsor are counted by $\mu_{12}/\operatorname{im}\varsigma$.

(a) $d' = \gcd(12, 4, 6, 8) = 2$, $m' = 6$, $\alpha' = (2^{324}, 3^6, 4^{324}) \bmod 6$. (b) $(\mathbb{Z}/12)^\times = \{1, 5, 7, 11\}$, reducing to $1, 5, 1, 5 \bmod 6$. For $t \equiv 1$, $t\alpha' = \alpha'$; for $t \equiv 5$, $t\alpha' = (4^{324}, 3^6, 2^{324})$. Both are fully paired: each 2 with a 4, and the six 3's in three pairs $(3 + 3 \equiv 0)$. (c) The torsor $F^\circ \times^{G_F,\varsigma} \mu_{12}$ has $[\mu_{12} : \operatorname{im}\varsigma] = [\mu_{12} : \langle\sigma^2\rangle] = 2$ components. $\ker\varsigma = \{(g_i) \in G_F : \sum_i a_i g_i \equiv 0 \bmod 12\}$; since $a_i g_i = 2a'_i g_i$ with $g_i \in \mathbb{Z}/6$, this is $\{\sum_i a'_i g_i \equiv 0 \bmod 6\} = \ker\alpha'$, of index 6 in G_F (α' is onto μ_6, as its entries $2, 3$ generate $\mathbb{Z}/6$). Each component is a copy of $F^\circ/\ker\alpha'$, a connected μ_6-torsor over U_R. (d) $(\psi^t \circ \varsigma)(e_i) = \zeta_{12}^{ta_i} = \zeta_6^{ta'_i}$ depends only on t mod 6, so ψ and ψ^7 use the same Fermat line $V(\alpha')$ and the same algebraic class on F. They differ in the projector ϵ_{ψ^t} of step 7 and in the eigenspace W_{ψ^t}; Theorem 8.1(a) is applied to each ψ^t separately and only needs $\pm\alpha' \in \mathfrak{C}$, so nothing is lost. □

Chapter 10

Solution to Exercise 10.1. Use $g_X - g = (m-1)(g-1)$ and count characters.

(a) Piece 5: $(24-1)\cdot 2340 = 53820 = 56161 - 2341$ and $\varphi(24)\cdot 2340 = 8\cdot 2340 = 18720$. Piece 7: $23\cdot 4212 = 96876 = 101089 - 4213$ and $8\cdot 4212 = 33696$. Piece 8: $11\cdot 4680 = 51480 = 56161 - 4681$ and $\varphi(12)\cdot 4680 = 4\cdot 4680 = 18720$. In general $g_X - g = m(g-1) + 1 - g = (m-1)(g-1)$.

(b) $H^1(\mathrm{Prym}^{\mathrm{prim}}, \mathbb{C}) = \bigoplus_{\chi \text{ primitive}} H^1(X, \mathbb{C})_\chi$ has dimension $\varphi(m)\cdot d$ by Proposition 10.1, and the complex dimension of an abelian variety is half the dimension of its H^1, so $\dim \mathrm{Prym}^{\mathrm{prim}} = \varphi(m)\, d/2$. Since $\mathrm{Prym}^{\mathrm{prim}} \subseteq \mathrm{Prym}$ are abelian varieties, they are equal if and only if their dimensions agree, i.e. $\varphi(m) = m - 1$ ($d > 0$ as $g \geq 2$), i.e. every non-trivial character is faithful, i.e. μ_m has no subgroups other than 1 and itself: m is prime. □

Solution to Exercise 10.2. Compute $H^\bullet(Y, L_\chi)$ with a cellular cochain complex and use that H^0 and H^2 vanish.

Since $\pi\colon X \to Y$ is a finite covering map, $H^1(X, \mathbb{C}) = H^1(Y, \pi_*\mathbb{C})$ and $\pi_*\mathbb{C} = \bigoplus_\chi L_\chi$, the sum over the characters of μ_m of rank-one local systems with monodromy χ (or $\bar\chi$, depending on the convention), so $H^1(X, \mathbb{C})_\chi \cong H^1(Y, L_\chi)$. Triangulate Y; the cochain complex computing $H^\bullet(Y, L_\chi)$ has one copy of $\mathbb{C}$ per cell, so $\sum_i (-1)^i \dim H^i(Y, L_\chi) = \sum_i (-1)^i \#\{i\text{-cells}\} = 2 - 2g$. $H^0(Y, L_\chi)$ is the space of flat global sections, i.e. of vectors fixed by the monodromy, which is 0 for $\chi \neq 1$. By Poincaré duality for local systems $H^2(Y, L_\chi) \cong H^0(Y, L_{\bar\chi})^\vee = 0$. Hence $-\dim H^1(Y, L_\chi) = 2 - 2g$, i.e. $\dim H^1(Y, L_\chi) = 2g - 2$. □

Solution to Exercise 10.3. Compute the character of $H^{0,1}$ from the holomorphic Lefschetz formula and compare with the regular character.

