ON THE IRRATIONALITY OF THE 3-ADIC ZETA VALUE $\zeta_3(5)$

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Abstract. We give a proof that the 3-adic zeta value $\zeta_3(5)$ is irrational. The proof adapts to $p = 3$ a Hankel-determinant method that a recent preprint applies to $\zeta_2(7)$. A Volkenborn integral of an explicit rational function defines a linear functional with values in $\mathbb{Q} + \mathbb{Q}\,\zeta_3(5)$, and its Hankel determinants are the values at $\zeta_3(5)$ of polynomials with rational coefficients. We bound the 3-adic valuation of these values and the size and the denominators of the coefficients. The new ingredients at $p = 3$ are a triangular basis of the polynomials that are integer valued on $3^{-1}\mathbb{Z}_3^\times$, a local estimate at primes of the order of n in which a pole of order 3 coming from harmonic sums cancels exactly against a zero of order 3 of the rational function, and a non-vanishing theorem at an auxiliary prime, in which Wilson's theorem makes one block of a reduced Gram matrix invertible. Here n is the parameter of the rational function. Two constants of the estimates are bounded by finite computations, and with these bounds the 3-adic valuation exceeds the size and the denominators together by more than 4.5 bits per n^2. An appendix gives a second proof, independent of the first, which adapts to $\zeta_3(5)$ and the modular curve $X_0(3)$ a determinant argument that a preprint of Yi gives for $\zeta_7(3)$, and which rests on published theorems on overconvergent modular forms.

1. Introduction

Apéry proved that $\zeta(3)$ is irrational [1], and Ball and Rivoal proved that infinitely many of the numbers $\zeta(3), \zeta(5), \zeta(7), \dots$ are irrational [2]. Both results have p-adic counterparts. For a prime p and an odd integer $s \geq 3$, the p-adic zeta value is $\zeta_p(s) = L_p(s, \omega^{1-s})$, the value at s of a Kubota–Leopoldt p-adic L-function, where ω is the Teichmüller character: for odd p the character of $\mathbb{Z}_p^\times$ with $\omega(a)^{p-1} = 1$ and $\omega(a) \equiv a \pmod p$, and for $p = 2$ the character with $\omega(a) = \pm 1 \equiv a \pmod 4$ (Section 2.1 recalls the definitions). Calegari proved that $\zeta_2(3)$ and $\zeta_3(3)$ are irrational, using overconvergent p-adic modular forms [3], and Beukers gave another proof [4]. The irrationality of $\zeta_2(5)$ was proved by Calegari, Dimitrov and Tang [5, Theorem 5.2], with the arithmetic holonomy bounds of [6], and Lai, Sprang and Zudilin gave a second proof [7].

Other results concern several values at once. Lai proved that at least one of $\zeta_2(7)$, $\zeta_2(9)$, $\zeta_2(11)$, $\zeta_2(13)$ is irrational [8]. Lai and Sprang proved that, for every prime p and every $\epsilon > 0$, at least $(c_p - \epsilon)\sqrt{s/\log s}$ of the values $\zeta_p(3), \zeta_p(5), \dots, \zeta_p(s)$ are irrational once the odd integer s is large enough, with an explicit constant $c_p > 0$ [9, Theorem 1.1]. Lai, Lupu and Sprang proved that for every prime $p \geq 5$ some $\zeta_p(i)$ with odd $i \in [3,\, p + p/\log p + 5]$ is irrational [10]. At $p = 3$ these results give the irrationality of $\zeta_3(3)$, and of unspecified values among $\zeta_3(3), \zeta_3(5), \dots, \zeta_3(s)$ for large s; as far as we know, no refereed result singles out $\zeta_3(5)$. In this note we give a proof of the following statement.

Theorem 1.1. *The number $\zeta_3(5)$ is irrational.*

Theorem 1.1 has been claimed before. Two manuscripts of Long, posted on GitHub in August 2026, state the irrationality of twenty-two p-adic zeta values, among them $\zeta_3(s)$ for $s = 3, 5, 7, 9, 11$ [11, 12]. An issue opened in the repository of [11] on 27 September 2026 reports a possible gap in its Proposition 5.10, and as of 28 October 2026 the issue has no reply. Our argument uses a different method and nothing from [11, 12]. Several other preprints claiming the irrationality of

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individual p-adic zeta values appeared in August and September 2026, among them [13] for $\zeta_2(7)$, [14] for $\zeta_5(3)$ and [15] for $\zeta_7(3)$.

After the first version of this note was made public on 4 October 2026, Kilani, the author of [13], posted a proof of Theorem 1.1 on 7 October 2026 [16]. It applies the Hankel-determinant method of [13] to a rational function that has poles of order 4 at the integers of an interval symmetric about 0 and no zeros, and it proves non-vanishing at an auxiliary prime just above the right end of that interval. The preprint states that its proof is formalized in Lean 4, apart from the identification of the Volkenborn integral with the Kubota–Leopoldt value. Our proofs do not use [16], and the second proof (Appendix F) uses a different method.

Use of large language models and status of the proofs. This note was produced by large language models, instances of the Claude models of Anthropic running as autonomous agents, in September and October 2026. They adapted the method of the preprint [13] to $p = 3$ and, for the second proof (Appendix F), the method that the preprint [15] applies to $\zeta_7(3)$, found and wrote the new steps, wrote and ran the programs of Appendices E and F, and wrote the text. The arguments were checked only by other instances of the same models, acting as referees and recomputing the numbers with programs of their own; such checks may share the blind spots of the models that wrote the proof. No human expert has checked either proof.

The note has not been refereed. It is made public together with this account of how it was produced and checked.

Code availability. The programs for the computations of Appendices E and F, and for the exact computations of Section 6, are available at `https://github.com/ahippelainen/zeta-3-5-irrational-public` under the MIT licence.

Conventions. In Sections 1–6 and Appendices A–E the letter n always denotes an odd positive integer; Appendix F has its own notation. We write log for the natural logarithm; $\log_2$ and $\log_3$ count bits and trits, a trit being a factor 3, that is, $\log_2 3$ bits. For a prime p, v_p is the p-adic valuation, extended to polynomials $F = \sum_j f_j X^j \in \mathbb{Q}_p[X]$ as the Gauss valuation $v_p(F) = \min_j v_p(f_j)$, which satisfies $v_p(F+G) \geq \min(v_p(F), v_p(G))$ and $v_p(FG) \geq v_p(F) + v_p(G)$. We put $\|F\|_1 = \sum_j |f_j|$, which satisfies $\|FG\|_1 \leq \|F\|_1 \|G\|_1$. We write $[z^a]F$ for the coefficient of z^a in F, $x^+ = \max(x, 0)$, and $[\,\cdot\,]$ for the indicator of a condition. Implied constants in $O(\cdot)$ and $o(\cdot)$ are absolute unless a dependence is indicated, and all asymptotic statements refer to $n \to \infty$ (in Appendix F, to $d \to \infty$). Limits of 3-adic quantities are taken in the 3-adic absolute value $|x|_3 = 3^{-v_3(x)}$; for instance $3^N \to 0$ in $\mathbb{Z}_3$ as $N \to \infty$, because $|3^N|_3 = 3^{-N}$. For a set E, $\mathbf{1}_E$ is its indicator function; without a subscript, $\mathbf{1}$ is the trivial Dirichlet character.

The method. Our proof adapts to $p = 3$ a method that the preprint [13] applies to $\zeta_2(7)$, and which uses the full Hankel determinant for a single odd zeta value, as proposed for $\zeta(5)$ in Fauzan's preprint [17]. For an odd integer n let

$$W(y) = \frac{\left(y + \frac{n}{2}\right) \prod_{j=0}^{n-1} \left(y + j + \frac{1}{3}\right)^3 \left(y + j + \frac{2}{3}\right)^3}{\prod_{k=-n}^{2n} (y+k)^4} \,. \tag{1.2}$$

The numerator is a polynomial of degree $6n+1$ and the denominator one of degree $4(3n+1)$. With the Pochhammer symbol $(x)_n = x(x+1)\cdots(x+n-1)$ the numerator is $(y+\frac{n}{2})(y+\frac{1}{3})_n^3 (y+\frac{2}{3})_n^3$, and we use this shorter form below. For polynomials $P \in \mathbb{Q}[y]$ put

$$L(P) = \sum_{r=1,2} \int_{\mathbb{Z}_3} (PW)'\Big(t + \frac{r}{3}\Big) \mathrm{d}t \,, \tag{1.3}$$

where the integral is the Volkenborn integral (Section 2.1). The values of L lie in $\mathbb{Q} + \mathbb{Q}\,\zeta_3(5)$ (Lemma 2.19). Hence, for $K \le 3n+1$, the Hankel determinant $\det\big[L(y^{i+j})\big]_{i,j<K}$ is the value at $X = \zeta_3(5)$ of a polynomial $\Delta_K(X) \in \mathbb{Q}[X]$ of degree at most K (Definition 2.23).

Suppose that $\zeta_3(5) = a/b$ and that $\Delta_K(a/b) \neq 0$. Clearing the denominators of $\Delta_K(a/b)$ gives a nonzero integer, and comparing its 3-adic valuation with its absolute value gives (Lemma 3.1)

$$v_3(\Delta_K(\zeta_3(5)))\log_2 3 \le \log_2 \|\Delta_K\|_1 + \sum_{p\neq 3} (-v_p(\Delta_K))^+ \log_2 p + K\log_2 \max(|a|,b)\ . \tag{1.4}$$

We call the left side the *gain*; the first two terms on the right are the *size* and the *denominators* of Δ_K. With $K = \kappa n$, and up to terms of lower order, the gain is at least $G(\kappa)n^2$ (Theorem 3.5), the size is at most $(\kappa^2 - 6\kappa)n^2\log_2 n + F^\dagger n^2$ (Theorem 3.8), and the denominators are at most $\kappa(6-\kappa)n^2\log_2 n + c\,n^2$ (Theorem 3.10). The terms of order $n^2\log n$ cancel, and the last term of (1.4) is $O(n)$. The constants $F^\dagger$ and c depend on κ, and on the range $\kappa \in [2.8, 3]$ that we need, $G - F^\dagger - c$ is at least 4.58 bits (Table 1). For n large this contradicts (1.4).

The inequality (1.4) needs $\Delta_K(a/b) \neq 0$. For the Hankel determinants of a positive measure on $\mathbb{R}$ this would follow from positivity, but L is a 3-adic functional, and no such argument is available. Following [13], we choose an auxiliary prime $p \nmid b$ with $2.1n + 2 \le p \le 2.159n$, and we show that the constant coefficient of Δ_K has strictly smaller p-adic valuation than all its other coefficients (Theorem 3.13). The choice of p fixes K, and the range of p confines κ to $[2.8, 3]$.

What is new. The overall structure and several of the estimates are those of [13]. We give our own proofs of the statements that we take from there (Appendices A–C), so that the argument should not depend on the correctness of [13]. Three steps do not carry over from $p = 2$. The first is the change of basis in the 3-adic estimate: at $p = 2$ it contributes a positive term of order K^2, while at $p = 3$ the natural basis of the polynomials that are integer valued on the set $3^{-1}\mathbb{Z}_3^\times$ of integration points (Lemma 4.1) costs about $K^2/4$ trits. The second is a local estimate at the primes of the order of n (Lemma 5.24), which determine the constant c. At such a prime, the harmonic sums that the Volkenborn integral produces behave like poles of order 3 at points of $\frac{p}{3}\mathbb{Z}$, and when such a point is a zero of W, which has order 3 as well, the two cancel exactly. In a numerical evaluation at $\kappa = 3$ with the cruder local bound of [13] in its place, the comparison no longer leads to a contradiction (Remark 5.27). The third is the non-vanishing theorem (Section 6). There, after a change of basis and a rescaling by powers of $\sqrt{p}$, the constant coefficient of Δ_K becomes, modulo $\sqrt{p}$ in the ring $\mathbb{Z}_p[\sqrt{p}]$, a product of explicit block determinants. The block that belongs to a pair of nodes (the poles $y = -k$, $-n \le k \le 2n$, of W) congruent modulo p is invertible because the leading coefficients of W at the two nodes have the ratio -8 modulo p, by Wilson's theorem (Lemma 6.10); the ratio 1 would make it singular (Remark 6.22).

The constants $F^\dagger$ and c are upper bounds obtained by finite computations: $F^\dagger$ in ball arithmetic, in which every real number is enclosed in an interval given by a midpoint and a radius and every operation returns an interval that contains the exact result [18], and c in exact rational arithmetic, up to a final evaluation of Euler's constant and of three logarithms in ball arithmetic (Appendix E). The computations evaluate expressions that the lemmas of Section 5 show to be upper bounds. All other steps are proved in the text, and the coefficients and determinants needed in Section 6 are small exact computations that can be checked by hand.

Section 2 writes $\zeta_3(5)$ as a Volkenborn integral and defines the functional and the Hankel polynomial. Section 3 states the criterion and the four estimates and deduces Theorem 1.1 from them, and Sections 4, 5 and 6 prove the estimates. The longer proofs are in the appendices: Appendix A treats the analytic step of the 3-adic estimate, Appendix B the Heine expansion, the tree bound and the size, Appendix C the local bounds and the prime sum, Appendix D the case analysis of the non-vanishing theorem, and Appendix E the computations. Appendix F gives a second proof of Theorem 1.1 by a different method, the determinant argument that [15] uses for

$\zeta_7(3)$ on $X_0(7)$, carried over to the modular curve $X_0(3)$. It uses from the rest of the note only the statement of the theorem, the normalization (2.4) and some notation, so that a reader can check the theorem along two independent routes, but it depends on published theorems on overconvergent modular forms that the main proof does not need.

2. The functional and its Hankel polynomial

This section fixes the normalization of $\zeta_3(5)$, writes it as a Volkenborn integral, and shows that the functional (1.3) takes values in $\mathbb{Q}+\mathbb{Q}\,\zeta_3(5)$. The Volkenborn integral is not translation invariant, and the defect of a translation by an integer is a harmonic sum (Lemma 2.15). These harmonic sums are harmless at the prime 3, but at the other primes they produce the virtual poles of Section 5.3.

2.1. **The Volkenborn integral and $\zeta_3(5)$.** We use the following standard result, for which we recall the notation. $\mathbb{C}_p$ is the completion of an algebraic closure of $\mathbb{Q}_p$, and we fix embeddings of the algebraic numbers into $\mathbb{C}$ and into $\mathbb{C}_p$. A Dirichlet character χ modulo N is a homomorphism $(\mathbb{Z}/N\mathbb{Z})^\times\to\mathbb{C}^\times$, extended to $\mathbb{Z}$ by $\chi(a)=0$ for $\gcd(a,N)>1$. Its *conductor* is the smallest divisor f of N such that χ factors through $(\mathbb{Z}/f\mathbb{Z})^\times$, and we always use the primitive character modulo f that induces χ; the trivial character $\mathbf{1}$ has conductor 1. The Bernoulli numbers B_j are defined by $t/(e^t-1)=\sum_j B_j t^j/j!$, and the *generalized Bernoulli numbers* of a character χ of conductor f_χ by $\sum_{a=1}^{f_\chi}\chi(a)\,te^{at}/(e^{f_\chi t}-1)=\sum_k B_{k,\chi}t^k/k!$, see e.g. [19]. The *Teichmüller character* $\omega:\mathbb{Z}_p^\times\to\mathbb{Z}_p^\times$ sends a to the unique $(p-1)$-st root of unity congruent to a modulo p; it depends only on a modulo p, so it is a Dirichlet character of conductor p, and at $p=3$ it is the nontrivial character modulo 3, $\omega(a)=\pm1\equiv a \pmod 3$. Finally $\langle a\rangle=a/\omega(a)\in 1+p\mathbb{Z}_p$ for $a\in\mathbb{Z}_p^\times$.