For $1 \neq \tau \in G$ the action on X is free, so the holomorphic Lefschetz number $\operatorname{tr}(\tau^*|H^0(X, \mathcal{O})) - \operatorname{tr}(\tau^*|H^1(X, \mathcal{O}))$ is a sum over fixed points and vanishes. $H^0(X, \mathcal{O}) = \mathbb{C}$ with trivial action, so $\operatorname{tr}(\tau^*|H^1(X, \mathcal{O})) = 1$. Via $H^1(X, \mathcal{O}) \cong H^{0,1} = \overline{H^{1,0}}$, the trace on $H^{1,0}$ is $\bar 1 = 1$ too, for every $\tau \neq 1$ (so the choice between τ^* and $(\tau^{-1})^*$ does not matter). At $\tau = 1$ the trace is g_X. The class function with value g_X at 1 and 1 elsewhere is $1_G + \frac{g_X - 1}{|G|}\mathrm{reg}_G$, and $g_X - 1 = |G|\,(g-1)$ by Riemann–Hurwitz for the étale cover. Hence $H^{1,0} \cong \mathbb{C} \oplus \mathbb{C}[G]^{\oplus(g-1)}$; since $\mathbb{C}[G]$ contains each irreducible ρ' with multiplicity $\dim\rho'$, $\dim \mathrm{Hom}_G(\rho', H^{1,0}) = (g-1)\dim\rho' + \delta_{\rho',1}$. The same holds for $H^{0,1}$ (its character is the complex conjugate one, and $\mathbb{C}[G]$ is self-conjugate). So for $\rho' \neq 1$, $M_{\rho'} = \mathrm{Hom}_G(\rho', H^1)$ has Hodge numbers $((g-1)\dim\rho', (g-1)\dim\rho')$: balanced. For $G = \mu_m$ and $\dim\rho' = 1$ this is Proposition 10.1. □

Solution to Exercise 10.4. The Galois group permutes the lines Λ_χ and fixes rational vectors.

(a) The decomposition $U'' \otimes \mathbb{Q}(\mu_m) = \bigoplus_\chi \Lambda_\chi$ (primitive χ, each Λ_χ the subspace on which $a \in \mathbb{Q}(\mu_m)$ acts through the embedding χ) is defined over $\mathbb{Q}(\mu_m)$, and $\gamma \in \mathrm{Gal}(\mathbb{Q}(\mu_m)/\mathbb{Q})$, acting on the coefficients, maps Λ_χ to $\Lambda_{\gamma\circ\chi}$. Let $0 \neq x \in U''$ lie in Λ_χ. Since x is rational, $x = \gamma x \in \Lambda_{\gamma\circ\chi}$ for every γ. For $m > 2$ there are $\varphi(m) \geq 2$ primitive characters and the Galois group acts transitively on them, so some $\gamma \circ \chi \neq \chi$; but $\Lambda_\chi \cap \Lambda_{\gamma\circ\chi} = 0$, so $x = 0$, a contradiction.

(b) Here $m = 3$, the Galois group is $\{1, c\}$ with c complex conjugation, and $c(\Lambda_\chi) = \Lambda_{\bar\chi}$. Choose u spanning Λ_χ inside $U'' \otimes \mathbb{Q}(\zeta_3)$; then $\bar u$ spans $\Lambda_{\bar\chi}$. The vectors $u + \bar u$ and $(u - \bar u)/\sqrt{-3}$ are fixed by c, hence lie in U'', and they are linearly independent because $u, \bar u$ are. As $\dim_{\mathbb{Q}} U'' = \varphi(3) = 2$, they form a basis. In the cycle picture of Example 10.9 this is the statement that $z_\chi + z_{\bar\chi} = 2[Q_0] - [Q_1] - [Q_2]$ and $(z_\chi - z_{\bar\chi})/\sqrt{-3} = [Q_2] - [Q_1]$ (up to the sign fixed by the choice of ζ_3) are rational. □

Solution to Exercise 10.5. Apply Proposition 10.7, then expand the generating function.

(a) For $g = 3$, $d = 4$: $\mathrm{Sym}^4 Y$ has dimension 4 and $J(Y)$ dimension 3; the fibres are $\mathbb{P}^1$ over $L \ncong K_Y$ and $\mathbb{P}^2 = |K_Y|$ over K_Y (for a non-hyperelliptic Y, embedded by $|K_Y|$ as a plane quartic, $|K_Y|$ is the plane of lines in $\mathbb{P}^2$, each cutting out four points). With ξ as in the proof of Proposition 10.8, $H^4(\mathrm{Sym}^4 Y) = H^4(J) \oplus \xi H^2(J) \oplus \mathbb{Q}[|K_Y|]$, of dimension $15 + 15 + 1 = 31$, which is the s^4-coefficient of Macdonald's formula; the last summand is the class of the jumping fibre.

(b) $(1 + st)^4 = 1 + 4st + 6s^2t^2 + \dots$ and $1/((1-t)(1-s^2t)) = \sum_n t^n(1 + s^2 + \dots + s^{2n})$. The coefficient of t^2 is $(1 + s^2 + s^4) + 4s(1 + s^2) + 6s^2 = 1 + 4s + 7s^2 + 4s^3 + s^4$. The blow-up of $J(Y)$ (Poincaré polynomial $(1 + s)^4 = 1 + 4s + 6s^2 + 4s^3 + s^4$) at a point adds the exceptional class in degree 2: the same polynomial.

(c) At $s = -1$ the generating function is $(1 - t)^{2g}/(1 - t)^2 = (1 - t)^{2g-2}$, whose t^{2g-2}-coefficient is $(-1)^{2g-2} = 1$. From the fibres: the Euler characteristic with compact supports is additive over $J(Y) = (J(Y) \smallsetminus \{[K_Y]\}) \sqcup \{[K_Y]\}$ and multiplicative for fibre bundles, so $e(\mathrm{Sym}^{2g-2} Y) = e(\mathbb{P}^{g-2})(e(J(Y)) - 1) + e(\mathbb{P}^{g-1}) = (g - 1)(0 - 1) + g = 1$. □

Solution to Exercise 10.6. Use $\sigma_*[Q_t] = [Q_{t+1}]$, character orthogonality, and disjointness.

(a) $\sigma_* z_\chi = \sum_t \chi(\sigma^t)^{-1}[Q_{t+1}] = \sum_s \chi(\sigma^{s-1})^{-1}[Q_s] = \chi(\sigma) z_\chi$. Summing over all m characters, $\sum_\chi z_\chi = \sum_t (\sum_\chi \chi(\sigma^t)^{-1})[Q_t] = m[Q_0]$, since the inner sum is m for $t = 0$ and 0 otherwise.

(b) The Q_t are pairwise disjoint, so $[Q_s] \cdot [Q_t] = 0$ for $s \neq t$, and $[Q_t] \cdot [Q_t] = [Q] \cdot [Q] = (-1)^{g-1}$ because σ^t is an automorphism carrying Q to Q_t. If $\sum_t a_t[Q_t] = 0$, intersecting with $[Q_s]$ gives $(-1)^{g-1} a_s = 0$. So the m classes are independent, and $[Q] \cdot z_\chi = [Q] \cdot [Q]$ follows from the same computation.

(c) $(1 + H)^{-1} = \sum_k (-1)^k H^k$, so $c_{g-1} = (-1)^{g-1} H^{g-1}$ and $\int_{\mathbb{P}^{g-1}} c_{g-1} = (-1)^{g-1}$. For $g = 2$ this is -1; the normal bundle of the exceptional curve $E \cong \mathbb{P}^1$ of a blown-up surface is $\mathcal{O}(-1)$, so $E \cdot E = -1$: the two descriptions of $\mathrm{Sym}^2 Y$ agree. □

Solution to Exercise 10.7. Write $2g(X) - 2$ in two ways; list characters by order; follow the degrees through the transfer.

(a) $24 \cdot 2340 + 1 = 56161$, $24 \cdot 4212 + 1 = 101089$, $12 \cdot 4680 + 1 = 56161$. Pieces 5 and 8 have the same $m(g(Y) - 1) = 56160$. A more conceptual reason: $X_j = C/\ker\psi_j$ with $|\ker\psi_j| = |H_j|/m = 9, 5, 9$, and $2g(C) - 2 = 1010880 = 9 \cdot 112320 = 9(2 \cdot 56161 - 2) = 5(2 \cdot 101089 - 2)$, so in all three cases $\ker\psi_j$ acts freely on C and $g(X_j)$ is fixed by $|\ker\psi_j|$ alone.

(b) The characters of μ_{12} are $\chi_k(\zeta) = \zeta^k$, $k \in \mathbb{Z}/12$. Primitive: $k \in \{1, 5, 7, 11\}$ ($\varphi(12) = 4$). Non-trivial, not primitive: $k \in \{2, 3, 4, 6, 8, 9, 10\}$, of orders $6, 4, 3, 2, 3, 4, 6$. Schoen's Theorem 10.10 covers the four primitive lines; Theorem 10.11 covers all eleven non-trivial ones. The proof needs only ψ_8, which is primitive.

(c) $d = 9360$, $g_X = 56161$. The preimage of $Q \cong \mathbb{P}^{4680}$ in X^d has dimension 4680, so its image under Ξ is a cycle of dimension 4680 (if non-zero) in the 56161-dimensional $J(X_8)$: codimension 51481, class in H^{102962}. The inverse of $\cup\Theta^{g_X - d}$, $g_X - d = 46801$, lowers the

degree by $2 \cdot 46801$ to $102962 - 93602 = 9360 = d$: Λ_{ψ_8} lives in $H^{9360}(J(X_8))$, as classes of codimension $4680 = d/2$. □

Chapter 11

Solution to Exercise 11.1. Everything is diagonal, so read off the eigenvalues.

(a) $x = \operatorname{diag}(1+t, 1)$ acts on H^1 by $P = \operatorname{diag}(1+\omega, 1)$, $Q = \operatorname{diag}(1+\bar\omega, 1)$. Note $1+\omega = -\omega^2$ and $1+\bar\omega = -\omega$, both of modulus 1 (indeed $1+t = -t^2$ is an automorphism of E). On the basis of H^2: $dz_1 \wedge dz_2 \mapsto \det P = -\omega^2$; $dz_1 \wedge d\bar z_1 \mapsto (1+\omega)(1+\bar\omega) = 1$; $dz_1 \wedge d\bar z_2 \mapsto 1+\omega = -\omega^2$; $dz_2 \wedge d\bar z_1 \mapsto 1+\bar\omega = -\omega$; $dz_2 \wedge d\bar z_2 \mapsto 1$; $d\bar z_1 \wedge d\bar z_2 \mapsto \det Q = -\omega$. Each basis vector of $H^{1,1}$ is mapped to a multiple of itself, so $H^{1,1}$ is preserved; this must be so because x^* is a pull-back by a homomorphism, which preserves Hodge types and algebraic classes.