Theorem 2.1 (see e.g. [19, 20, 21])**.** *Let p be an odd prime, χ a Dirichlet character of conductor f_χ, and F a common multiple of p and f_χ. There is a p-adic meromorphic function $L_p(s,\chi)$ on $\{s\in\mathbb{C}_p : |s|_p<p^{(p-2)/(p-1)}\}$, analytic if $\chi\neq\mathbf{1}$ and with only a simple pole at $s=1$ if $\chi=\mathbf{1}$, such that*

$$L_p(1-k,\chi)=-\Bigl(1-\chi\omega^{-k}(p)\,p^{k-1}\Bigr)\frac{B_{k,\chi\omega^{-k}}}{k}\qquad(k\geq 1)\ , \tag{2.2}$$

and

$$L_p(s,\chi)=\frac{1}{F}\,\frac{1}{s-1}\sum_{\substack{1\leq a\leq F\\ p\nmid a}}\chi(a)\langle a\rangle^{1-s}\sum_{j\geq 0}\binom{1-s}{j}\left(\frac{F}{a}\right)^j B_j\ . \tag{2.3}$$

In (2.2), $\chi\omega^{-k}$ denotes the character of smallest conductor attached to the product, so that $\chi\omega^{-k}(p)=0$ if p divides its conductor. The interpolation property (2.2) determines $L_p(s,\chi)$ and is the reason why L_p is called the p-adic L-function of χ; in the proof we use only the formula (2.3). The p-adic zeta values are $\zeta_p(s)=L_p(s,\omega^{1-s})$. At $p=3$ the Teichmüller character has order 2, so $\omega^{-4}=\mathbf{1}$ and

$$\zeta_3(5)=L_3(5,\mathbf{1})\ . \tag{2.4}$$

A function $f:\mathbb{Z}_3\to\mathbb{Q}_3$ is *strictly differentiable* if the difference quotients $(f(x)-f(y))/(x-y)$, $x\neq y$, converge as $(x,y)\to(a,a)$, for every $a\in\mathbb{Z}_3$; the limit is $f'(a)$. Polynomials are strictly differentiable, and so is every power series $\sum_m e_m t^m$ that converges on a disc of radius larger than 1, because its difference quotient $\sum_m e_m\sum_{i<m}x^i y^{m-1-i}$ converges uniformly on $\mathbb{Z}_3^2$. For such f the *Volkenborn integral*

$$\int_{\mathbb{Z}_3} f(t)\,\mathrm{d}t=\lim_{N\to\infty}3^{-N}\sum_{t=0}^{3^N-1}f(t) \tag{2.5}$$

exists [22]; see e.g. [9]. We need three of its properties.

Lemma 2.6. *Let $f : \mathbb{Z}_3 \to \mathbb{Q}_3$ be strictly differentiable.*

(V1) $\int_{\mathbb{Z}_3} f(t+1)\,\mathrm{d}t = \int_{\mathbb{Z}_3} f(t)\,\mathrm{d}t + f'(0)$. *Hence, for $N \geq 1$, $\int_{\mathbb{Z}_3} f(t+N)\,\mathrm{d}t = \int_{\mathbb{Z}_3} f + \sum_{i=0}^{N-1} f'(i)$ and $\int_{\mathbb{Z}_3} f(t-N)\,\mathrm{d}t = \int_{\mathbb{Z}_3} f - \sum_{i=1}^{N} f'(-i)$.*

(V2) $\int_{\mathbb{Z}_3} \binom{t}{c}\,\mathrm{d}t = (-1)^c/(c+1)$ *for $c \geq 0$, and $\int_{\mathbb{Z}_3} t^c\,\mathrm{d}t = B_c$.*

(V3) *If $|f(0)|_3 \leq A$ and $|f(x) - f(y)|_3 \leq A|x-y|_3$ for all $x, y \in \mathbb{Z}_3$, then $|\int_{\mathbb{Z}_3} f|_3 \leq 3A$.*

Proof. (V1) The N-th Riemann sum of $f(t+1) - f(t)$ telescopes to $3^{-N}(f(3^N) - f(0)) = \frac{f(3^N)-f(0)}{3^N-0}$. This is a difference quotient of f at the two points 3^N and 0, and $3^N \to 0$ in $\mathbb{Z}_3$, since $|3^N|_3 = 3^{-N}$. So it tends to $f'(0)$ by strict differentiability. Apply this to $f(t+i)$, $i = 0, \ldots, N-1$, and add. For the second formula apply the first to $g(t) = f(t-1)$, which gives $\int_{\mathbb{Z}_3} f = \int_{\mathbb{Z}_3} g(t+1)\,\mathrm{d}t = \int_{\mathbb{Z}_3} f(t-1)\,\mathrm{d}t + f'(-1)$, and iterate.

(V2) By $\sum_{t<T} \binom{t}{c} = \binom{T}{c+1}$, the N-th Riemann sum is $3^{-N}\binom{3^N}{c+1} = \frac{1}{c+1}\binom{3^N-1}{c}$. As $N \to \infty$, $3^N - 1 \to -1$ in $\mathbb{Z}_3$, and $x \mapsto \binom{x}{c}$ is continuous, so the limit is $\frac{1}{c+1}\binom{-1}{c} = \frac{(-1)^c}{c+1}$. For the monomials, Faulhaber's formula gives $3^{-N}\sum_{t<3^N} t^c = \sum_{i=0}^{c} \binom{c+1}{i} B_i\, 3^{N(c-i)}/(c+1)$, and only the term $i = c$ survives the limit.

(V3) Let Σ_N be the N-th Riemann sum. Then

$$\Sigma_{N+1} - \Sigma_N = 3^{-N-1} \sum_{t<3^N} \sum_{a=0}^{2} \big(f(t + a3^N) - f(t)\big)\ , \tag{2.7}$$

and every bracket has absolute value at most $A3^{-N}$. By the ultrametric inequality $|\Sigma_{N+1} - \Sigma_N|_3 \leq 3^{N+1}A3^{-N} = 3A$, and $|\Sigma_0|_3 = |f(0)|_3 \leq A$. So $|\Sigma_N|_3 \leq 3A$ for all N, and also in the limit. □

Property (V3) is the form in which we use the integral 3-adically: a function whose values and Lipschitz constant are 3-adically small has a small integral, with the loss of one factor 3. We shall apply it to power series $\sum_m e_m t^m$ whose coefficients tend to 0 fast enough for convergence on a disc of radius larger than 1. For such a series, $|f(0)|_3$ and the Lipschitz constant on $\mathbb{Z}_3$ are at most $\max_m |e_m|_3$, since $|x^m - y^m|_3 \leq |x-y|_3$ for $x, y \in \mathbb{Z}_3$.

The integration points of our functional are $t + \frac{r}{3}$ with $t \in \mathbb{Z}_3$ and $r \in \{1, 2\}$, that is, the points of

$$S = 3^{-1}\mathbb{Z}_3^\times = \{y \in \mathbb{Q}_3 :\ v_3(y) = -1\}\ . \tag{2.8}$$

The substitution $t \mapsto -1 - t$ maps $t + \frac{r}{3}$ to $-(t + \frac{3-r}{3})$, so it acts on S as $y \mapsto -y$ and exchanges the two classes $\frac{1}{3} + \mathbb{Z}_3$ and $\frac{2}{3} + \mathbb{Z}_3$; this removes the even zeta values.

Lemma 2.9 (reflection)**.** *Let $f(t) = \sum_{i \geq 0} e_i t^i$ with $v_3(e_i) \geq i + c_0$ for a constant c_0. Then $\int_{\mathbb{Z}_3} f(-1-t)\,\mathrm{d}t = \int_{\mathbb{Z}_3} f(t)\,\mathrm{d}t$. Consequently, for every odd $M \geq 1$,*

$$\sum_{r=1,2} \int_{\mathbb{Z}_3} \Big(t + \frac{r}{3}\Big)^{-M} \mathrm{d}t = 0\ . \tag{2.10}$$

Proof. For $f = t^k$, (V2) gives $\int_{\mathbb{Z}_3} (t+x)^k\,\mathrm{d}t = \sum_i \binom{k}{i} x^{k-i} B_i = B_k(x)$, the Bernoulli polynomial, so $\int_{\mathbb{Z}_3} (-1-t)^k\,\mathrm{d}t = (-1)^k B_k(1) = B_k(0) = \int_{\mathbb{Z}_3} t^k\,\mathrm{d}t$ by $B_k(1-x) = (-1)^k B_k(x)$. By linearity the identity holds for polynomials. For the series, split $f = f_{<J} + f_{\geq J}$ at a degree J. The tail $f_{\geq J}(t)$ and $f_{\geq J}(-1-t)$ are power series with coefficients of valuation at least $J + c_0$ (the coefficients of $(-1-t)^i$ are integers), so by (V3) both have integrals of absolute value at most 3^{1-J-c_0}. Letting $J \to \infty$ proves the first claim. For the second, $g_r(t) = (t + \frac{r}{3})^{-M} = (\frac{3}{r})^M (1 + \frac{3t}{r})^{-M}$ is a series of

this type, and $g_r(-1-t) = (-1)^M g_{3-r}(t)$. So the sum $\mathcal{J}_M = \sum_r \int_{\mathbb{Z}_3} g_r$ satisfies $\mathcal{J}_M = (-1)^M \mathcal{J}_M$, and $\mathcal{J}_M = 0$ for odd M. □

Lemma 2.11 (the value)**.** *For even $m \geq 2$,*

$$\sum_{r=1,2} \int_{\mathbb{Z}_3} \Big(t + \frac{r}{3}\Big)^{-m} \mathrm{d}t = m\, 3^{m+1} L_3(m+1, \mathbf{1}) \ . \tag{2.12}$$

In particular $\zeta_3(5) = \frac{1}{972} \sum_{r=1,2} \int_{\mathbb{Z}_3} (t + \frac{r}{3})^{-4}\, \mathrm{d}t$.

Proof. Take $p = F = 3$, $\chi = \mathbf{1}$ and $s = m+1$ in (2.3). For $a \in \{1,2\}$ we have $\omega(a) = \pm 1$, and m is even, so $\langle a \rangle^{-m} = a^{-m}$ and

$$3m\, L_3(m+1, \mathbf{1}) = \sum_{a=1,2} a^{-m} \sum_{c \geq 0} \binom{-m}{c} \Big(\frac{3}{a}\Big)^c B_c \ . \tag{2.13}$$

On the other side, $(t + \frac{a}{3})^{-m} = (\frac{3}{a})^m (1 + \frac{3t}{a})^{-m} = (\frac{3}{a})^m \sum_{c \geq 0} \binom{-m}{c} (\frac{3}{a})^c t^c$. We integrate term by term with $\int_{\mathbb{Z}_3} t^c\, \mathrm{d}t = B_c$. This is legitimate: the tail $\sum_{c \geq J}$ of the series has coefficients of valuation at least $m + J$, so by (V3) its integral has absolute value at most 3^{1-m-J}, which tends to 0. Hence $\sum_a \int_{\mathbb{Z}_3} (t + \frac{a}{3})^{-m}\, \mathrm{d}t = 3^m \cdot 3m\, L_3(m+1, \mathbf{1})$. Finally, $4 \cdot 3^5 = 972$ and $L_3(5, \mathbf{1}) = \zeta_3(5)$ by (2.4). □

For $k \in \mathbb{Z}$ and $M \geq 2$ put

$$\mathrm{HS}_3(k, M) = \sum_{\substack{0 < u < 3k \\ 3 \nmid u}} \Big(\frac{3}{u}\Big)^M \quad (k > 0) \ , \qquad \mathrm{HS}_3(k, M) = -\sum_{\substack{3k < u < 0 \\ 3 \nmid u}} \Big(\frac{3}{u}\Big)^M \quad (k < 0) \ , \tag{2.14}$$

and $\mathrm{HS}_3(0, M) = 0$. Every term has 3-adic valuation M, but at a prime $p \neq 3$ the terms with $p \mid u$ have p-adic valuation $-M$.

Lemma 2.15 (translation)**.** *For every $k \in \mathbb{Z}$ and $m \geq 1$,*

$$\sum_{r=1,2} \int_{\mathbb{Z}_3} \Big(t + \frac{r}{3} + k\Big)^{-m} \mathrm{d}t = \begin{cases} m\, 3^{m+1} L_3(m+1, \mathbf{1}) - m\, \mathrm{HS}_3(k, m+1) \ , & m \text{ even} \ , \\ -m\, \mathrm{HS}_3(k, m+1) \ , & m \text{ odd} \ . \end{cases} \tag{2.16}$$

Proof. Apply (V1) to $f_r(t) = (t + \frac{r}{3})^{-m}$, whose derivative is $f_r'(i) = -m(i + \frac{r}{3})^{-m-1} = -m\, 3^{m+1} (3i + r)^{-m-1}$. For $k > 0$, (V1) adds $\sum_{i<k} f_r'(i)$, and as $r \in \{1,2\}$ and $0 \leq i < k$ vary, $u = 3i + r$ runs exactly once over the integers $0 < u < 3k$ prime to 3; this gives $-m\mathrm{HS}_3(k, m+1)$. For $k < 0$, (V1) subtracts $\sum_{i=1}^{|k|} f_r'(-i)$, and $u = -3i + r$ runs over $3k < u < 0$, $3 \nmid u$, which explains the sign in (2.14). The value at $k = 0$ is Lemma 2.11 for even m and Lemma 2.9 for odd m. □

2.2. **The functional and the Hankel polynomial.** Lemma 2.15 shows why the functional (1.3) takes values in $\mathbb{Q} + \mathbb{Q}\zeta_3(5)$. Let W be a rational function whose poles are integers, all of order at most 4. If PW decays fast enough at infinity, it is the sum of its principal parts, and after the derivative in (1.3) a pole of order i of PW becomes a pole of order $i+1$. By Lemma 2.15 with $m = i+1$, such a term contributes a harmonic sum and, for odd i, a multiple of $L_3(i+2, \mathbf{1})$. So only $\zeta_3(3) = L_3(3, \mathbf{1})$, from the simple poles, and $\zeta_3(5)$, from the poles of order 3, can occur. The coefficient of $\zeta_3(3)$ is a multiple of the sum of the residues of PW, which vanishes when PW decays fast enough at infinity; this is the reason for the derivative. So only $\zeta_3(5)$ survives (Lemma 2.19). The zeros of W do not affect this, and we choose them for the estimates of Sections 4 and 5.

The *nodes* are the integers $k \in \mathcal{N} := \{-n, -n+1, \dots, 2n\}$, and the *weight* is the rational function W of (1.2). The node k sits at $y = -k$ and is a pole of order exactly 4; since n is odd, the zero $y = -\frac{n}{2}$ is not a node. The other zeros are the points $y = -(j + \frac{r}{3})$, $0 \leq j < n$, $r \in \{1,2\}$, each of order 3. The degree of W, the degree of its numerator minus that of its denominator, is $1 + 6n - 4(3n+1) = -6n - 3$. The functional L of (1.3) is well defined: the poles of $t \mapsto (PW)(t + \frac{r}{3})$

are the points $t = -k - \frac{r}{3}$, all of absolute value 3, so its Taylor series at $t = 0$ converges on $|t|_3 < 3$, and the same holds for its derivative.

The weight is that of [13] with the prime 2 replaced by 3: there the poles have order 4 at the same nodes, and the zeros lie on the class $\frac{1}{2} + \mathbb{Z}$. Here the zeros, of order 3, lie on both classes $\frac{1}{3} + \mathbb{Z}$ and $\frac{2}{3} + \mathbb{Z}$. Then on S one of the two Pochhammer symbols becomes an integer-valued polynomial times $n!$, which produces the factor $(n!)^3$ in the 3-adic gain (Section 4), and at the primes $p \neq 3$ the zeros cancel the virtual poles produced by the harmonic sums (Section 5.3). We keep the factor $y + \frac{n}{2}$ of [13], which makes W antisymmetric about the centre of the nodes, $W(-n-y) = -W(y)$. Our argument does not use this symmetry, and the factor affects the estimates only through terms of lower order and through the two nodes $\frac{n \pm p}{2}$ of Section 6, where p is the auxiliary prime of Theorem 3.13.

For a node k put

$$\widetilde{W}_k(z) = z^4\, W(-k+z)\ , \qquad H_k[b] = [z^b]\,\widetilde{W}_k(z)\ , \qquad h_k = H_k[0] = \widetilde{W}_k(0) \neq 0\ , \tag{2.17}$$

so that $\widetilde{W}_k$ is holomorphic at $z = 0$ and its Taylor coefficients give the principal part of W at the node. Explicitly,

$$h_k = \frac{\left(\frac{n}{2} - k\right) \prod_{0 \le j < n,\, r=1,2} \left(j + \frac{r}{3} - k\right)^3}{\prod_{m \in \mathcal{N} \setminus \{k\}} (m-k)^4}\ , \qquad \prod_{m \in \mathcal{N} \setminus \{k\}} (m-k) = (-1)^{k+n} (k+n)!\,(2n-k)!\ . \tag{2.18}$$

For a polynomial P let $P_a(k) = [z^a]P(-k+z)$ be its Taylor coefficients at the node k.

Lemma 2.19 (purity)**.** *If* $\deg(PW) \le -2$*, then*

$$L(P) = \sum_{k \in \mathcal{N}} \sum_{a=0}^{3} c_{k,a}\, P_a(k)\ , \qquad c_{k,a} = \beta_{k,a} + \zeta_3(5)\, \alpha_{k,a}\ , \tag{2.20}$$

with the rational local constants

$$\beta_{k,a} = \sum_{i=1}^{4-a} i(i+1)\, \mathrm{HS}_3(k, i+2)\, H_k[4-i-a]\ , \qquad \alpha_{k,a} = \begin{cases} -2916\, H_k[1-a] & (a \le 1)\ , \\ 0 & (a \ge 2)\ . \end{cases} \tag{2.21}$$

In particular, $L(y^m) \in \mathbb{Q} + \mathbb{Q}\,\zeta_3(5)$ *for every* $m \le 6n+1$.

Proof. Since $\deg(PW) < 0$, the rational function PW is the sum of its principal parts at the nodes:

$$P(y)W(y) = \sum_{k \in \mathcal{N}} \sum_{i=1}^{4} \frac{\mathrm{res}_{k,i}}{(y+k)^i}\ , \qquad \mathrm{res}_{k,i} = \sum_{b \le 4-i} P_b(k)\, H_k[4-i-b]\ , \tag{2.22}$$

the formula for $\mathrm{res}_{k,i}$ being the coefficient of z^{-i} in $P(-k+z)W(-k+z) = z^{-4}P(-k+z)\widetilde{W}_k(z)$. Differentiate once, put $y = t + \frac{r}{3}$, integrate, sum over r and apply Lemma 2.15 with $m = i+1$. The term (k,i) contributes $i(i+1)\,\mathrm{HS}_3(k,i+2)\,\mathrm{res}_{k,i}$ and, for odd i, also $-i(i+1)\,3^{i+2} L_3(i+2, \mathbf{1})\,\mathrm{res}_{k,i}$. For $i = 1$ the latter is $-54\,\zeta_3(3)\,\mathrm{res}_{k,1}$, and $\sum_k \mathrm{res}_{k,1}$ is the sum of all residues of PW, which is zero because $PW(y) = O(y^{-2})$ as $y \to \infty$. For $i = 3$ it is $-3 \cdot 4 \cdot 3^5 \zeta_3(5)\,\mathrm{res}_{k,3} = -2916\,\zeta_3(5)\,\mathrm{res}_{k,3}$, and $\mathrm{res}_{k,3} = P_0(k)H_k[1] + P_1(k)H_k[0]$. Collecting the coefficient of each $P_a(k)$ gives (2.21). Finally, $\deg(y^m W) = m - 6n - 3$, which is at most -2 when $m \le 6n+1$. □

Lemma 2.19 writes $L(P) = \beta(P) + \zeta_3(5)\,\alpha(P)$ with rational $\beta(P) = \sum_{k,a} \beta_{k,a} P_a(k)$ and $\alpha(P) = \sum_{k,a} \alpha_{k,a} P_a(k)$. If $\zeta_3(5)$ were rational, the value $L(P)$ alone would not determine $\alpha(P)$ and $\beta(P)$, and the polynomial we work with must not depend on such a choice. We therefore define it by the local constants, with an indeterminate X in place of $\zeta_3(5)$.

Definition 2.23 (the Hankel polynomial)**.** For $k \in \mathcal{N}$ put $c_{k,a}(X) = \beta_{k,a} + X\alpha_{k,a} \in \mathbb{Q}[X]$ for $0 \le a \le 3$, and $c_{k,a} = 0$ for $a \ge 4$. Define $\mathbb{Q}[X]$-linear maps on $\mathbb{Q}[X][y]$ by

$$\Lambda_k(P) = \sum_{a=0}^{3} c_{k,a}(X)\, P_a(k)\ , \qquad \Lambda = \sum_{k\in\mathcal{N}} \Lambda_k\ , \qquad \Lambda(P) = \beta(P) + X\,\alpha(P)\ . \tag{2.24}$$

For $K \ge 1$ the *Hankel polynomial* is

$$\Delta_K(X) = \det\left[\Lambda(y^{i+j})\right]_{0\le i,j<K} = \det\left[\beta(y^{i+j}) + X\,\alpha(y^{i+j})\right]_{0\le i,j<K} \in \mathbb{Q}[X]\ . \tag{2.25}$$

Every entry of the matrix in (2.25) is affine in X, so $\deg \Delta_K \le K$. If $K \le 3n+1$, then $2K-2 \le 6n$, so by Lemma 2.19 the entries at $X = \zeta_3(5)$ are the moments $L(y^{i+j})$, and

$$\Delta_K(\zeta_3(5)) = \det\left[L(y^{i+j})\right]_{0\le i,j<K} \qquad (K \le 3n+1)\ . \tag{2.26}$$

The 3-adic estimate concerns the right side, a determinant of 3-adic numbers; the other estimates concern the coefficients of $\Delta_K(X)$.

By the Leibniz rule for Taylor coefficients, $(y^{i+j})_m(k) = \sum_{a+b=m} (y^i)_a(k)(y^j)_b(k)$, so

$$\Lambda(y^{i+j}) = \sum_{k\in\mathcal{N}} \sum_{a+b\le 3} c_{k,a+b}\,(y^i)_a(k)\,(y^j)_b(k)\ , \qquad (y^i)_a(k) = \binom{i}{a}(-k)^{i-a}\ . \tag{2.27}$$

The Hankel matrix is therefore a sum of $3n+1$ matrices of rank at most 4, one for each node, and the node k enters only through its 4×4 matrix of local constants $C_k = [c_{k,a+b}]_{0\le a,b\le 3}$, which vanishes below the anti-diagonal. The Heine expansion of Section 5.1 expands Δ_K over the ways of distributing its K rows among the nodes, the denominator estimates bound each node separately, and the non-vanishing theorem chooses a basis of polynomials in which the nodes decouple modulo the auxiliary prime.

3. The criterion and the proof of Theorem 1.1

In this section we state the criterion, state the four estimates that the rest of the note proves, and deduce Theorem 1.1 from them. The criterion is the product formula for the integer obtained by clearing the denominators of $\Delta_K(a/b)$.

Lemma 3.1 (criterion)**.** *Let $F \in \mathbb{Q}[X]$ be nonzero with $\deg F \le K$, and let $a, b \in \mathbb{Z}$ be coprime with $b > 0$ and $F(a/b) \ne 0$. Then*

$$v_3(F(a/b))\log_2 3 \le \log_2 \|F\|_1 + \sum_{p\ne 3} (-v_p(F))^+ \log_2 p + K\log_2 \max(|a|, b)\ , \tag{3.2}$$

where the sum runs over all primes $p \ne 3$, including $p = 2$.

Proof. Only finitely many $v_p(F)$ are nonzero. Let $c_F = \prod_p p^{v_p(F)}$ and $Q = F/c_F$. Then $v_p(Q) = 0$ for every p, so $Q \in \mathbb{Z}[X]$, and $N = b^K Q(a/b) = \sum_j q_j a^j b^{K-j}$ is an integer, nonzero because $F(a/b) \ne 0$. Since $v_3(b) \ge 0$,

$$v_3(F(a/b)) - v_3(F) = v_3(Q(a/b)) \le v_3(N) \le \log_3 |N| \le \log_3 \|Q\|_1 + K\log_3 \max(|a|, b)\ , \tag{3.3}$$

and $\log_3 \|Q\|_1 = \log_3 \|F\|_1 - v_3(F) - \sum_{p\ne 3} v_p(F)\log_3 p$. The terms $-v_3(F)$ on the two sides cancel. Multiply by $\log_2 3$ and use $-v_p(F) \le (-v_p(F))^+$. □

We apply the lemma with $F = \Delta_K$, under the assumption $\zeta_3(5) = a/b$. Then $F(a/b) = \Delta_K(\zeta_3(5))$, and the left side of (3.2) is the 3-adic valuation of the determinant (2.26). The lemma needs $\Delta_K(a/b) \ne 0$, and we obtain it from the following observation at an auxiliary prime.

Lemma 3.4 (dominance)**.** *Let $F = \sum_j f_j X^j \in \mathbb{Q}[X]$, and let p be a prime with $v_p(f_0) < v_p(f_j)$ for every $j \ge 1$. Then $F(a/b) \ne 0$ for every $a/b \in \mathbb{Q}$ with $p \nmid b$.*

Proof. If $p \nmid b$, then $v_p(f_j(a/b)^j) \geq v_p(f_j) > v_p(f_0)$ for $j \geq 1$, so $v_p(F(a/b)) = v_p(f_0) < \infty$. □

The same idea, a coefficient that is strictly dominant at auxiliary primes, replaces the non-vanishing hypothesis in the irrationality criterion of Lai and Sprang [9, Lemma 2.2], and [13] uses it in the form of Lemma 3.4. The prime p can be chosen after b: if dominance holds at every prime of a window whose number of primes grows with n, then for n large one of these primes does not divide b.

We now state the four estimates, writing $\kappa = K/n$. The first estimate is proved in Section 4.

Theorem 3.5 (3-adic gain)**.** *For $K \leq 3n+1$,*

$$v_3(\Delta_K(\zeta_3(5)))\log_2 3 \geq G(\kappa)\,n^2 - O(n\log n)\ , \qquad G(\kappa) = \left(10.5\,\kappa - \frac{\kappa^2}{4}\right)\log_2 3\ . \tag{3.6}$$

The second and the third estimate bound the size and the denominators of Δ_K. Their main terms are of order $n^2\log n$, and they cancel: $(\kappa^2 - 6\kappa) + \kappa(6-\kappa) = 0$. Roughly, both estimates measure the same products $\prod_k h_k^{s_k}\prod_{k\neq m}(k-m)^{s_k s_m}$ coming from the Heine expansion of Section 5.1, where $s_k \leq 4$ is the number of variables of that expansion placed at the node k, once with the absolute value and once with the p-adic valuations for $p \neq 3$, and by the product formula the two agree up to the 3-adic part and up to terms of order n^2. What remains are two terms of order n^2, whose constants $F^\dagger$ and c depend on κ. We bound them separately on the five intervals

$$I_i = [\kappa_{1,i}, \kappa_{2,i}] = [2.80 + 0.04(i-1),\ 2.84 + 0.04(i-1)] \qquad (i = 1, \dots, 5)\ , \tag{3.7}$$

which cover $[2.8, 3]$, and on each interval the bounds hold uniformly in κ.

Theorem 3.8 (size)**.** *There are constants $F_i^\dagger$, defined in* (5.10) *and bounded in Table 1, such that for $K/n \in I_i$*

$$\log_2 \|\Delta_K\|_1 \leq (\kappa^2 - 6\kappa)\,n^2\log_2 n + F_i^\dagger\,n^2 + O(n\log n)\ , \tag{3.9}$$

uniformly for $K/n \in I_i$, and with an implied constant that does not depend on i.

Theorem 3.10 (denominators)**.** *There are constants c_i, defined in* (5.39) *and bounded in Table 1, such that*

$$\sum_{p\neq 3} (-v_p(\Delta_K))^+ \log_2 p \leq \kappa(6-\kappa)\,n^2\log_2 n + c_i\,n^2 + o(n^2)\ , \tag{3.11}$$

uniformly for $K/n \in I_i$.

Theorems 3.8 and 3.10 are proved in Section 5. The fourth estimate supplies the non-vanishing and fixes K. For a prime p with $2.1n + 2 \leq p < 3n$ let

$$K(n,p) = \begin{cases} 10n - \dfrac{10(p-1)}{3}\ , & p \equiv 1 \pmod 3\ , \\[2ex] 10n - \dfrac{10p-14}{3}\ , & p \equiv 2 \pmod 3\ . \end{cases} \tag{3.12}$$

It is the number of Taylor conditions that the non-vanishing argument imposes (Section 6.2).

Theorem 3.13 (non-vanishing)**.** *Let p be a prime with $2.1n + 2 \leq p < 3n$, and let $K = K(n,p)$. Then $K \leq 3n-2$, and $\Delta_K(a/b) \neq 0$ for every $a/b \in \mathbb{Q}$ with $p \nmid b$. If moreover $p \leq 2.159n$, then $K > 2.8n$.*

The lower limit $2.1n + 2$ for p is what gives $K \leq 3n-2$, so that (2.26) applies, and the upper limit $2.159n$ gives $K > 2.8n$, the left end of the range covered by the intervals (3.7). The theorem is proved in Section 6, where we show that the constant coefficient of Δ_K is strictly p-adically dominant and apply Lemma 3.4.

I_i	$G(\kappa_{1,i})$	$F_i^\dagger$	c_i	$G(\kappa_{1,i}) - F_i^\dagger - c_i$
$[2.80, 2.84]$	43.49	8.43	30.48	4.58
$[2.84, 2.88]$	44.06	8.46	30.79	4.81
$[2.88, 2.92]$	44.64	8.52	31.10	5.02
$[2.92, 2.96]$	45.21	8.58	31.41	5.22
$[2.96, 3.00]$	45.78	8.65	31.72	5.41

TABLE 1. The constants of Theorems 3.5–3.10, in bits per n^2. The values of G are rounded down, the bounds for $F_i^\dagger$ and c_i of Propositions 5.11 and 5.40 are rounded up, and the last column is computed from the printed values. The function G is increasing on $[2.8, 3]$, so $G(\kappa) \geq G(\kappa_{1,i})$ on I_i. The proof of Theorem 1.1 needs only that the last column is positive.

Proof of Theorem 1.1. Suppose, for a contradiction, that $\zeta_3(5) = a/b$ with coprime integers a and $b > 0$. From now on a and b are fixed, and only the odd integer n varies.

Step 1: the auxiliary prime. Let N_b be the number of distinct prime factors of b. By the prime number theorem, the interval $[2.1n + 2,\ 2.159n]$ contains $(0.059 + o(1))\, n/\log n$ primes, a number that tends to infinity with n. So there is N_1 such that for every $n \geq N_1$ the interval contains more than N_b primes, and hence a prime $p \nmid b$. This is the only use of the prime number theorem in this proof, and it concerns large n only: for small n the interval may contain no prime at all, but small n are never used.

Step 2: one exact inequality. Fix an odd $n \geq N_1$, a prime $p \nmid b$ in the interval, and $K = K(n, p)$. By Theorem 3.13, $2.8n < K \leq 3n - 2$ and $\Delta_K(a/b) \neq 0$, so $\kappa = K/n$ lies in one of the intervals I_i, and Lemma 3.1 applies with $F = \Delta_K$. It gives the inequality (3.2) for this n, without any error term.

Step 3: the three estimates. Theorems 3.5, 3.8 and 3.10 bound the three quantities in (3.2). Each bound consists of a main term and an error term, and what these theorems assert about the error terms is the following: there are functions $E_1(n)$, $E_2(n)$ and $E_3(n)$, which depend neither on a, b and p nor on K in the range $K/n \in I_i$, such that $E_j(n)/n^2 \to 0$ as $n \to \infty$, and the gain is at least $G(\kappa)n^2 - E_1(n)$, the size is at most $(\kappa^2 - 6\kappa)n^2 \log_2 n + F_i^\dagger n^2 + E_2(n)$, and the denominators are at most $\kappa(6 - \kappa)n^2 \log_2 n + c_i n^2 + E_3(n)$. (There are five intervals, so we may take the same functions for all i.) Inserting these bounds into (3.2), the terms of order $n^2 \log n$ cancel, because $(\kappa^2 - 6\kappa) + \kappa(6 - \kappa) = 0$ for every κ. Using $G(\kappa) \geq G(\kappa_{1,i})$ and $K \leq 3n$, we obtain

$$\Bigl(G(\kappa_{1,i}) - F_i^\dagger - c_i\Bigr)\, n^2 \leq E_1(n) + E_2(n) + E_3(n) + 3n \log_2 \max(|a|, b)\ . \tag{3.14}$$

Step 4: the contradiction. By Table 1, the bracket on the left is at least 4.58 for every i. Dividing (3.14) by n^2 gives

$$4.58 \leq \frac{E_1(n) + E_2(n) + E_3(n)}{n^2} + \frac{3 \log_2 \max(|a|, b)}{n}\ .$$

The right side tends to 0 as $n \to \infty$, because a and b are fixed, so there is N_2 such that it is smaller than 4.58 for all $n \geq N_2$. For an odd $n \geq \max(N_1, N_2)$, Steps 2–4 give a contradiction. Hence $\zeta_3(5)$ is irrational. □

The argument needs a single, sufficiently large n. Nothing is claimed about the inequalities for small n, where the error terms can be larger than the main terms. In particular, the prime number theorem, which also enters the proof of Theorem 3.10, is used there only in asymptotic form: it replaces sums over primes by their main terms with an error that, divided by n^2, tends to 0, and no statement is made for small n.