(b) e_χ maps to $(0,1,0)$ under $(\text{trivial}, \chi, \bar\chi)$, so $\Psi = e_\chi \mathbf{1}_2$ has $P = \mathbf{1}_2$, $Q = 0$. Then $\bigwedge^2 \Psi^*$ is $\det P = 1$ on $dz_1 \wedge dz_2$, $P \otimes Q = 0$ on $H^{1,1}$ and $\det Q = 0$ on $H^{0,2}$: the projector onto $H^{2,0}$. Proposition 11.5 only says that $\operatorname{Alg} \otimes \mathbb{C} = H^{1,1}$ is mapped into $\operatorname{Alg} \otimes \mathbb{C}$, and here it is mapped to 0. A complex combination of correspondences may map transcendental classes to transcendental classes (the identity does); what it cannot do is map algebraic classes to non-algebraic ones. □

Solution to Exercise 11.2. Use $\bigwedge^D (LL') = \bigwedge^D L \bigwedge^D L'$ and the top degree.

(a) The action of matrices over $\mathbb{C}[G]$ on H^1 is compatible with products, $(\Psi\Psi')^* = \Psi^* \circ \Psi'^*$ (Remark 11.2), and $\bigwedge^D$ is multiplicative by Lemma 11.3. If $\bigwedge^D \Psi^*$ and $\bigwedge^D \Psi'^*$ both map $\operatorname{Alg} \otimes \mathbb{C}$ into $\operatorname{Alg} \otimes \mathbb{C}$, so does their composite, which is $\bigwedge^D$ of the composite action.

(b) On $H^{2n} = \bigwedge^{2n} H^1$ a linear map L of H^1 acts by $\det L$. So $f + f'$ acts by $\det(f^* + f'^*)$, and the cycle $\Gamma_f + \Gamma_{f'}$ (as ${}^t\Gamma$, i.e. by pull-back) by $\det f^* + \det f'^*$, the sum of the degrees. For $f = f' = \mathrm{id}_E$: $f + f' = [2]$ acts on $H^2(E)$ by $\det(2 \cdot \mathbf{1}_2) = 4 = \deg[2]$, while $2[\Delta]$ acts by 2. □

Solution to Exercise 11.3. Changes of basis are linear.

(a) In coordinates with respect to a $\mathbb{Q}$-basis of $V_\mathbb{Q}$, $V_\mathbb{Q} = \mathbb{Q}^N \subset \mathbb{C}^N = V$, and a polynomial vanishing on $\mathbb{Q}^N$ is zero by Proposition 3.26(a). Polynomial functions on V do not depend on the basis used to define them, since a change of basis is a linear substitution, which maps polynomials to polynomials; and $V_\mathbb{Q}$ is described as $\mathbb{Q}^N$ in the coordinates of any $\mathbb{Q}$-basis. So the statement is basis-free.

(b) Write $P = X_1 + \mathrm{i}X_2$ with $X_1 = \mathbf{1}_2$, $X_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. By (11.1) for $D = 2$,

$$\det P = \det X_1 + \mathrm{i}^2 \det X_2 + \mathrm{i}\big(\det(X_1 + X_2) - \det X_1 - \det X_2\big) = 1 + 1 + \mathrm{i}\,(0 - 1 + 1) = 2,$$

using $\det X_2 = -1$ and $\det \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = 0$; directly, $\det P = 1 - \mathrm{i}^2 = 2$. □

Solution to Exercise 11.4. Use $\sum_{\rho'} v_{H,\psi}(\rho') \dim \rho' = \dim \operatorname{Ind}_H^G \psi$.

(a) Since $n_+ - n_- = e_\rho$, the first sum is $\dim M_\rho = 6 = 28374 - 28368$, the second is $\dim \rho = 4$. By Frobenius reciprocity (Theorem 3.10), $v_j(\rho')$ is the multiplicity of ρ' in $\operatorname{Ind}_{H_j}^G \psi_j$, so $\sum_{\rho'} v_j(\rho') \dim \rho' = [G : H_j]$, and the second sum is $\sum_j c_j [G : H_j]$. With $[G : H_j] = 155520/|H_j|$: the plus pieces give $320 + 540 + 720 + 1080 + 1440 + 40 + 160 = 4300$, the minus pieces $360 + 480 + 1296 + 2160 = 4296$, difference 4. This check uses the subgroup orders, not the character table.

(b) In the étale model $\dim M_{\rho'} = 2 \dim \rho' = 4$ for the three 2-dimensional irreducibles (positions $4, 5, 6$). Then $\dim W_{Z,\mathrm{sgn}} = 2 \cdot 4 \cdot 3 = 24 = D_+$, $\dim W_{C_4,\psi_4} = 4 \cdot 3 = 12$, $\dim W_{C_6,\psi_6} = 4 \cdot 2 = 8$, so $D_- = 20$ and $D_+ - D_- = 4 = \dim M_{\chi_5}$. □

Solution to Exercise 11.5. Apply invariance to a group element on which $\kappa\kappa' \neq 1$.

(a) For $x \in V_\kappa$, $y \in V_{\kappa'}$ and $z \in Z$, $\langle x, y\rangle = \langle zx, zy\rangle = \kappa(z)\kappa'(z)\langle x, y\rangle$; if $\kappa\kappa' \neq 1$, pick z with $\kappa(z)\kappa'(z) \neq 1$ to get $\langle x, y\rangle = 0$. If κ is not real-valued, $\kappa^2 \neq 1$ (since $\kappa^{-1} = \bar\kappa \neq \kappa$), so $\langle V_\kappa, V_\kappa\rangle = 0$.