Remark 3.15 (the numerical constants). The decimal numbers that enter the proof are not approximations whose validity depends on n. (The numbers introduced by "about" in Remark 5.27 and Appendix E are quoted for orientation only and are not used.) They are of two kinds. Some are choices: the window $2.1n+2 \le p \le 2.159n$ for the auxiliary prime, the five intervals I_i, the trial density σ_0 of (5.9), the number $\epsilon = \frac{1}{20}$ of Section 5.2, the partition and the rationals ξ_ν and the discretization parameters of Appendix E, and the circles of Table 3. Every statement in which they enter remains true for every admissible choice; another choice changes the numerical margin, and the argument closes only if the margin stays positive. The others are bounds for constants that are defined exactly. The coefficient 10.5 in $G(\kappa)$ is exact, being $9+\frac{3}{2}$ from the $9n+3$ trits of (4.5) and from $3v_3(n!) = \frac{3}{2}n + O(\log n)$. The constants $F_i^\dagger$ and c_i are defined by (5.10) and (5.39), and Propositions 5.11 and 5.40 bound them from above by rigorous computations, in exact rational arithmetic and in ball arithmetic of which only the upper ends are used, with the discretization error bounded by Lemma B.14; the printed values are rounded up once more. So the last column of Table 1 is a lower bound for the true margin, and the proof needs only that it is positive.

4. The 3-adic gain

In this section we prove Theorem 3.5. Following [13], we proceed in two steps. First we write the determinant (2.26) in a basis of polynomials that are integer valued on the set S of integration points; this changes its valuation by an amount that depends only on S and K. Then we show that L maps every polynomial that is integer valued on S to a number divisible by about $3^{10.5n}$. The second step rests on a feature of the 3-adic place that has no archimedean counterpart: by the ultrametric inequality, every node $-k$ is at distance exactly 3 from every point of S, since $|t+\frac{r}{3}+k|_3 = |\frac{r}{3}|_3 = 3$. So each pole of order 4 multiplies the weight by 3^4 uniformly on S.

Let $\mathrm{Int}(S) = \{B \in \mathbb{Q}_3[y] :\ B(S) \subseteq \mathbb{Z}_3\}$. For the general theory of integer-valued polynomials on subsets we refer to Bhargava [23]; we need only the explicit polynomials of the following lemma.

Lemma 4.1. *For $m \ge 0$ put*

$$F_{2m}(y) = 3^m\, m! \binom{y-\frac{1}{3}}{m}\binom{y-\frac{2}{3}}{m}\,, \qquad F_{2m+1}(y) = 3y\, F_{2m}(y)\ . \tag{4.2}$$

Then $F_i \in \mathrm{Int}(S)$, $\deg F_i = i$, and the leading coefficient of F_i has 3-adic valuation $-\nu_i$, where $\nu_i = v_3(m!) - m - e$ for $i = 2m+e$, $e \in \{0,1\}$. Consequently, for every $\mathbb{Q}_3$-linear functional Λ' on $\mathbb{Q}_3[y]$,

$$v_3 \det\big[\Lambda'(y^{i+j})\big]_{i,j<K} \ge 2\sum_{i<K}\nu_i + K \min_{i,j<K} v_3\,\Lambda'(F_iF_j)\ . \tag{4.3}$$

Proof. Let $y = \frac{1}{3}+t$ with $t \in \mathbb{Z}_3$. Then $\binom{y-1/3}{m} = \binom{t}{m} \in \mathbb{Z}_3$, and $3^m m!\binom{y-2/3}{m} = \prod_{i<m}(3y-2-3i) = \prod_{i<m}(3t-1-3i)$ is a product of 3-adic units. So $F_{2m}(y) \in \mathbb{Z}_3$. On $\frac{2}{3}+\mathbb{Z}_3$ the two binomial coefficients exchange their roles, and on S the factor $3y$ is a unit, so also $F_{2m+1}(S) \subseteq \mathbb{Z}_3$. The leading coefficient of F_{2m} is $3^m m!/(m!)^2 = 3^m/m!$, of valuation $m - v_3(m!)$, and that of F_{2m+1} is $3^{m+1}/m!$. For (4.3), write $F_i = \sum_{j\le i} f_{ij}y^j$. The matrix $T = [f_{ij}]$ is lower triangular with diagonal entries f_{ii}, and $[\Lambda'(F_iF_j)] = T\,[\Lambda'(y^{i+j})]\,T^{\mathsf T}$. Hence $\det\big[\Lambda'(y^{i+j})\big] = \prod_{i<K} f_{ii}^{-2}\det[\Lambda'(F_iF_j)]$, and every term of the Leibniz expansion of the last determinant is a product of K entries. □

The first term of (4.3) is negative at $p=3$. By Legendre's formula $v_3(m!) = (m - s_3(m))/2$, where $s_3(m) \le 2\log_3(3m)$ is the sum of the ternary digits of m, so

$$2\sum_{i<K}\nu_i = -2\sum_{m<K/2} m + O(K\log K) = -\frac{K^2}{4} + O(K\log K)\ . \tag{4.4}$$

At $p = 2$, where the integration set of [13] is a translate of $\mathbb{Z}_2$ and the binomial basis has leading coefficients $1/i!$, the corresponding term is positive and of size about K^2 bits. At $p = 3$ the change of basis costs rather than gains, and the whole gain has to come from the weight.

We now turn to the weight on S. Fix $r \in \{1, 2\}$ and put $y = t + \frac{r}{3}$ with $t \in \mathbb{Z}_3$. The factors of W behave as follows.

- Of the two Pochhammer symbols, the one with shift $\frac{3-r}{3}$ becomes $(t+1)_n = n!\binom{t+n}{n}$, an integer-valued polynomial times $n!$.
- The other becomes $\prod_{i<n}(t + \frac{2r}{3} + i) = 3^{-n}\prod_{i<n}(3t + 2r + 3i)$, and every factor $3t + 2r + 3i$ is a unit.
- $y + \frac{n}{2} = \frac{1}{6}(6t + 2r + 3n)$, where $6t + 2r + 3n$ is a unit, so this factor has valuation -1.
- Every node contributes $(y + k)^{-4} = 3^4(3t + r + 3k)^{-4}$, where $3t + r + 3k$ is a unit.

The powers of 3 add up to $-3n - 1 + 4(3n + 1) = 9n + 3$, and therefore

$$W\Big(t + \frac{r}{3}\Big) = 3^{9n+3}(n!)^3\binom{t+n}{n}^3 Q_r(t)\ , \tag{4.5}$$

where Q_r is a unit constant times a product of factors $(a + 3t)^{\pm 1}$ and $a + 6t$ with integers a prime to 3. Each such factor is $a^{\pm 1}(1 + 3t/a)^{\pm 1}$ or $a(1 + 6t/a)$, a power series whose coefficient of t^i has valuation at least i. Hence $Q_r(t) = \sum_i q_i t^i$ with $v_3(q_i) \geq i$.

Thus the poles contribute $4(3n + 1)$ trits, the non-integral Pochhammer symbol and the linear factor cost $3n + 1$ trits, and the integral Pochhammer symbol adds $3v_3(n!)$, about $1.5n$ trits. It remains to see how much is lost when the derivative and the integral are taken.

The following lemma is proved in Appendix A.

Lemma 4.6 (after [13, Lemmas 4.2 and 4.3])**.** *Let $d \geq 1$, let P be a polynomial of degree at most d with $P(\mathbb{Z}_3) \subseteq \mathbb{Z}_3$, and let $Q(t) = \sum_i q_i t^i$ with $v_3(q_i) \geq i$. Then PQ is strictly differentiable on $\mathbb{Z}_3$, and*

$$v_3 \int_{\mathbb{Z}_3} (PQ)'(t)\,\mathrm{d}t \geq -\log_3 d - \log_3(d + 1)\ . \tag{4.7}$$

The two losses come from the binomial basis. Writing PQ in the basis $\binom{t}{c}$, the derivative of $\binom{t}{c}$ is $\sum_{i=1}^{c} \frac{(-1)^{i-1}}{i}\binom{t}{c-i}$, which costs at most $\log_3 d$, and the integral of $\binom{t}{c}$ is $(-1)^c/(c + 1)$ by (V2), which costs at most $\log_3(d+1)$. The factor Q costs nothing, because its coefficients in the binomial basis decay fast enough to compensate for the shift of degrees.

Proof of Theorem 3.5. Let $A_3 = 9n + 3 + 3v_3(n!)$ and $d_K = 2K - 2 + 3n$. By (2.26) and Lemma 4.1 with $\Lambda' = L$, it suffices to show that

$$v_3 L(B) \geq A_3 - \log_3 d_K - \log_3(d_K + 1) \tag{4.8}$$

for every $B \in \mathrm{Int}(S)$ of degree at most $2K - 2$, since each product $F_i F_j$ with $i, j < K$ is such a B. Fix $r \in \{1, 2\}$. By (4.5), $(BW)(t + \frac{r}{3}) = 3^{9n+3}(n!)^3 P(t) Q_r(t)$ with $P(t) = B(t + \frac{r}{3})\binom{t+n}{n}^3$. Both factors of P map $\mathbb{Z}_3$ into $\mathbb{Z}_3$, and $\deg P \leq d_K$. Since $(BW)'(t + \frac{r}{3}) = \frac{\mathrm{d}}{\mathrm{d}t}(BW)(t + \frac{r}{3})$, Lemma 4.6 gives (4.8) for the r-th term of $L(B)$, and the sum over r follows from the ultrametric inequality. Hence

$$v_3\,\Delta_K(\zeta_3(5)) \geq 2\sum_{i<K} \nu_i + K(A_3 - \log_3 d_K - \log_3(d_K + 1))\ . \tag{4.9}$$

By Legendre's formula $3v_3(n!) = 1.5n - O(\log n)$, so $KA_3 = 10.5\,\kappa n^2 - O(n \log n)$; by (4.4) the first term is $-\kappa^2 n^2/4 + O(n \log n)$; and $K \log_3(d_K(d_K + 1)) = O(n \log n)$. Multiplying by $\log_2 3$ gives (3.6). □

5. The size and the denominators

In this section we prove Theorems 3.8 and 3.10. Both concern the coefficients of $\Delta_K(X)$, and both start from an expansion of the determinant over the nodes, which we state first.

5.1. **The Heine expansion and the size.** Heine's formula writes the Hankel determinant of the moments of a positive measure on $\mathbb{R}$ as $\frac{1}{K!}$ times the integral of the squared Vandermonde $\prod_{i<i'}(y_{i'}-y_i)^2$ against the K-fold product of the measure, see e.g. [24]. The identity is algebraic and holds for every linear functional (Lemma B.1). Our functional Λ is a sum of the local functionals Λ_k, and Λ_k sees only the Taylor coefficients of order at most 3 at the node k. So the K variables $y_1,\dots,y_K$ are distributed among the nodes, and a node that receives five or more variables contributes nothing (Step 6 of the proof in Appendix B); the determinant becomes a finite sum over these distributions. For a node k and a subset $R\subseteq\{0,1,2,3\}$ with elements $R_0<\dots<R_{|R|-1}$, write $C_k[R]=[c_{k,R_b+b'}]_{0\le b,b'<|R|}$ for the square submatrix of C_k with row set R and the first $|R|$ columns.

Proposition 5.1 (Heine expansion; after [13, Proposition 5.1])**.** *Let $K\ge 1$. Then*

$$\Delta_K(X)=\sum_{s}\sum_{\lambda}\operatorname{coef}_\lambda(s)\prod_{k\in\mathcal{N}}M_{k,\lambda_k}(X)\ , \tag{5.2}$$

where:

(a) *$s=(s_k)_{k\in\mathcal{N}}$ runs over the profiles, the integer vectors with $0\le s_k\le 4$ and $\sum_k s_k=K$, and $\lambda=(\lambda_k)_{k\in\mathcal{N}}$ over the tuples of partitions with at most s_k parts and $|\lambda_k|\le s_k(4-s_k)$;*

(b) *for each profile fix a map $\iota:\{1,\dots,K\}\to\mathcal{N}$ with $|\iota^{-1}(k)|=s_k$, and put*

$$Q_s(w)=\prod_{\substack{i<i'\\ \iota(i)\neq\iota(i')}}\big(\iota(i')-\iota(i)+w_i-w_{i'}\big)^2\in\mathbb{Z}[w_1,\dots,w_K]\ ; \tag{5.3}$$

then $\operatorname{coef}_\lambda(s)\in\mathbb{Z}$ is the coefficient in Q_s of any monomial whose exponents at the variables w_i with $\iota(i)=k$ are, for every k, a rearrangement of the parts of λ_k, padded with zeros;

(c) *$M_{k,\emptyset}=1$, and for $s_k\ge 1$, M_{k,λ_k} is an integer combination of the minors $\det C_k[R]$ with $|R|=s_k$ and $\sum R-\binom{s_k}{2}=|\lambda_k|$, whose coefficients depend only on s_k, λ_k and R.*

The proof is in Appendix B. In a term of (5.2), the local factors M_{k,λ_k} carry the local constants, hence the numbers $h_k^{s_k}$, and the cross factor $\operatorname{coef}_\lambda(s)$ carries the node differences $(k-m)^{2s_ks_m}$, up to the corrections that come from the variables w. The size estimate applies the triangle inequality to (5.2), and the denominator estimate applies the ultrametric inequality.

For the size, in the rescaled variable $x=k/n$, in which the node k, sitting at $y=-k$, has the position $x\in[-1,2]$, a profile s becomes in the limit a density σ on $[-1,2]$ with $0\le\sigma\le 4$ and total mass κ. For integrable f on $[-1,2]$ let

$$\mathcal{L}f(x)=\int_{-1}^{2}\log|x-y|\,f(y)\,\mathrm{d}y\ ,\qquad \langle f,g\rangle=\int_{-1}^{2}f(x)g(x)\,\mathrm{d}x\ . \tag{5.4}$$

In the same variable, the zeros $y=-(j+\frac{r}{3})$ of W sit at $x=(j+\frac{r}{3})/n\in(0,1)$, with density $3+3=6$ on $I_{\mathrm{in}}=(0,1)$, and its poles have density 4 on $[-1,2]$. Accordingly, the *external field* is the potential of the zeros minus that of the poles,

$$\mathcal{V}=6\,\mathcal{L}\mathbf{1}_{I_{\mathrm{in}}}-4\,\mathcal{L}\mathbf{1}_{[-1,2]}=2\,\mathcal{L}\mathbf{1}_{I_{\mathrm{in}}}-4\,\mathcal{L}\mathbf{1}_{I_{\mathrm{out}}}\ ,\qquad I_{\mathrm{out}}=[-1,0]\cup[1,2]\ . \tag{5.5}$$

A measurable function σ on $[-1,2]$ is *feasible* for κ if $0\le\sigma\le 4$ and $\int\sigma=\kappa$. Put

$$\mathcal{E}[\sigma]=\langle\sigma,\mathcal{V}\rangle+\langle\sigma,\mathcal{L}\sigma\rangle\ ,\qquad P_\sigma=\mathcal{V}+2\mathcal{L}\sigma\ ,\qquad \operatorname{gap}(\sigma)=\sup_{\sigma'\ \text{feasible}}\langle P_\sigma,\sigma'-\sigma\rangle\ge 0\ . \tag{5.6}$$

The functional $\mathcal{E}$ is the energy of the density in the field $\mathcal{V}$ plus its logarithmic self-energy, and P_σ is its first variation. The kernel $\log|x-y|$ is conditionally negative definite, that is, $\iint \log|x-y|\,\mathrm{d}\nu(x)\,\mathrm{d}\nu(y) \le 0$ for every signed measure ν of total mass 0 for which the integral exists (Lemma B.10 proves the discrete form that we need), so $\mathcal{E}$ is concave on the feasible densities, and $\mathcal{E}[\sigma]+\mathrm{gap}(\sigma)$ is an upper bound for $\sup\mathcal{E}$ obtained from a single trial density σ. The following theorem is the discrete form of this statement.

Theorem 5.7 (after [13, Theorem 6.1])**.** *Let $J \ge 1$. For $n \le K \le 3n+1$ and every feasible σ that is piecewise constant with at most J pieces,*

$$\log\|\Delta_K\|_1 \le (\kappa^2-6\kappa)\,n^2\log n + n^2(\mathcal{E}[\sigma]+\mathrm{gap}(\sigma)) + O_J(n\log n)\ , \tag{5.8}$$

where the implied constant depends only on J.

The proof is in Appendix B. The term $-6\kappa n^2\log n$ comes from $\sum_k s_k\log|h_k|$, as $|h_k|$ is about n^{-6n} by (2.18), and the term $\kappa^2 n^2\log n$ comes from the cross factors $|k-m|^{2s_ks_m}$. The details of the weight that are not visible in the densities, namely the zero $y=-\frac{n}{2}$, the distance $\frac{1}{3}$ of the closest zeros to a node, the harmonic sums and the weights $i(i+1)$ and 2916 in (2.21), affect only the error term. Since the zeros and the poles of W have the same densities as in [13], $\mathcal{E}$ is the functional of [13].

To prove Theorem 3.8 we use the trial density of [13, Table 2]. Let σ_0 be the step function on the intervals $[\frac{j}{10},\frac{j+1}{10})$, $-10\le j\le 19$, with

$$64\,\sigma_0 = 95,\ 112,\ 113,\ 113,\ 112,\ 109,\ 104,\ 95,\ 70,\ 3 \quad \text{on} \quad [-1,-0.9),\ \dots,\ [-0.1,0)\ , \tag{5.9}$$

$\sigma_0=0$ on $[0,1)$, and $\sigma_0(x)=\sigma_0(1-x)$ on $[1,2]$. Its mass is $m_0=\int\sigma_0=\frac{463}{160}$. For $\kappa\in[2.8,3]$ put $\sigma_\kappa=(\kappa/m_0)\,\sigma_0$. Since $\max\sigma_\kappa<1.84$, σ_κ is feasible for κ, and it has 30 pieces. The density vanishes on the interval I_{in}, where the zeros of W repel the variables, and puts its mass on $[-1,0]\cup[1,2]$. Define

$$F_i^\dagger = \frac{1}{\log 2}\sup_{\kappa\in I_i}\left(\mathcal{E}[\sigma_\kappa]+\mathrm{gap}(\sigma_\kappa)\right) \qquad (i=1,\dots,5)\ . \tag{5.10}$$

Proposition 5.11. *The constants $F_1^\dagger,\dots,F_5^\dagger$ are at most* 8.425174*,* 8.458313*,* 8.510836*,* 8.576728 *and* 8.647895*, respectively.*

The proof is a finite computation in ball arithmetic, described in Appendix E. It uses that $\mathcal{E}[\sigma_\kappa]$ is a quadratic polynomial in κ with explicit coefficients, and that $\mathrm{gap}(\sigma_\kappa)$ is bounded by the value of a linear optimization problem whose data are values of the explicit function P_{σ_κ}.

Proof of Theorem 3.8. For $K/n=\kappa\in I_i$ apply Theorem 5.7 with $\sigma=\sigma_\kappa$ and $J=30$, divide by $\log 2$, and use $\mathcal{E}[\sigma_\kappa]+\mathrm{gap}(\sigma_\kappa)\le F_i^\dagger\log 2$. The implied constant depends only on $J=30$, hence not on i or κ. □

5.2. **The tree bound and the small primes.** We now turn to the denominators. We bound $-v_p(\Delta_K)$ for each prime $p\ne 3$ separately and then sum over p. The bound for a single prime is a combinatorial maximum over the profiles of Proposition 5.1. Fix $\epsilon=\frac{1}{20}$. At the primes $p\le\epsilon n$ a crude evaluation of this maximum suffices, and its sum over p produces the main term $\kappa(6-\kappa)n^2\log n$ through Mertens' theorem. The primes $p>\epsilon n$ give the positive part of the constant c_i, and there we need the refined local lemma of Section 5.3 and the circle model of Section 5.4. Until the end of this section $p\ne 3$ is a prime, $p=2$ included, and $v=v_p$ is the Gauss valuation on $\mathbb{Q}_p[X]$.

For a node k put $\eta_p(k)=\max_{m\in\mathcal{N}\setminus\{k\}} v_p(k-m)$. For $0\le s\le 4$ and $\ell\ge 0$ put

$$\mathsf{e}_p(k,s,\ell) = \max\left\{-v(\det C_k[R]):\ R\subseteq\{0,1,2,3\},\ |R|=s,\ \textstyle\sum R-\binom{s}{2}=\ell\right\}\ , \tag{5.12}$$

with $\mathsf{e}_p = -\infty$ if there is no such R. Let $f_p(k,0) = 0$, and for $s \geq 1$ let $f_p(k,s)$ be the maximum of $\mathsf{e}_p(k,s,\ell) + \ell\,\eta_p(k)$ over $0 \leq \ell \leq s(4-s)$.

Theorem 5.13 (tree bound; after [13, Theorem 7.1])**.** *For every prime p,*

$$-v_p(\Delta_K) \leq T_p := \max_s \left[\sum_{k\in\mathcal{N}} f_p(k,s_k) - \sum_{j\geq 1} \sum_{\mathfrak{c} \bmod p^j} \left(S_{\mathfrak{c}}^2 - \sum_{k\in\mathfrak{c}} s_k^2 \right) \right], \tag{5.14}$$

where s runs over the profiles, $\mathfrak{c}$ over the residue classes of the nodes modulo p^j, and $S_{\mathfrak{c}} = \sum_{k\in\mathfrak{c}} s_k$.

The proof is in Appendix B. The first sum in (5.14) is the loss from the local factors of (5.2). The second is what the cross factors save: variables placed at two nodes that are congruent modulo p^j produce node differences divisible by p^j. The residue classes modulo $p, p^2, p^3, \ldots$ form a tree, whose leaves are the single nodes, and the maximum is an optimization over this tree.

The nonzero roots and poles of $\widetilde{W}_k$ are $d = k - j - \frac{r}{3} = \frac{3k-3j-r}{3}$ for $0 \leq j < n$, $r \in \{1,2\}$ (zeros of order 3), $d = k - \frac{n}{2}$ (a zero of order 1), and $d = k - m$ for $m \in \mathcal{N} \setminus \{k\}$ (poles of order 4). Write e_d for the order, positive for zeros and negative for poles, so that $\widetilde{W}_k(z) = h_k \prod_d (1 - z/d)^{e_d}$. Let

$$D_k = -v_p(h_k) = 4 \sum_{m\neq k} v_p(k-m) - 3 \sum_{0\leq j<n,\, r=1,2} v_p(3k-3j-r) - v_p\left(\frac{2k-n}{2}\right), \tag{5.15}$$

where the last term equals $+1$ for $p = 2$, since $2k - n$ is odd, and $-v_p(2k-n)$ for odd p. Since the coefficient of z^b in $(1-z/d)^e$ is an integer times d^{-b}, we have the following.

Lemma 5.16. *Let $\mu_k = \max_d v_p(d)$ over the nonzero roots and poles d of $\widetilde{W}_k$. Then $v_p(H_k[b]) \geq -D_k - b\mu_k$ for $b \geq 0$, and $v_p(\mathrm{HS}_3(k,M)) \geq -M \max_u v_p(u)$, the maximum over the u in (2.14).*

Let $\mathfrak{l}_p = \lceil \log_p(6n) \rceil$. All integers whose valuations enter the local constants have absolute value less than $6n$, so these valuations are at most $\mathfrak{l}_p$. The following three lemmas are the analogues in our setting of [13, Lemmas 8.2, 8.3 and Proposition 8.5], and their proofs are in Appendix C. The first bounds the local loss at a single node, a leaf of the tree.

Lemma 5.17 (leaf lemma)**.** *For every node k and $1 \leq s \leq 4$, $f_p(k,s) \leq sD_k + 9s\mathfrak{l}_p$.*

Lemma 5.18 (small primes)**.** *For every prime $p \neq 3$, $T_p \leq K(6n+4-K)/(p-1) + 9K\mathfrak{l}_p + 150n + K$.*

Lemma 5.19 (middle primes)**.** *If $p \geq 5$, $p^2 > 6n$ and $p \leq 0.4n$, then $T_p \leq K(6n+4-K)/p + 3K + 25p$.*

The leading terms $K(6n+4-K)/(p-1)$ and $K(6n+4-K)/p$, summed with weight $\log p$ over $p \leq \epsilon n$, give $\kappa(6-\kappa)n^2 \log n$ plus a term of order n^2, by Mertens' theorem. For $p > \epsilon n$ these bounds are too weak, and we treat these primes more precisely.

5.3. **Single-level primes.** A prime $p \geq 5$ with $p^2 > 6n$ is called *single-level*. Every integer that occurs in the local constants, namely $k - m$, $3k - 3j - r$, the u of (2.14) and $2k - n$, has absolute value less than $6n < p^2$, so its p-adic valuation is 0 or 1. In particular the classes modulo p^j with $j \geq 2$ are singletons, and the tree (5.14) has only one level.

At such a prime, a nonzero root or pole d of $\widetilde{W}_k$ is *p-special* if $p \mid d$. There are three kinds of p-special points. The poles $d = k - m$ with $m \equiv k \pmod p$ come from the *partners* m of k, the other nodes congruent to k. The zero $d = k - \frac{n}{2}$ is p-special if $p \mid 2k - n$. And the zero $d = \frac{3k-3j-r}{3}$ is p-special exactly when $3k - 3j - r = pt$ for an integer t, necessarily prime to 3; as $3j + r$ runs over the integers in $[1, 3n-1]$ prime to 3, pt runs over the interval $(3k - 3n, 3k)$. Besides, the terms of $\mathrm{HS}_3(k,M)$ with $p \mid u$ are, writing $u = pt$, the numbers $(3/(pt))^M = \rho_t^{-M}$ with $\rho_t = pt/3$. The following definition records these data.

Definition 5.20 (node data)**.** Let p be single-level and k a node. Let o_k be the number of partners of k, let $g_k = 1$ if $p \mid 2k - n$ and $g_k = 0$ otherwise, and let

(5.21)
$$\mathrm{VP}_k = \{t \in \mathbb{Z} \setminus 3\mathbb{Z} :\ 0 < pt < 3k\} \quad (k > 0)\ , \qquad \mathrm{VP}_k = \{t \in \mathbb{Z} \setminus 3\mathbb{Z} :\ 3k < pt < 0\} \quad (k < 0)\ ,$$
$$\mathrm{VP}_0 = \emptyset\ , \qquad \mathcal{Z}_k = \{t \in \mathbb{Z} \setminus 3\mathbb{Z} :\ 3k - 3n < pt < 3k\}\ .$$

For $t \in \mathrm{VP}_k$ the point $\rho_t = pt/3$ is a *virtual pole* of the node, and it is *covered* if $t \in \mathcal{Z}_k$. Put $z_k = 3|\mathcal{Z}_k|$. For $t \in \mathrm{VP}_k$ let $\mu_{k,t} = 0$ if t is covered and $o_k + z_k + g_k = 3$, and $\mu_{k,t} = 1$ otherwise.

Counted with multiplicity, the p-special points of $\widetilde{W}_k$ are $4o_k$ poles and $z_k + g_k$ zeros, so $D_k = 4o_k - z_k - g_k$ by (5.15), and the μ_k of Lemma 5.16 equals 1 if $o_k + z_k + g_k > 0$ and 0 otherwise. The condition $\mu_{k,t} = 0$ means that t is covered and that the covering zero is the only p-special point of the node: $o_k = 0$, $g_k = 0$ and $\mathcal{Z}_k = \{t\}$.

Split $\beta_{k,a} = \beta^{\mathrm{HS}}_{k,a} + \beta^{\mathrm{rest}}_{k,a}$, where $\beta^{\mathrm{HS}}_{k,a}$ collects the terms of the harmonic sums with $p \mid u$. The next lemma says that these terms act like additional poles of order 3 at the points $z = \rho_t$, which are not poles of W at all.

Lemma 5.22 (virtual poles)**.** *For every node k,*

$$\beta^{\mathrm{HS}}_{k,a} = 2\,\mathrm{sgn}(k) \sum_{t \in \mathrm{VP}_k} [z^{3-a}]\, \widetilde{W}_k(z)\,(\rho_t - z)^{-3}\ . \tag{5.23}$$

Proof. For $k > 0$ the part of $\mathrm{HS}_3(k, M)$ with $p \mid u$ is $\sum_{t \in \mathrm{VP}_k} \rho_t^{-M}$, and for $k < 0$ it is $-\sum_{t \in \mathrm{VP}_k} \rho_t^{-M}$. From $\sum_{i \ge 1} i(i+1)x^{i-1} = 2(1-x)^{-3}$ we get $2(\rho - z)^{-3} = \sum_{i \ge 1} i(i+1)\rho^{-(i+2)} z^{i-1}$. So the coefficient of z^{3-a} in $\widetilde{W}_k(z) \cdot 2(\rho_t - z)^{-3}$ is $\sum_{i \ge 1} i(i+1)\rho_t^{-(i+2)} H_k[4 - a - i]$, which is the contribution of ρ_t to $\beta_{k,a}$ in (2.21); the sum stops at $i = 4 - a$ because $H_k[b] = 0$ for $b < 0$. □

A virtual pole costs p-adic valuation: ρ_t has $v_p(\rho_t) = 1$, so the factor $(\rho_t - z)^{-3}$ contributes p^{-3-c} to the coefficient of z^c. The bound of [13, Section 8.3] charges every virtual pole in this way. The next lemma shows that a covered virtual pole costs less, because the zero of W that covers it has the same order 3 as the virtual pole.

Lemma 5.24 (refined local lemma)**.** *Let p be single-level. For every node k and $0 \le m \le 3$,*

$$v_p(c_{k,m}) \ge \hat{v}_m(k) := \min\left(-D_k - (3-m)\mu_k\ ,\ \min_{t \in \mathrm{VP}_k} [-D_k - 3 - (3-m)\mu_{k,t}]\ ,\ \hat{a}_m(k)\right)\ , \tag{5.25}$$

where $\hat{a}_m(k) = -D_k - (1-m)\mu_k$ for $m \le 1$, $\hat{a}_m(k) = +\infty$ for $m \ge 2$, and the minimum over t is $+\infty$ if $\mathrm{VP}_k = \emptyset$. The three terms bound $\beta^{\mathrm{rest}}_{k,m}$, $\beta^{\mathrm{HS}}_{k,m}$ and $\alpha_{k,m}$, respectively.

Proof. Since all valuations are 0 or 1, $v_p(h_k) = -D_k$, and every d has $v_p(d) \le \mu_k$, so $v(H_k[b]) \ge -D_k - b\mu_k$ by Lemma 5.16.

The part β^{rest}. The weights $i(i+1)$ are integers and the terms of the harmonic sums with $p \nmid u$ are p-integral, so $v(\beta^{\mathrm{rest}}_{k,m}) \ge \min_{i \ge 1} v(H_k[4 - i - m]) \ge -D_k - (3-m)\mu_k$.

The part α. Here $\alpha_{k,m} = -2916\, H_k[1-m]$, and $2916 = 2^2 \cdot 3^6$ is prime to p, which gives the third term.

The part β^{HS}. We use (5.23), one virtual pole at a time. If t is not covered, then $(\rho_t - z)^{-3} = \rho_t^{-3} \sum_{c \ge 0} \binom{c+2}{2} (z/\rho_t)^c$, so the coefficient of z^{3-m} in $\widetilde{W}_k(z)(\rho_t - z)^{-3}$ has valuation at least $-D_k - 3 - (3-m)$, and this is $-D_k - 3 - (3-m)\mu_{k,t}$ because $\mu_{k,t} = 1$. If t is covered, $\widetilde{W}_k$ contains the factor $(1 - z/\rho_t)^3$ with exponent exactly 3. Indeed, the zeros $k - j - \frac{r}{3}$ are pairwise distinct, ρ_t is not a pole because the poles are integers, and ρ_t is not the zero $k - \frac{n}{2}$, because $v_3(\rho_t) = -1$

while $v_3(k-\frac{n}{2}) \geq 0$. Since $1 - z/\rho_t = (\rho_t - z)/\rho_t$,

$$\left(1 - \frac{z}{\rho_t}\right)^3 (\rho_t - z)^{-3} = \rho_t^{-3} \ , \tag{5.26}$$

and therefore $\widetilde{W}_k(z)(\rho_t - z)^{-3} = h_k \rho_t^{-3} \prod_{d \neq \rho_t} (1 - z/d)^{e_d}$. The virtual pole has disappeared, and only the constant ρ_t^{-3} remains. If $\mu_{k,t} = 0$, every remaining d is a p-adic unit, and if $\mu_{k,t} = 1$, every remaining d has valuation at most 1. So the coefficient of z^b of the remaining product has valuation at least $-b\mu_{k,t}$, and the coefficient of z^{3-m} of $\widetilde{W}_k(z)(\rho_t - z)^{-3}$ has valuation at least $-D_k - 3 - (3-m)\mu_{k,t}$. □

Stated otherwise, a covered virtual pole that is the only p-special point of its node contributes $v_p \geq -D_k - 3$ in every Taylor order m, whereas the bound of [13, Section 8.3], transcribed to our weight, charges it $-D_k - 6 + m$, with D_k already lowered by the covering zero. The cancellation (5.26) rests on the coincidence of the order 3 of the zeros of W with the order 3 of the virtual poles, which comes from the single derivative in (1.3). At $p = 2$ in [13] the zeros have order 6 and the virtual poles order 5.

Remark 5.27. Without the refined lemma our argument does not close, at least at $\kappa = 3$. For primes $p > 3n$ no node has a partner, and every virtual pole is covered and alone at its node, so the lemma gives $\hat{v} \geq 0$ and these primes cost nothing (Lemma 5.31). The cruder bound, by contrast, gives a positive value of the relaxed tree bound T_p^+ of Lemma 5.28 at every single-level prime up to $6n$. Evaluating the model of Section 5.4 numerically at $\kappa = 3$, with the cruder bound in place of $\hat{v}$, gives a denominator constant of about 44 bits per n^2, against about 31.5 with $\hat{v}$ (the bound $c_5 \leq 31.72$ of Table 1 is larger because it holds on all of I_5). Since $G(3) - F_5^\dagger$ is about 37.7, the comparison would then give no contradiction. This evaluation is not part of the proof; we quote it only to show that the cancellation (5.26) is needed in our argument.