(b) $t^*dz = \omega\, dz$, $t^*d\bar z = \bar\omega\, d\bar z$, and $\langle x, y\rangle = \int_E x \cup y$ is invariant. Indeed $\langle dz, dz\rangle = \langle d\bar z, d\bar z\rangle = 0$ (as $\omega^2 \neq 1$), while $\langle dz, d\bar z\rangle = \int_E dz \wedge d\bar z = -2\mathrm{i}\,\mathrm{area}(E) \neq 0$, consistent with $\omega\bar\omega = 1$.

(c) Let $\mu_2 = \{\pm 1\}$ act on $H^1(E, \mathbb{C})$ by $[-1]^* = -1$. Every bilinear form is invariant under -1, the whole of H^1 is the sgn-eigenspace, and it is not isotropic for $\int_E x \cup y$. □

Solution to Exercise 11.6. Here $\Theta^{g-D} = \Theta$, and $dz_1 \wedge d\bar z_1 \wedge \Theta_1 = 0$.

With $g = 2$, $D = 1$: $\Lambda \cup \bar\Lambda \cup \Theta = dz_1 \wedge d\bar z_1 \wedge (\Theta_1 + \Theta_2) = dz_1 \wedge d\bar z_1 \wedge \Theta_2$, since $dz_1 \wedge d\bar z_1 \wedge dz_1 = 0$. This is $\mathrm{pr}_1^*(dz \wedge d\bar z) \cup \mathrm{pr}_2^*\Theta$, so its integral is $\int_E dz \wedge d\bar z \cdot \int_E \Theta = -2\mathrm{i}\,\mathrm{Im}\,\tau \cdot 1 \neq 0$ (with $dz \wedge d\bar z = -2\mathrm{i}\, dx \wedge dy$). This is the mechanism of Theorem 11.8(b) in the smallest case: $U = \mathbb{C}dz_1$ is isotropic, $U \oplus \bar U = H^1(E_1)$ is non-degenerate, and Θ_2 fills the complement. For $U = \mathbb{C}(dz_1 + d\bar z_1)$ one has $\bar U = U$, so $\Lambda \cup \bar\Lambda = (dz_1 + d\bar z_1) \wedge (dz_1 + d\bar z_1) = 0$ and the integral vanishes: a subspace meeting its conjugate gives no contraction of this form, which is why the non-real character matters. □

Solution to Exercise 11.7. Coordinates with respect to a $\mathbb{C}$-basis are unique.

(a) Let $a_1, \dots, a_r$ be a $\mathbb{Q}$-basis of $\mathrm{Alg}^p(X)$ and extend it by $b_1, \dots, b_s$ to a $\mathbb{Q}$-basis of $H^{2p}(X, \mathbb{Q})$; it is also a $\mathbb{C}$-basis of $H^{2p}(X, \mathbb{C})$, and $\mathrm{Alg}^p(X) \otimes \mathbb{C} = \mathrm{span}_{\mathbb{C}}(a_i)$. A rational $x = \sum \alpha_i a_i + \sum \beta_j b_j$ $(\alpha_i, \beta_j \in \mathbb{Q})$ that lies in $\mathrm{span}_{\mathbb{C}}(a_i)$ has $\beta_j = 0$ by uniqueness of coordinates, so $x \in \mathrm{Alg}^p(X)$. Apply this to every $x \in S$.

(b) The lemma concerns rational subspaces. The line $\Lambda_{u'}$ is not defined over $\mathbb{Q}$: its conjugate $\overline{\Lambda_{u'}}$ lies in $\bigwedge^6(u'' \otimes M_{\bar\rho})$, a different line, so $\Lambda_{u'}$ contains no nonzero rational class. Only the conjugation-stable plane $\Lambda_{u'} \oplus \overline{\Lambda_{u'}} = p^*\mathrm{HW}(B_u) \otimes \mathbb{C}$ is the complexification of a rational space, and the lemma applies to it. □

Solution to Exercise 11.8. A rational map commutes with complex conjugation.

Every $m \in M_\rho$ is $f|_\rho$ for some $f \in \mathcal{B}_C \otimes \mathbb{C}$, since $\mathcal{B}_C \otimes \mathbb{C} = M_\rho \oplus M_{\bar\rho}$ (Proposition 3.27(d)). Write $f = \sum_k z_k f_k$ with $f_k \in \mathcal{B}_C$ rational and put $\tilde f = \sum_k \bar z_k f_k$. Each f_k commutes with complex conjugation, so $\overline{m(u')} = \overline{f(u')} = \sum_k \bar z_k f_k(\bar u') = \tilde f(u'') = \tilde f|_{\bar\rho}(u'') \in u'' \otimes M_{\bar\rho}$. Hence conjugation maps $u' \otimes M_\rho$ into $u'' \otimes M_{\bar\rho}$; both have dimension 6 and conjugation is injective, so it is onto. Conjugation is an antilinear ring automorphism of $H^\bullet(P, \mathbb{C})$, so it maps $\bigwedge^6$ of a subspace of H^1 onto $\bigwedge^6$ of the conjugate subspace: $\overline{\Lambda_{u'}} = \bigwedge^6(u'' \otimes M_{\bar\rho})$. □

Chapter 12

Solution to Exercise 12.1. Compare the first term of the binomial expansion of $(1 + q^r X)^p$ with the others, modulo one power of q more than it carries.