5.4. **The circle model and the prime sum.** For the single-level primes we now evaluate the maximum T_p asymptotically. The plan is the following. First, T_p is bounded by a sum of independent maxima, one for each residue class modulo p (Lemma 5.28). For $p = \varrho n$ with ϱ fixed, a residue class modulo p corresponds to a point of the circle $\mathbb{R}/\varrho\mathbb{Z}$, and the data of its nodes depend only on that point (Lemma 5.31). So T_p is about n times an integral over the circle, which is piecewise affine in ϱ (Lemma 5.34). Summing over p with the prime number theorem turns it into an integral over ϱ (Proposition 5.36).

Lemma 5.28 (relaxation)**.** *Let p be single-level and let $\check{v}_m(k)$, $0 \leq m \leq 3$, be integers with $v(c_{k,m}) \geq \check{v}_m(k)$ for all k and m, for instance $\hat{v}_m(k)$. Put $f^+(k,0) = 0$ and, for $1 \leq s \leq 4$,*

$$f^+(k,s) = \max\left(0,\ \max_{R,\pi}\left[\sum_{b<s} -\check{v}_{R_b+\pi(b)}(k) + \left(\sum R - \binom{s}{2}\right)[o_k > 0]\right]\right) , \tag{5.29}$$

the maximum over $R = \{R_0 < \cdots < R_{s-1}\} \subseteq \{0,1,2,3\}$ and permutations π of $\{0,\dots,s-1\}$ with $R_b + \pi(b) \leq 3$ for all b. For a residue class $\mathfrak{c}$ modulo p let

$$\begin{aligned} V_{\mathfrak{c}}(S) &= \max\left\{\sum_{k\in\mathfrak{c}} \left(f^+(k,s_k) + s_k^2\right) :\ 0 \leq s_k \leq 4,\ \sum_{k\in\mathfrak{c}} s_k = S\right\} - S^2 \ , \\ \psi_{\mathfrak{c}}(\xi) &= \max_{0\leq S\leq 4|\mathfrak{c}|} \left(V_{\mathfrak{c}}(S) - \xi S\right) \ . \end{aligned} \tag{5.30}$$

Then $T_p \leq T_p^+ := \max\{\sum_{\mathfrak{c}} V_{\mathfrak{c}}(S_{\mathfrak{c}}) : \sum_{\mathfrak{c}} S_{\mathfrak{c}} = K\}$, and $T_p^+ \leq \xi K + \sum_{\mathfrak{c}} \psi_{\mathfrak{c}}(\xi)$ for every real ξ.

The multiplier ξ prices the constraint $\sum_{\mathfrak{c}} S_{\mathfrak{c}} = K$, so that each class is optimized separately. With $\varrho = p/n$ and $x = k/n$, a class modulo p consists of the nodes $k \equiv k_0 \pmod p$, and it

corresponds to the point $x_0 = k_0/n$ of the circle $\mathbb{R}/\varrho\mathbb{Z}$. All node data of Definition 5.20, except g_k, are functions of the positions (x, ϱ): $pt < 3k$ if and only if $t\varrho < 3x$, $3k - 3n < pt$ if and only if $3x - 3 < t\varrho$, and the node range is $x \in [-1, 2]$. For $\varrho > 0$ and $x \in \mathbb{R}/\varrho\mathbb{Z}$ the nodes of x are the points $y \in (x + \varrho\mathbb{Z}) \cap [-1, 2]$, and the *type* of x is the multiset of the pairs $(\hat{v}(y), [o > 0])$ over these nodes. Here $\hat{v}(y) = (\hat{v}_0, \dots, \hat{v}_3)$ is given by (5.25), with the data computed from (y, ϱ) and with $g = 0$, and $o + 1$ is the number of nodes of x. For a type τ let ψ_τ be the function $\psi_{\mathfrak{c}}$ of (5.30) for a class with these data.

Lemma 5.31 (cells)**.** *Let $\mathcal{B} = \{3j/N : \ j \in \{1, 2, 3\}, \ N \geq 1\}$. On each open interval of $(0, \infty) \setminus \mathcal{B}$, which we call a cell, the cyclic order of the seven points*

$$-1, \quad 2, \quad 0, \quad \frac{\varrho}{3}, \quad \frac{2\varrho}{3}, \quad 1 + \frac{\varrho}{3}, \quad 1 + \frac{2\varrho}{3} \tag{5.32}$$

of $\mathbb{R}/\varrho\mathbb{Z}$ and the type of each open arc between consecutive points are constant. If each of the seven points is represented by a real number $\alpha + \beta\varrho + w\varrho$, with an integer w that is kept fixed on the whole cell, the lengths of the arcs are differences of such numbers, hence affine in ϱ. Hence the measure $\mathrm{meas}_\tau(\varrho)$ of the set of points of type τ is affine in ϱ on each cell; for a cell (a, b) we write $\mathrm{meas}_\tau^{[a,b]}$ for its affine extension to the closed cell $[a, b]$. Moreover, for $\varrho > 3$ every node has $\hat{v} \geq 0$, and $T_p^+ = 0$ for every single-level prime $p > 3n$.

The seven points are where a node enters or leaves the node range (the points -1 and 2), where the range of the virtual poles changes sign (0), and where a virtual pole or an end of the zero range appears ($3y \in \varrho\mathbb{Z}$ or $3y - 3 \in \varrho\mathbb{Z}$, with the quotient prime to 3). The proof is in Appendix C. With these measures put

$$\mathcal{G}_{[a,b]}(\xi, \varrho, \kappa) = \xi\kappa + \sum_\tau \mathrm{meas}_\tau^{[a,b]}(\varrho)\, \psi_\tau(\xi) \qquad (\varrho \in [a, b]) \ . \tag{5.33}$$

For fixed ξ it is affine in ϱ and in κ.

Lemma 5.34 (counting; after [13, Lemma 8.7])**.** *Fix a finite set $\mathcal{A} \subset \mathbb{R}$. There are C and n_0 such that for all $n \geq n_0$, all primes $p \in [\epsilon n, 3n]$, every closed cell $[a, b] \ni p/n$ and every $\xi \in \mathcal{A}$,*

$$T_p \leq n\, \mathcal{G}_{[a,b]}(\xi, p/n, K/n) + C \ . \tag{5.35}$$

The proof is in Appendix C. It rests on the fact that a residue class modulo p is a point of the grid $\frac{1}{n}\mathbb{Z}$ on the circle $\mathbb{R}/\varrho\mathbb{Z}$, so that, up to $O(1)$ exceptional classes, the number of classes of type τ is $n\,\mathrm{meas}_\tau(\varrho) + O(1)$. Summing (5.35) over the primes in $(\epsilon n, 3n]$ with the prime number theorem, and the bounds of Lemmas 5.18 and 5.19 over the smaller primes with Mertens' theorem, gives the following.

Proposition 5.36 (the prime sum)**.** *Let $\epsilon = a_0 < a_1 < \dots < a_M = 3$ be a partition of $[\epsilon, 3]$ that refines $\mathcal{B} \cap [\epsilon, 3]$, let $\xi_1, \dots, \xi_M$ be rational numbers, and let $[\kappa_1, \kappa_2] \subseteq [2.8, 3]$. For $1 \leq \nu \leq M$ put*

$$\bar{\mathcal{G}}_\nu = \max\big\{\mathcal{G}_{[a,b]}(\xi_\nu, \varrho, \kappa) : \ \varrho \in \{a_{\nu-1}, a_\nu\}, \ \kappa \in \{\kappa_1, \kappa_2\}\big\} \ , \tag{5.37}$$

where $[a, b]$ is the closed cell that contains $[a_{\nu-1}, a_\nu]$. Then, uniformly for $K/n \in [\kappa_1, \kappa_2]$,

$$\begin{aligned}\sum_{p \neq 3} \max(T_p, 0) \log p \leq\ & \kappa(6 - \kappa)\, n^2 \log n - \kappa_1(6 - \kappa_1)\left(\gamma + \frac{\log 3}{2} + \log\frac{1}{\epsilon}\right) n^2 \\ & + \left(\sum_{\nu=1}^{M} \max(\bar{\mathcal{G}}_\nu, 0)(a_\nu - a_{\nu-1}) + 3\kappa_2\epsilon + 12.5\,\epsilon^2\right) n^2 + o(n^2) \ ,\end{aligned} \tag{5.38}$$

where γ is Euler's constant.

The proof is in Appendix C. The first two terms come from the primes $p \le \epsilon n$, the sum over ν from the primes in $(\epsilon n, 3n]$, and the primes $p > 3n$ contribute nothing by Lemma 5.31. For $i = 1, \dots, 5$ let c_i be the value of

$$\frac{1}{\log 2}\left[-\kappa_{1,i}(6-\kappa_{1,i})\left(\gamma + \frac{\log 3}{2} + \log 20\right) + \sum_{\nu=1}^{M} \max(\bar{\mathcal{G}}_\nu, 0)(a_\nu - a_{\nu-1}) + \frac{3\kappa_{2,i}}{20} + \frac{12.5}{400}\right] \tag{5.39}$$

with $[\kappa_1, \kappa_2] = I_i$ in (5.37), for the partition and the rationals ξ_ν described in Appendix E. By Proposition 5.36, any other admissible choice would also give a valid bound.

Proposition 5.40. *The constants $c_1, \dots, c_5$ are at most* 30.471829, 30.783002, 31.093465, 31.403655 *and* 31.713600, *respectively.*

The sum over ν in (5.39) is an exact rational number; its computation and the evaluation of the remaining terms are described in Appendix E.

Proof of Theorem 3.10. By Theorem 5.13, $(-v_p(\Delta_K))^+ \le \max(T_p, 0)$ for every $p \ne 3$. Apply Proposition 5.36 with $[\kappa_1, \kappa_2] = I_i$ and with the partition and the ξ_ν that define c_i, and divide by $\log 2$. □

6. Non-vanishing at an auxiliary prime

In this section we prove Theorem 3.13. Throughout, n is odd and p is a prime with $2.1n + 2 \le p < 3n$. This forces $n \ge 5$ and $p \ge 13$ (for $n = 3$ the interval $[8.3, 9)$ contains no prime), and p is single-level. By Lemma 3.4 it suffices to show that the constant coefficient of Δ_K is strictly p-adically dominant, and we shall compute its p-adic valuation exactly. The structure of the argument is that of [13, Theorem 9.1]: we choose a basis of polynomials adapted to the nodes that are p-adically special, scale its elements by suitable powers of p, and reduce the Gram matrix of the functional modulo $\sqrt{p}$ (Section 6.2), where it becomes block diagonal with explicit blocks.

6.1. **The nodes and their local constants.** Two distinct nodes are congruent modulo p exactly when they differ by p, because $|k - m| \le 3n < 2p$. So the nodes $k \in [p - n, 2n]$, which we call *R nodes*, have the partner $k - p$; the nodes $k \in [-n, 2n - p]$, the *L nodes*, are these partners; and the nodes in $(2n - p, p - n)$, the *U nodes*, have no partner. The last interval is nonempty because $2p > 3n$. In Definition 5.20 only $t \in \{\pm 1, \pm 2\}$ occur, since $|pt| < 6n < 3p$.

Lemma 6.1. *The nodes fall into the types of Table 2. The partner of an Ra (Rb, Rb_g, Rc) node is an La (Lb, Lb_g, Lc) node, and the two nodes $\frac{n \pm p}{2}$ are partners.*

Proof. By (5.21), $t = 1$ lies in VP_k if and only if $k > \frac{p}{3}$, $t = 2$ if and only if $k > \frac{2p}{3}$, $t = -1$ if and only if $k < -\frac{p}{3}$, and $t = -2$ never, since $k \ge -n > -\frac{2p}{3}$. Similarly $t \in \mathcal{Z}_k$ if and only if $\frac{pt}{3} < k < n + \frac{pt}{3}$. Finally $g_k = 1$ if and only if $2k - n = \pm p$, because $2k - n$ is odd and $|2k - n| \le 3n < 2p$. Since $2.1n < p < 3n$, the relevant thresholds are ordered as

$$-n < -\frac{p}{3} < n - \frac{2p}{3} < 2n - p < p - n < \frac{2p}{3} < n + \frac{p}{3} < 2n\ , \tag{6.2}$$

and moreover $n - \frac{p}{3} < p - n$ and $2n - p < \frac{p}{3}$. Each of these inequalities is equivalent to $p < 3n$ or to $2p > 3n$. Reading off the conditions in each interval of (6.2) gives the R and L rows of the table. For instance, an Ra node has $k \ge p - n > \frac{p}{3}$, so $1 \in \mathrm{VP}_k$, and $k < \frac{2p}{3}$, so $2 \notin \mathrm{VP}_k$; it has $\frac{p}{3} < k < n + \frac{p}{3}$, so $1 \in \mathcal{Z}_k$, while $2 \notin \mathcal{Z}_k$ because $k < \frac{2p}{3}$, and $-1, -2 \notin \mathcal{Z}_k$ because $k > n - \frac{p}{3}$. Then $D_k = 4o_k - z_k - g_k = 4 - 3 - 0 = 1$. The other R and L rows are obtained in the same way. At a U node, $o_k = 0$, and there are three cases: if $\frac{p}{3} < k < p - n$, then $\mathrm{VP}_k = \mathcal{Z}_k = \{1\}$ and $D_k = -3$; if $2n - p < k < n - \frac{p}{3}$, then $\mathrm{VP}_k = \emptyset$, $\mathcal{Z}_k = \{-1\}$ and $D_k = -3$; otherwise $\mathrm{VP}_k = \mathcal{Z}_k = \emptyset$ and $D_k = 0$. The first two ranges are disjoint, as $n - \frac{p}{3} < \frac{p}{3}$. Finally, $\frac{n+p}{2}$ lies in the Rb range and $\frac{n-p}{2}$

type	nodes k	data	D_k
Ra	$p-n \le k < \frac{2p}{3}$	$\mathrm{VP}_k=\{1\}$, covered; $\mathcal{Z}_k=\{1\}$	1
Rb	$\frac{2p}{3} < k < n+\frac{p}{3}$, $k \ne \frac{n+p}{2}$	$\mathrm{VP}_k=\mathcal{Z}_k=\{1,2\}$, both covered	-2
Rb_g	$k=\frac{n+p}{2}$	as Rb, and $g_k=1$	-3
Rc	$n+\frac{p}{3} < k \le 2n$	$\mathrm{VP}_k=\{1,2\}$, $\mathcal{Z}_k=\{2\}$	1
La	$-n \le k < -\frac{p}{3}$	$\mathrm{VP}_k=\{-1\}$, not covered; $\mathcal{Z}_k=\{-2\}$	1
Lb	$-\frac{p}{3} < k < n-\frac{2p}{3}$, $k \ne \frac{n-p}{2}$	$\mathrm{VP}_k=\emptyset$, $\mathcal{Z}_k=\{-1,-2\}$	-2
Lb_g	$k=\frac{n-p}{2}$	as Lb, and $g_k=1$	-3
Lc	$n-\frac{2p}{3} < k \le 2n-p$	$\mathrm{VP}_k=\emptyset$, $\mathcal{Z}_k=\{-1\}$	1
U	$2n-p < k < p-n$	no partner; three cases, see the proof	-3, -3, 0

TABLE 2. The nodes at a prime p with $2.1n+2 \le p < 3n$, in the notation of Definition 5.20. All R and L nodes have exactly one partner, and $g_k=0$ except at the two nodes $\frac{n\pm p}{2}$.

in the Lb range, both being integers since n and p are odd, and the map $k \mapsto k-p$ sends each R range onto the L range of the same letter. □

Let N_{Ra} and N_{Rc} be the numbers of Ra and Rc nodes, and N_{Rb} the number of Rb and Rb_g nodes together. Counting the integers in the ranges of Table 2,

$$\begin{aligned} &N_{\mathrm{Ra}}=N_{\mathrm{Rb}}=N_{\mathrm{Rc}}=n-\frac{p-1}{3} \qquad (p\equiv 1 \bmod 3)\ , \\ &N_{\mathrm{Ra}}=N_{\mathrm{Rc}}=n-\frac{p-2}{3}\ , \quad N_{\mathrm{Rb}}=n-\frac{p+1}{3} \qquad (p\equiv 2 \bmod 3)\ , \end{aligned} \tag{6.3}$$

and in both cases the number $K(n,p)$ of (3.12) is

$$K(n,p)=4(N_{\mathrm{Ra}}+N_{\mathrm{Rc}})+2N_{\mathrm{Rb}}\ . \tag{6.4}$$

From (3.12) and $2.1n+2 \le p$ we get $K \le 10n-\frac{10(2.1n+2)-14}{3}=3n-2$, and if $p \le 2.159n$, then $K \ge 10n-\frac{10(2.159n-1)}{3} > 2.8n$. This proves the first and the last statement of Theorem 3.13.