If $\gamma \neq 1$ has finite order, some power γ' of it has prime order p, and $\gamma' \equiv 1 \pmod N$ still; so assume $\gamma^p = 1$, $\gamma \neq 1$. Since $N \geq 3$, either N has an odd prime factor q, or $4 \mid N$ and we take $q = 2$. Let r be the largest integer with $\gamma \equiv 1 \pmod{q^r}$ (it exists since $\gamma \neq 1$) and $X = (\gamma - 1)/q^r$, an integral matrix with $q \nmid X$; then $r \geq 1$, and $r \geq 2$ if $q = 2$. Expand

$$0 = \gamma^p - 1 = pq^r X + \sum_{i=2}^{p} \binom{p}{i} q^{ir} X^i.$$

If $p \neq q$, every term with $i \geq 2$ is divisible by q^{2r}, hence by q^{r+1}, so $pq^r X \equiv 0 \pmod{q^{r+1}}$; as p is invertible modulo q, $q \mid X$, a contradiction. If $p = q$, the first term is $p^{r+1}X$; for $2 \leq i \leq p-1$ the binomial coefficient is divisible by p, so the term is divisible by p^{1+ir} with $1 + ir \geq r + 2$; the last term is $p^{pr}X^p$ and $pr \geq r + 2$ means $(p-1)r \geq 2$, true for $p \geq 3$ and for $p = 2$ because then $r \geq 2$. So $p^{r+1}X \equiv 0 \pmod{p^{r+2}}$ and $p \mid X$, a contradiction. Hence $\gamma = 1$. For $N = 2$ the only choice is $q = 2$, $r \geq 1$, and for $p = 2$, $r = 1$ the last term $p^{pr}X^p = 4X^2$ is only divisible by $p^{r+1} = 4$: the argument fails, and $\gamma = -1 = 1 + 2 \cdot (-1)$ is a counterexample. □

Solution to Exercise 12.2. Write a constructible set as a finite union of sets $U_i \cap F_i$ with U_i open and F_i closed irreducible.

(a) For $x \neq 0$, (x, w) is the image of $(x, w/x)$; for $x = 0$ the image is $(0, 0)$ only. The image is not open (it contains $(0, 0)$ but no neighbourhood of it) and not closed (its closure is $\mathbb{C}^2$); it contains the dense open set $\{x \neq 0\}$. (b) $\overline{Z} = \bigcup_i \overline{U_i \cap F_i} \subseteq \bigcup_i F_i$. Since $\dim \overline{Z} = \dim S$, some i has $\dim \overline{U_i \cap F_i} = \dim S$; in particular $U_i \cap F_i \neq \emptyset$, so it is dense in the irreducible F_i and $\dim F_i = \dim S$. As $F_i \subseteq S$ is closed and S is irreducible, $F_i = S$, and $Z \supseteq U_i \cap S = U_i$, a non-empty open subset of S, dense because S is irreducible. (c) If $\mathcal{P}$ has rank 9 at t, its derivative $T_t\mathcal{H}' \to T_{\mathcal{P}(t)}S'$ is onto, so in local coordinates $\mathcal{P}$ is a holomorphic submersion near t; by the holomorphic implicit function theorem it is locally a projection $\mathbb{C}^{13} \to \mathbb{C}^9$, which is open. So the image contains an open neighbourhood of $\mathcal{P}(t)$ in the analytic topology, without any algebraicity. □

Solution to Exercise 12.3. On $\bigwedge^k_K$ a K-linear map acts by its k-th exterior power; on the top power, by the determinant.

(a) The proof of Lemma 12.3 applies verbatim with $V_{\mathbb{Z}} = \mathbb{Z}[\omega]^6$: γ has entries in $\mathbb{Z}[\omega]$, so $\det_K \gamma \in \mathbb{Z}[\omega]$, and $\det(\gamma^{\mathsf{T}} h_2 \bar{\gamma}) = \det h_2$ gives $\mathrm{Nm}(\det_K \gamma) = 1$; the norm-one elements of $\mathbb{Z}[\omega]$ are the six units. The matrix $\mathrm{diag}(-1, 1, 1, 1, 1, 1)$ is unitary for h_2, preserves $\mathbb{Z}[\omega]^6$ and has determinant -1. (b) $u \cdot 1_6$ acts on $\bigwedge^k_K$ by u^k: by $u^6 = 1$ for $k = 6$ and by u^3 for $k = 3$, which is -1 for $u = \zeta$. (c) $u = (2 + \theta)/(2 - \theta) = (1 + 4\theta)/7$ has $\mathrm{Nm}(u) = 7/7 = 1$ and is not one of the six roots of unity in K, so it has infinite order; $\mathrm{diag}(u, 1, 1, 1, 1, 1)$ is unitary for h_2 over $\mathbb{Q}$ with determinant u. It does not preserve $\mathbb{Z}[\omega]^6$, since $u \notin \mathbb{Z}[\omega]$: integrality is exactly what forces the determinant into μ_6. □

Solution to Exercise 12.4. A finite étale map is open and closed, which is all that (c) uses.

(a) The trivial double cover $S \sqcup S \to S$. (b) $Z = S \sqcup \emptyset$ is a union of connected components, hence closed and algebraic, and it is open, so it has non-empty interior; it is not all of S''. In Proposition 12.8 the last step uses that S is irreducible; S'' is not. (c) A connected component S''_0 is smooth and connected, hence irreducible, and Proposition 12.8 applies to it. Its image in S' is open and closed (the map is finite étale) and non-empty, hence all of S'. For $s \in S''_0$ the fibre of $\mathcal{A}''$ at s is the fibre of $\mathcal{A}$ at $\pi(s)$, so algebraicity of the Weil classes at every point of S''_0 gives it at every point of S'. □

Solution to Exercise 12.5. A real-analytic function vanishing on an open set vanishes identically.

(a) A proper closed algebraic subset $Z \subsetneq \mathbb{C}^2$ lies in the zero set of some non-zero polynomial P. If Z contained a ball B, all partial derivatives of P at the centre of B would vanish, so its Taylor expansion there, which is P itself, would be 0. (b) $\mathbb{C}^2$ is a complete metric space, so by Baire a countable union of closed sets with empty interior has empty interior; by (a) the union is not $\mathbb{C}^2$. Since $\mathbb{Q}(\mathrm{i}) \cap \mathbb{R} = \mathbb{Q}$, neither $\sqrt{2}$ nor $\sqrt{3}$ lies in $\mathbb{Q}(\mathrm{i})$, so $(z, w) = (\sqrt{2}, \sqrt{6})$ has $z \notin \mathbb{Q}(\mathrm{i})$ and $w/z = \sqrt{3} \notin \mathbb{Q}(\mathrm{i})$: it lies on no L_q and on no line $z = q$. (c) $\bigcup_{a \in \overline{\mathbb{Q}}} \{z = a\}$: countably many lines, since $\overline{\mathbb{Q}}$ is countable. □

Solution to Exercise 12.6. Work in $\mathrm{Hom}(\cdot, \cdot) \otimes \mathbb{Q}$, where φ is invertible.

(a) From $\psi \circ \varphi = [m]$, $\psi = m\varphi^{-1}$. The hypothesis $\varphi\iota(a) = \iota'(a)\varphi$ gives $\varphi^{-1}\iota'(a) = \iota(a)\varphi^{-1}$, so $\psi\iota'(a) = m\varphi^{-1}\iota'(a) = m\iota(a)\varphi^{-1} = \iota(a)\psi$. (b) $\varphi \circ \psi \circ \varphi = \varphi \circ [m] = [m] \circ \varphi$, and φ is surjective, so $\varphi \circ \psi = [m]_{A'}$. Hence $\psi^*\varphi^* = (\varphi \circ \psi)^* = [m]^* = m$ on $H^1(A', \mathbb{Q})$, and on $H^6 = \bigwedge^6 H^1$ it is m^6, i.e. $t = m^{-6}\psi^*\varphi^* t$. (c) Antilinearity says that φ^* is K-linear from $(H^1(A', \mathbb{Q}), \iota')$ to $(H^1(A, \mathbb{Q}), \bar{\iota})$, so by the proposition it maps $\mathrm{HW}(A', \iota')$ onto $\mathrm{HW}(A, \bar{\iota})$. And $\mathrm{HW}(A, \bar{\iota}) = \mathrm{HW}(A, \iota)$: both are the rational subspace whose complexification is $\bigwedge^6 H^1_\iota \oplus \bigwedge^6 H^1_{\bar{\iota}}$, a description symmetric in ι and $\bar{\iota}$. □

Solution to Exercise 12.7. Compute $h_1(x, y) = 2x_1\bar{y}_1 - x_2\bar{y}_2$ on the columns x, y of P.

(a) For $P = \left(\begin{smallmatrix} 2 & \theta \\ \theta & -4 \end{smallmatrix}\right)$: $h_1(x,x) = 8-3 = 5$, $h_1(x,y) = 2\cdot 2\cdot(-\theta) - \theta\cdot(-4) = 0$, $h_1(y,y) = 6-16 = -10$. For $P = \left(\begin{smallmatrix} 2 & c \\ 1 & 4c \end{smallmatrix}\right)$, $c = (2-\theta)/7$: $h_1(x,x) = 8-1 = 7$, $h_1(x,y) = 4\bar{c} - 4\bar{c} = 0$, $h_1(y,y) = (2-16)\,\mathrm{Nm}(c) = -14/7 = -2$. For $P = \left(\begin{smallmatrix} 2+\theta & 1 \\ 2+\theta & 2 \end{smallmatrix}\right)$: $h_1(x,x) = 14-7 = 7$, $h_1(x,y) = 2(2+\theta) - 2(2+\theta) = 0$, $h_1(y,y) = 2-4 = -2$. (b) $\det(P^{\mathsf{T}} h_1 \bar{P}) = \det P \cdot \det h_1 \cdot \overline{\det P} = \mathrm{Nm}(\det P)\det h_1$. For $\langle 1,-1\rangle$ the ratio would be $(-1)/(-2) = \frac{1}{2}$, which is not a norm since 2 appears to an odd power. (c) All forms have signature $(1,1)$. $\langle -1,2\rangle$: same determinant -2, isometric (swap the basis vectors). $\langle 10,-5\rangle$: $-50 = -2\cdot 25$, isometric. $\langle 3,-6\rangle$: $-18 = -2\cdot 9$, isometric. $\langle 3,-3\rangle$: $-9 \equiv -1$, not isometric; it is 3 times the hyperbolic plane, with isotropic vector $(1,1)$. □

Solution to Exercise 12.8. Scaling a form of rank n by c multiplies its determinant by c^n.