At the R nodes and at the La nodes, the leading term of $\beta^{\mathrm{HS}}_{k,m}$ can be computed exactly. For such a node let $\Phi_k(w)=\mathrm{sgn}(k)\prod(1-w/\delta)^e\sum_{t\in\mathrm{VP}_k}(\frac{t}{3}-w)^{-3}$, where the product runs over the p-special points $d=p\delta$ of $\widetilde{W}_k$, with their orders e. From Table 2,

$$\begin{aligned} &\Phi_{\mathrm{Ra}}(w)=(1-w)^{-4}(1-3w)^3\big(\tfrac13-w\big)^{-3}=27(1-w)^{-4}\ , \\ &\Phi_{\mathrm{Rb}}(w)=(1-w)^{-4}(1-3w)^3\big(1-\tfrac32 w\big)^3\Sigma(w)\ , \qquad \Phi_{\mathrm{Rb}_g}(w)=(1-2w)\,\Phi_{\mathrm{Rb}}(w)\ , \\ &\Phi_{\mathrm{Rc}}(w)=(1-w)^{-4}\big(1-\tfrac32 w\big)^3\Sigma(w)\ , \qquad \Phi_{\mathrm{La}}(w)=(1+w)^{-4}\big(1+\tfrac32 w\big)^3\big(\tfrac13+w\big)^{-3}\ , \end{aligned} \tag{6.5}$$

with $\Sigma(w)=(\frac13-w)^{-3}+(\frac23-w)^{-3}$. For example, at an Ra node the p-special points of $\widetilde{W}_k$ are the partner pole $d=k-(k-p)=p$, of order -4, and the covering zero $d=\frac{p}{3}$, of order 3, and $\mathrm{VP}_k=\{1\}$. At an La node they are the partner pole $d=-p$ and the zero $d=-\frac{2p}{3}$, the virtual pole is $t=-1$, and $\mathrm{sgn}(k)=-1$. Let $\phi_{k,b}=[w^b]\Phi_k(w)$. These coefficients depend only on the type. Expanding (6.5), the ones that we need are

$$\begin{aligned} &\text{Ra: } 27,\ 108,\ 270,\ 540\ , \qquad \text{La: } 27,\ -\tfrac{459}{2},\ \tfrac{5211}{4},\ -\tfrac{49761}{8} \qquad (b=0,1,2,3)\ , \\ &\text{Rb: } -\tfrac{243}{8},\ -\tfrac{243}{8},\ 0\ , \qquad \mathrm{Rb}_g\text{: } -\tfrac{729}{8},\ \tfrac{243}{8},\ \tfrac{243}{4} \qquad (b=1,2,3)\ , \qquad \text{Rc: } \phi_{\mathrm{Rc},0}=\tfrac{243}{8}\ . \end{aligned} \tag{6.6}$$

All coefficients of the Φ_k are p-integral, their denominators being powers of 2.

Lemma 6.7. *Let $\varepsilon_k = 2h_k p^{D_k}$, a p-adic unit. For the nodes of types Ra, Rb, Rb$_g$, Rc and La,*

$$\beta^{\mathrm{HS}}_{k,m} = \varepsilon_k\, p^{-D_k+m-6}(\phi_{k,3-m} + O(p)) \qquad (0 \le m \le 3)\ , \tag{6.8}$$

where $O(p)$ denotes an element of $p\mathbb{Z}_p$. The other local constants satisfy the following bounds, which are those of Lemma 5.24 with the data of Table 2: $v_p(\beta^{\mathrm{rest}}_{k,m}) \ge -4+m$ and $v_p(\alpha_{k,m}) \ge -2+m$ at the nodes of types Ra, Rc, La and Lc; $v_p(\beta^{\mathrm{rest}}_{k,m}) \ge -1+m$ and $v_p(\alpha_{k,m}) \ge 1+m$ at Rb and Lb; and $v_p(\beta^{\mathrm{rest}}_{k,m}) \ge m$ and $v_p(\alpha_{k,m}) \ge 2+m$ at Rb$_g$ and Lb$_g$. At the U nodes, all local constants $\beta^{\mathrm{HS}}_{k,m}$, $\beta^{\mathrm{rest}}_{k,m}$ and $\alpha_{k,m}$ have $v_p \ge 0$.

Proof. Substitute $z = pw$ in (5.23). Then $[z^{3-m}]f(z) = p^{-(3-m)}[w^{3-m}]f(pw)$ for any power series f, and

$$\begin{aligned} &\widetilde{W}_k(pw) = h_k \prod_{p\mid d} \Big(1 - \frac{w}{\delta}\Big)^{e_d} \Upsilon_k(pw)\ , \qquad \Upsilon_k(z) = \prod_{p\nmid d} \Big(1 - \frac{z}{d}\Big)^{e_d}\ , \\ &(\rho_t - pw)^{-3} = p^{-3}\bigg(\frac{t}{3} - w\bigg)^{-3}\ . \end{aligned} \tag{6.9}$$

The coefficient of z^b in $\Upsilon_k(z)$ is p-integral, so the coefficient of w^b in $\Upsilon_k(pw) - 1$ lies in $p\mathbb{Z}_p$ for every b. Hence $\beta^{\mathrm{HS}}_{k,m} = 2h_k p^{-(3-m)-3}[w^{3-m}](\Phi_k(w)\Upsilon_k(pw)) = \varepsilon_k p^{-D_k+m-6}(\phi_{k,3-m} + O(p))$, using that Φ_k has p-integral coefficients. The bounds for β^{rest} and α are the first and the third term of (5.25) with $\mu_k = 1$ and the D_k of the table. At a U node with $\mathrm{VP}_k = \mathcal{Z}_k = \{1\}$ we have $D_k = -3$, $\mu_k = 1$ and $\mu_{k,1} = 0$, so the three terms of (5.25) are m, 0 and $2+m$; at a U node with $\mathcal{Z}_k = \{-1\}$ and no virtual pole they are m and $2+m$; and at the remaining U nodes $D_k = 0$ and $\mu_k = 0$. □

The next lemma relates the units ε_k at the two members of an Ra/La pair; this is where Wilson's theorem enters.

Lemma 6.10 (partner congruence)**.** *If k is an Ra node, then $\varepsilon_{k-p} \equiv -8\,\varepsilon_k \pmod p$.*

Proof. Both nodes have $D = 1$, so $\varepsilon_{k-p}/\varepsilon_k = h_{k-p}/h_k$. Write $W = N_W/\prod_m (y+m)^4$ with $N_W(y) = (y+\frac{n}{2})(y+\frac{1}{3})_n^3(y+\frac{2}{3})_n^3$, and $\Pi(k) = \prod_{m\ne k}(m-k) = (-1)^{k+n}(k+n)!\,(2n-k)!$ as in (2.18). Then $h_k = N_W(-k)/\Pi(k)^4$, and

$$\frac{\Pi(k)}{\Pi(k-p)} = (-1)^p\, \frac{\prod_{i=1}^{p}(n+k-p+i)}{\prod_{i=1}^{p}(2n-k+i)}\ . \tag{6.11}$$

Both products run over p consecutive integers in $[1, 3n]$. Since $2p > 3n$, the only multiple of p among them is p itself, and by Wilson's theorem the product of the other $p-1$ factors is $\equiv (p-1)! \equiv -1 \pmod p$ in each. Hence $\Pi(k)/\Pi(k-p) \equiv -1$, and its fourth power is $\equiv 1$. For each linear factor $y+c$ of N_W, the ratio of its values at $y = -k+p$ and $y = -k$ is $(c-k+p)/(c-k)$, which is $\equiv 1 \pmod p$ unless $p \mid c-k$. At an Ra node the only such factor is the covering zero, with $c-k = -\frac{p}{3}$ and multiplicity 3, which contributes $((-\frac{1}{3}+1)/(-\frac{1}{3}))^3 = (-2)^3 = -8$. □

6.2. **An adapted basis and the reduction.** The choice of the basis comes from the tree bound of Section 5.2 at the prime p. Lemma 5.28 bounds $-v_p(\Delta_K)$ by a sum over the residue classes modulo p, and here a class is either a single U node or a pair consisting of an R node and its partner. If a node with a partner has $\check{v}_m = m - D^*$ for $m \le 3$, then (5.29) gives $f^+(k,s) = \max(0, D^*s - s(s-1))$, because $\sum_b (R_b + \pi(b)) = \sum R + \binom{s}{2}$ for every row set R in (5.29). By Lemma 6.7, the bound (5.25) is $-7+m$ at the Ra, Rc and La nodes, $-4+m$ at Rb and Lc, $-3+m$ at Rb$_g$, $-1+m$ at Lb, and m at Lb$_g$. A short computation with (5.30) then gives the class values. For a pair of type a or c, $V_{\mathfrak{c}}(S) = 8S - S^2$ for $S \le 4$, so the first four rows placed in the class have the marginal values 7, 5, 3, 1, and further rows have negative marginal values. For a pair of type b the first two rows have the marginal values 4, 2, and for the pair of type b$_g$ the values 3, 1; further rows have marginal

values at most 0. A U node accepts no row at a positive marginal value, since $\hat{v} \geq 0$ there. So the relaxed tree bound attains its largest value, $16(N_{\mathrm{Ra}} + N_{\mathrm{Rc}}) + 6N_{\mathrm{Rb}} - 2$, when K is the number (6.4). We shall show that the constant coefficient of Δ_K attains this value exactly.

Accordingly, on polynomials of degree less than $K = K(n,p)$ we prescribe the Taylor coefficients of orders $0,1,2,3$ at the Ra and Rc nodes, and those of orders $0,1$ at the Rb and Rb_g nodes. We call these nodes the *condition nodes*; there are K conditions by (6.4). The conditions sit at the R nodes and not at their partners, because two partners are congruent modulo p, and the next lemma needs condition nodes that are pairwise incongruent. We index the conditions by $x = (R_x, j_x)$, where R_x is a condition node and j_x the order.

Lemma 6.12. *There are polynomials $E_x \in \mathbb{Z}_p[y]$ of degree less than K with $(E_x)_{j_{x'}}(R_{x'}) = \delta_{x,x'}$ for all conditions x and x', and the transition matrix from the monomials y^i, $i < K$, to the E_x has determinant in $\mathbb{Z}_p^\times$.*

Proof. Let Ξ be the $K \times K$ matrix of the conditions $P \mapsto P_j(R)$ on the monomials y^i, $i < K$. Its entries $\binom{i}{j}(-R)^{i-j}$ are integers. We claim that Ξ is invertible modulo p. The condition nodes lie in $[p-n, 2n]$, an interval of length $3n - p < p$, so they are pairwise incongruent modulo p. Over the field $\mathbb{F}_p$, a polynomial of degree less than K is determined by its residues modulo the coprime polynomials $(y+R)^{d_R}$, where $d_R \in \{2,4\}$ is the number of conditions at R and $\sum_R d_R = K$ (Chinese remainder theorem), and the residue modulo $(y+R)^{d_R}$ is determined by the Taylor coefficients of orders less than d_R at $-R$. So the reduction of Ξ is injective, hence invertible, and $\det \Xi \in \mathbb{Z}_p^\times$. The polynomials E_x are the dual basis, whose coefficients are the entries of Ξ^{-1}. $\square$

Let T be the matrix of coefficients of the E_x, so that $[\Lambda(E_x E_{x'})]_{x,x'} = T\,[\Lambda(y^{i+j})]_{i,j<K}\,T^{\mathsf{T}}$ and $\det T \in \mathbb{Z}_p^\times$. Hence $\det[\Lambda(E_x E_{x'})] = (\det T)^2 \Delta_K(X)$, and the coefficients of $\det[\Lambda(E_x E_{x'})]$ and of $\Delta_K(X)$ have the same p-adic valuations. Write

$$\begin{aligned} &\Lambda(E_x E_{x'}) = \Gamma^{\beta}_{xx'} + X\,\Gamma^{\alpha}_{xx'}\,, \qquad \Gamma^{\beta} = \Gamma^{\mathrm{HS}} + \Gamma^{\mathrm{rest}}\,, \\ &\Gamma^{\bullet}_{xx'} = \sum_{k \in \mathcal{N}} \sum_{a+b \leq 3} c^{\bullet}_{k,a+b}\,(E_x)_a(k)\,(E_{x'})_b(k)\,, \end{aligned} \tag{6.13}$$

where $c^{\mathrm{HS}} = \beta^{\mathrm{HS}}$, $c^{\mathrm{rest}} = \beta^{\mathrm{rest}}$ and $c^{\alpha} = \alpha$, by (2.27). Then $[X^0]\det[\Lambda(E_x E_{x'})] = \det \Gamma^{\beta}$.

To evaluate the terms of (6.13) we need the Taylor coefficients of E_x at every node. Since $E_x \in \mathbb{Z}_p[y]$, all of them lie in $\mathbb{Z}_p$. At a condition node R the prescribed ones are given by Lemma 6.12. At the partner $\bar{R} = R - p$ of a condition node R, which sits at $y = -R + p$, Taylor's formula gives

$$(E_x)_b(\bar{R}) = \sum_{j \geq b} \binom{j}{b} (E_x)_j(R)\, p^{j-b}\,. \tag{6.14}$$

So if $R = R_x$ and $b \leq j_x$, the term $j = j_x$ gives $\binom{j_x}{b} p^{j_x - b}$, and $(E_x)_b(\bar{R})$ has valuation at least $j_x - b$. In all other cases only orders j that are not prescribed contribute, and $(E_x)_b(\bar{R})$ has valuation at least $4-b$ if R is of type Ra or Rc, and at least $\max(0, 2-b)$ if R is of type Rb or Rb_g. Accordingly, at the partner $\bar{R}$ of an Ra or Rc node R we write

$$(E_x)_b(\bar{R}) = \xi_{x,b} + \xi'_{x,b}\,, \qquad \xi_{x,b} = \binom{j_x}{b} p^{j_x - b} \quad (R_x = R,\ b \leq j_x)\,, \qquad \xi_{x,b} = 0 \quad \text{otherwise}\,, \tag{6.15}$$

so that $v_p(\xi'_{x,b}) \geq 4 - b$ in all cases. At such a partner we split each term of (6.13) into the four products of these parts, and we count them as separate contributions.

To each condition x we attach the weight θ_x, half of the marginal value of its row in the tree bound above: $\theta_x = \frac{7}{2} - j_x$ if R_x is of type Ra or Rc, $\theta_x = 2 - j_x$ if it is of type Rb, and $\theta_x = \frac{3}{2} - j_x$

if it is of type Rb_g. A *contribution* to $\Gamma^{\bullet}_{xx'}$ is one term $c^{\bullet}_{k,a+b}(E_x)_a(k)(E_{x'})_b(k)$ of (6.13), and its *weighted valuation* is

$$\varpi = v_p\big(c^{\bullet}_{k,a+b}\big) + v_p((E_x)_a(k)) + v_p((E_{x'})_b(k)) + \theta_x + \theta_{x'} \in \tfrac{1}{2}\mathbb{Z}\ . \tag{6.16}$$

With these weights the leading contributions at a condition node have $\varpi = 0$. For instance, at an Ra or Rc node R, by (6.8) with $D_R = 1$, the contribution of $\beta^{\mathrm{HS}}_{R,j_x+j_{x'}}$ to $\Gamma^{\mathrm{HS}}_{xx'}$ with $R_x = R_{x'} = R$ has valuation $j_x + j_{x'} - 7$, and $\theta_x + \theta_{x'} = 7 - j_x - j_{x'}$.

Lemma 6.17 (weighted valuations)**.** *Every contribution to Γ^{β} has $\varpi \geq 0$, and every contribution to Γ^{α} has $\varpi \geq 1$. If $R_x \neq R_{x'}$, every contribution to $\Gamma^{\beta}_{xx'}$ has $\varpi \geq \frac{1}{2}$. If $R_x = R_{x'} = R$, the contributions to $\Gamma^{\beta}_{xx'}$ with $\varpi = 0$ are the terms of β^{HS} at R with $(a,b) = (j_x, j_{x'})$ and, if R is of type Ra, the products $\beta^{\mathrm{HS}}_{\bar{R},a+b}\,\xi_{x,a}\,\xi_{x',b}$ at its partner $\bar{R}$.*

The proof is a case analysis over the types of nodes, given in Appendix D. Since the lemma allows contributions with $\varpi = \frac{1}{2}$, at the Rb nodes, the reduction below is made modulo $\sqrt{p}$ and not modulo p.