(a) $\det(-H) = (-1)^6 \det H$ for a 6×6 matrix. (b) $\det(2h) = 8\det h$ and $8 = 2\cdot 4 \equiv 2$, not a norm, so the class changes. The theorem only uses forms of even rank 6, for which scaling multiplies the determinant by a sixth power, a norm. □

Chapter 13

Solution to Exercise 13.1. Go through the eight implications and ask of each whether pieces 4 and 10 occur among its inputs; for (b), use that the set of k with ke_ρ in the span is an ideal.

(a) Implication (iii) loses its conclusion for $j = 4, 10$, and with it (v) produces only nine of the eleven lines. Implication (ii) is unaffected as a statement, but the relation it provides contains pieces 4 and 10 with coefficients $+1$ and -1, so (vi) loses its input: U_+ and U_- both contain a piece whose Weil line is not available. Implication (i) is unaffected, and (vii) and (viii) are unaffected as implications, but they have nothing to globalize: their input is the conclusion of (vi). So the chain would no longer reach the Main Theorem, while every other conclusion of the chain (the family, its dominance, the relation, the Weil lines of nine pieces) would stand.

(b) By Section 7.5, the 222 ramified pieces that satisfy the prime condition, together with the 69 étale pieces, reach e_ρ only with $k_{\min} = 2$. The set $I = \{k \in \mathbb{Z} : ke_\rho \in \mathbb{Z}\text{-span of the family}\}$ is an ideal, here $I = 2\mathbb{Z}$, and passing to a subfamily can only shrink I, to $2a\mathbb{Z}$ for some $a \geq 1$ or to 0. So no subfamily of these 291 pieces has $1 \in I$, and pieces 4 and 10 are not among them (the exponent 6 of an $m = 12$ cover is divisible by both primes of 12).

(c) With $2e_\rho = \sum_j c_j v_j$ the two sides of the isomorphism of step (vi) differ by two copies of $\rho \otimes M_\rho$ instead of one, so the transport would give $(\Lambda_u \boxtimes \Lambda_{u'}) \boxtimes \Lambda_-$ in $\mathrm{Alg}\otimes\mathbb{C}$, and after the contraction with $\bar{\Lambda}_- \cup \Theta^{g_- - D_-}$ the class $\Lambda_u \boxtimes \Lambda_{u'} \in \mathrm{Alg}(P\times P)\otimes\mathbb{C}$. To extract Λ_u one would contract once more with an algebraic class μ' on the second factor that pairs non-trivially with $\Lambda_{u'}$, for instance $\bar{\Lambda}_{u'} \cup \Theta^{\dim P - 6}$; but whether $\bar{\Lambda}_{u'}$ is in $\mathrm{Alg}\otimes\mathbb{C}$ is the very question being asked. This is Remark on the proof 11.1: we do not know whether the extraction is possible. □

Solution to Exercise 13.2. Read Section 12.1 with "dense Zariski-open subset of S'" replaced by "dense Zariski-open subset of the Zariski closure of the image".

(a) Steps (ii)–(vi) use no genericity, so the Weil classes of A_C are algebraic for every $C \in \mathcal{H}$, as before. Let $Z \subset S'$ be the Zariski closure of the image of the period map (from a finite cover of $\mathcal{H}$); it is irreducible, of dimension 8 if the rank is 8, and the image contains a dense Zariski-open subset of Z (Chevalley). The argument of step (vii) runs over Z instead of S', but Z may be singular, and Proposition 12.8 needs a smooth connected base on which the Weil classes are single-valued. So let Z'' be an irreducible component of the preimage of Z in the cover S_0'' of Lemma 12.3, which maps onto Z (the map π is finite and flat, so it maps each irreducible component of the preimage onto Z), and let $\tilde{Z} \to Z''$ be a resolution of singularities; pull the abelian scheme and the flat Weil classes back to the smooth connected quasi-projective $\tilde{Z}$. The preimage in $\tilde{Z}$ of the dense open subset of Z contained in the image is

non-empty and Zariski open, and the Weil classes are algebraic there; so by Proposition 12.8 they are algebraic at every point of $\tilde{Z}$, and since $\tilde{Z} \to Z$ is onto, at every point of Z, a proper closed subvariety of S' (Section 12.1).

(b) Baire's theorem needs the countable union to contain a non-empty open subset of the space on which one wants to conclude. The image is infinite, even of dimension 8, but it is contained in the proper closed subset Z, and a countable union of closed subsets containing Z may simply be Z itself; nothing forces it to contain an open subset of S'. An infinite set of points, or a positive-dimensional subvariety, gives no information off its Zariski closure.

(c) Yes, partly. K-isogeny invariance (Proposition 12.11) applies to any (A, ι) that is K-isogenous to a fibre over Z, so those sixfolds have algebraic Weil classes. The Landherr half of step (viii) still sends every (A, ι, λ) of the theorem to a point x_A of S', but that point need not lie in Z, so for a general sixfold of class 2 nothing would follow. □

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