Let Θ be the diagonal matrix with entries p^{θ_x}, viewed in the field $\mathbb{Q}_p(\sqrt{p})$. The polynomial $Y^2 - p$ is an Eisenstein polynomial, so this is a totally ramified quadratic extension of $\mathbb{Q}_p$, with valuation ring $\mathbb{Z}_p[\sqrt{p}]$ and residue field $\mathbb{Z}_p[\sqrt{p}]/(\sqrt{p}) = \mathbb{F}_p$, and the valuation v_p extends with $v_p(\sqrt{p}) = \frac{1}{2}$. By Lemma 6.17, the scaled matrix $\widetilde{\Gamma}^{\beta} = \Theta\,\Gamma^{\beta}\,\Theta$ has entries in $\mathbb{Z}_p[\sqrt{p}]$, its entries with $R_x \neq R_{x'}$ vanish modulo $\sqrt{p}$, and $\widetilde{\Gamma}^{\alpha} = \Theta\,\Gamma^{\alpha}\,\Theta$ has entries in $p\mathbb{Z}_p[\sqrt{p}]$. So modulo $\sqrt{p}$ the matrix $\widetilde{\Gamma}^{\beta}$ is block diagonal, with one block for each condition node, and Lemma 6.17, (6.8) and (6.14) give the blocks.

Lemma 6.18. *Modulo $\sqrt{p}$, the block of $\widetilde{\Gamma}^{\beta}$ at a condition node R is*

$$\begin{array}{l} \varepsilon_R\,[\phi_{\mathrm{Rc},3-a-b}]_{0\leq a,b\leq 3} \qquad (R \text{ of type Rc})\ , \\ \varepsilon_R \begin{pmatrix} \phi_{\mathrm{Rb},3} & \phi_{\mathrm{Rb},2} \\ \phi_{\mathrm{Rb},2} & \phi_{\mathrm{Rb},1} \end{pmatrix} \qquad (R \text{ of type Rb})\ , \qquad \varepsilon_R \begin{pmatrix} \phi_{\mathrm{Rb}_g,3} & \phi_{\mathrm{Rb}_g,2} \\ \phi_{\mathrm{Rb}_g,2} & \phi_{\mathrm{Rb}_g,1} \end{pmatrix} \qquad (R \text{ of type Rb}_g)\ , \\ \varepsilon_R\Big(A_{\mathrm{Ra}} - 8\,\mathrm{Bin}^{\mathsf{T}} A_{\mathrm{La}}\mathrm{Bin}\Big) \qquad (R \text{ of type Ra})\ , \end{array} \tag{6.19}$$

where $\phi_b = 0$ for $b < 0$, $A_{\bullet} = [\phi_{\bullet,3-a-b}]_{0\leq a,b\leq 3}$, and $\mathrm{Bin} = [\binom{j}{a}]_{0\leq a,j\leq 3}$.

Proof. At an Rc, Rb or Rb_g node R the only contributions with $\varpi = 0$ are those of $\beta^{\mathrm{HS}}_{R,j_x+j_{x'}}$ with $(E_x)_{j_x}(R) = (E_{x'})_{j_{x'}}(R) = 1$. By (6.8), multiplied by $p^{\theta_x+\theta_{x'}} = p^{D_R+6-j_x-j_{x'}}$, each becomes $\varepsilon_R(\phi_{R,3-j_x-j_{x'}} + O(p))$. For Rb and Rb_g, $j_x, j_{x'} \in \{0,1\}$, which gives the 2×2 blocks. At an Ra node R there is, in addition, the partner $\bar{R}$, of type La. By (6.15) and Lemma 6.17, the contributions of $\bar{R}$ with $\varpi = 0$ are the products with $\xi_{x,a}\xi_{x',b}$, and they sum, after scaling, to

$$\varepsilon_{\bar{R}} \sum_{a\leq j_x,\, b\leq j_{x'}} \binom{j_x}{a}\binom{j_{x'}}{b}\phi_{\mathrm{La},3-a-b} = \varepsilon_{\bar{R}}\Big(\mathrm{Bin}^{\mathsf{T}} A_{\mathrm{La}}\mathrm{Bin}\Big)_{j_x j_{x'}} \tag{6.20}$$

modulo p, and $\varepsilon_{\bar{R}} \equiv -8\,\varepsilon_R$ by Lemma 6.10. □

The determinants of the blocks are explicit. The Rc block vanishes below its anti-diagonal and has the anti-diagonal entries $\varepsilon_R\phi_{\mathrm{Rc},0} = \varepsilon_R \cdot \frac{3^5}{2^3}$, so its determinant is $\varepsilon_R^4\,3^{20}/2^{12}$. By (6.6) the Rb block is $\varepsilon_R \cdot \frac{243}{8}\left(\begin{smallmatrix} 0 & -1 \\ -1 & -1 \end{smallmatrix}\right)$, of determinant $-\varepsilon_R^2\,3^{10}/2^6$, and the Rb_g block is $\varepsilon_R \cdot \frac{243}{8}\left(\begin{smallmatrix} 2 & 1 \\ 1 & -3 \end{smallmatrix}\right)$, of determinant $-\varepsilon_R^2\,7\cdot 3^{10}/2^6$. For the Ra block, the matrix $\mathrm{Bin}^{\mathsf{T}} A_{\mathrm{La}}\mathrm{Bin}$ has the entries $\sum_m \binom{a+b}{m}\phi_{\mathrm{La},3-m}$, which

depend only on $a+b$, and a direct computation from (6.6) gives

$$A_{\mathrm{Ra}} - 8\,\mathrm{Bin}^{\mathsf{T}} A_{\mathrm{La}}\mathrm{Bin} = 3^5 \begin{pmatrix} 207 & 163 & 127 & 98 \\ 163 & 127 & 98 & 75 \\ 127 & 98 & 75 & 57 \\ 98 & 75 & 57 & 43 \end{pmatrix} , \tag{6.21}$$

which is 3^5 times an integer matrix of determinant 1, so its determinant is 3^{20}. All four determinants are products of units with powers of 2, 3 and 7, so they are p-adic units, as $p \geq 13$.

Remark 6.22. For a general value ϑ in place of -8 one finds $\det\bigl(A_{\mathrm{Ra}} + \vartheta\,\mathrm{Bin}^{\mathsf{T}} A_{\mathrm{La}}\mathrm{Bin}\bigr) = 3^{12}(\vartheta-1)^4$. So the Ra block is a unit precisely when $\vartheta \not\equiv 1 \pmod p$, and for $\vartheta = -8$ this holds since $p \nmid 9$. The number $-8 = (-2)^3$ comes from the covering zero of the Ra node: its linear factor takes the values $-\frac{p}{3}$ and $\frac{2p}{3}$ at the two members of the pair.

Proof of Theorem 3.13. The bounds on K were proved after (6.4). By Lemma 6.18 and the computation of the four block determinants, $\det\widetilde{\Gamma}^\beta$ is a unit of $\mathbb{Z}_p[\sqrt{p}]$. Since $\det\widetilde{\Gamma}^\beta = p^{2\sum_x \theta_x}\det\Gamma^\beta$, we get $v_p(\det\Gamma^\beta) = -2\sum_x \theta_x$. The sum of the weights is $\frac{7}{2}+\frac{5}{2}+\frac{3}{2}+\frac{1}{2} = 8$ at each Ra and Rc node, $2+1=3$ at each Rb node and $\frac{3}{2}+\frac{1}{2}=2$ at the single Rb_g node, so

$$v_p\bigl([X^0]\Delta_K\bigr) = v_p\Bigl(\det\Gamma^\beta\Bigr) = -(16(N_{\mathrm{Ra}}+N_{\mathrm{Rc}}) + 6N_{\mathrm{Rb}} - 2)\ . \tag{6.23}$$

The coefficient of X^j in $\det\Bigl(\widetilde{\Gamma}^\beta + X\widetilde{\Gamma}^\alpha\Bigr)$ is a sum of determinants of matrices with j columns taken from $\widetilde{\Gamma}^\alpha$ and the others from $\widetilde{\Gamma}^\beta$, so it lies in $p^j\mathbb{Z}_p[\sqrt{p}]$. Hence $v_p([X^j]\Delta_K) \geq v_p([X^0]\Delta_K) + j$ for every $j \geq 1$, and the constant coefficient of Δ_K is strictly dominant at p. By Lemma 3.4, $\Delta_K(a/b) \neq 0$ for every $a/b \in \mathbb{Q}$ with $p \nmid b$. □

Appendix A. Proof of Lemma 4.6

We expand only the polynomial part of PQ in the binomial basis and estimate the remainder directly, so that no infinite series is differentiated term by term.

A polynomial P of degree at most d with $P(\mathbb{Z}_3) \subseteq \mathbb{Z}_3$ can be written as $P = \sum_{c\leq d} \hat{P}_c\binom{t}{c}$ with

$$\hat{P}_c = \sum_{i=0}^{c} (-1)^{c-i}\binom{c}{i} P(i) \in \mathbb{Z}_3\ , \tag{A.1}$$

because the $\binom{t}{c}$, $c \leq d$, are a basis, and applying c times the difference operator $f \mapsto f(t+1)-f(t)$, which maps $\binom{t}{c}$ to $\binom{t}{c-1}$, and evaluating at 0 gives $\hat{P}_c$. This is the polynomial case of Mahler's expansion. The monomial coefficients of $\binom{t}{c}$ are integers divided by $c!$, so the monomial coefficients of P have valuation at least $-v_3(d!)$.

Let $\varsigma(m) = m + v_3(m!)$. It is increasing and superadditive, $\varsigma(a+b) \geq \varsigma(a)+\varsigma(b)$, since $(a+b)!/(a!\,b!)$ is an integer. The series Q converges on $|t|_3 < 3$, so PQ is strictly differentiable. Fix $M > d$ and split $Q = Q_{<M} + R_M$ with $R_M = \sum_{i\geq M} q_i t^i$.

The polynomial part. Using $t^i = \sum_{m\leq i} \left\{ {i \atop m} \right\} m!\binom{t}{m}$, with the Stirling numbers $\left\{ {i \atop m} \right\}$ of the second kind, which are integers, we get $Q_{<M} = \sum_m \hat{q}_m\binom{t}{m}$ with $\hat{q}_m = m!\sum_{m\leq i<M} \left\{ {i \atop m} \right\} q_i$, and so $v_3(\hat{q}_m) \geq v_3(m!) + m = \varsigma(m)$, uniformly in M. The identity

$$\binom{t}{a}\binom{t}{b} = \sum_{c=\max(a,b)}^{a+b} \binom{c}{a}\binom{a}{c-b}\binom{t}{c}\ , \tag{A.2}$$

which follows by counting pairs of subsets of a t-element set of sizes a and b according to their union, has integer coefficients. So $f = PQ_{<M} = \sum_c f_c\binom{t}{c}$, where only terms with $a \le d$ and $b \ge c - a \ge c - d$ contribute to f_c, and hence $v_3(f_c) \ge \varsigma((c-d)^+)$, ς being increasing.

One derivative. Differentiating $\binom{t+x}{c} = \sum_i \binom{x}{i}\binom{t}{c-i}$ with respect to x at $x = 0$, and using $\frac{\mathrm{d}}{\mathrm{d}x}\binom{x}{i}|_{x=0} = (-1)^{i-1}/i$ for $i \ge 1$, we get $\frac{\mathrm{d}}{\mathrm{d}t}\binom{t}{c} = \sum_{i=1}^{c} \frac{(-1)^{i-1}}{i}\binom{t}{c-i}$. Hence $f' = \sum_{c'} f'_{c'}\binom{t}{c'}$ with $f'_{c'} = \sum_{i\ge1} \frac{(-1)^{i-1}}{i} f_{c'+i}$. We claim that

$$v_3(f'_{c'}) \ge \varsigma((c'-d)^+) - \log_3 d\ . \tag{A.3}$$

The term i of $f'_{c'}$ has valuation at least $\varsigma((c'+i-d)^+) - v_3(i)$, and there are three cases. If $c' \ge d$, then $\varsigma(c'-d+i) - v_3(i) \ge \varsigma(c'-d) + \varsigma(i) - v_3(i) \ge \varsigma(c'-d)$, because $\varsigma(i) \ge i \ge v_3(i)$. If $c' + i \le d$, then $v_3(i) \le \log_3 i \le \log_3 d$. If $c' < d < c' + i$, put $e = c' + i - d \ge 1$; then $v_3(i) \le \log_3(d+e) \le \log_3 d + \log_3(1+e) \le \log_3 d + \varsigma(e)$. In all three cases the term is bounded as claimed.

The integral. By (V2), $\int_{\mathbb{Z}_3} f' = \sum_{c'} f'_{c'}(-1)^{c'}/(c'+1)$, a finite sum. In the same way as above, $v_3(c'+1) \le \log_3(d+1) + \varsigma((c'-d)^+)$: for $c' \le d$ this is clear, and for $c' = d + e$ we have $\log_3(d+1+e) \le \log_3(d+1) + \log_3(1+e)$. With (A.3), every term has valuation at least $-\log_3 d - \log_3(d+1)$, and so has $\int_{\mathbb{Z}_3} f'$.

The remainder. The monomial coefficients of P and P' have valuation at least $-v_3(d!)$, and those of R_M and R'_M at least $\max(M, m)$ at t^m. So $(PR_M)' = P'R_M + PR'_M = \sum_m e_m t^m$ with $v_3(e_m) \ge M - v_3(d!)$ and $v_3(e_m) \ge m - d - v_3(d!)$ for all m. This series converges on $|t|_3 < 3$, its value at 0 and its Lipschitz constant on $\mathbb{Z}_3$ are at most $\max_m |e_m|_3$, and (V3) gives $v_3 \int_{\mathbb{Z}_3}(PR_M)' \ge M - v_3(d!) - 1$.

By linearity, $\int_{\mathbb{Z}_3}(PQ)' = \int_{\mathbb{Z}_3} f' + \int_{\mathbb{Z}_3}(PR_M)'$. The left side does not depend on M, and letting $M \to \infty$ proves (4.7). □

Appendix B. The Heine expansion, the tree bound and the size

This appendix proves Proposition 5.1, Theorem 5.13 and Theorem 5.7. For a $\mathbb{Q}[X]$-linear map $\Lambda' : \mathbb{Q}[X][w] \to \mathbb{Q}[X]$ and $s \ge 1$ let $\Lambda'^{\otimes s} : \mathbb{Q}[X][w_1, \dots, w_s] \to \mathbb{Q}[X]$ be the $\mathbb{Q}[X]$-linear map with $w_1^{e_1} \cdots w_s^{e_s} \mapsto \prod_i \Lambda'(w^{e_i})$.

Lemma B.1 (Andréief)**.** *For polynomials $f_0, \dots, f_{s-1}, g_0, \dots, g_{s-1} \in \mathbb{Q}[X][w]$,*

$$\Lambda'^{\otimes s}\Big(\det[f_b(w_i)]_{i,b}\,\det[g_{b'}(w_i)]_{i,b'}\Big) = s!\,\det\big[\Lambda'(f_b g_{b'})\big]_{b,b'<s}\ . \tag{B.2}$$

In particular $\det\Big[\Lambda'(w^{b+b'})\Big]_{b,b'<s} = \frac{1}{s!}\Lambda'^{\otimes s}(V(w)^2)$ *with* $V(w) = \prod_{i<i'}(w_{i'} - w_i)$.

Proof. The product of the two determinants is $\sum_{\pi,\pi'} \operatorname{sgn}(\pi)\operatorname{sgn}(\pi') \prod_i f_{\pi(i)}(w_i) g_{\pi'(i)}(w_i)$, the sum over bijections π, π' from $\{1, \dots, s\}$ to $\{0, \dots, s-1\}$. The map $\Lambda'^{\otimes s}$ sends the product over i to $\prod_i \Lambda'(f_{\pi(i)} g_{\pi'(i)}) = \prod_b \Lambda'(f_b g_{\pi'\pi^{-1}(b)})$. For fixed π, the sum over π' gives $\operatorname{sgn}(\pi)^2 \det[\Lambda'(f_b g_{b'})]$, and the $s!$ choices of π contribute equally. □

Proof of Proposition 5.1. Step 1. By Lemma B.1, $\Delta_K = \frac{1}{K!}\Lambda^{\otimes K}(V(y_1, \dots, y_K)^2)$, where $y_1, \dots, y_K$ are independent variables.

Step 2. Since $\Lambda = \sum_k \Lambda_k$, we have $\Lambda^{\otimes K} = \sum_\iota \bigotimes_i \Lambda_{\iota(i)}$, the sum over all maps $\iota : \{1, \dots, K\} \to \mathcal{N}$. The polynomial V^2 is symmetric, so $\bigotimes_i \Lambda_{\iota(i)}(V^2)$ depends only on the occupancy $s = (s_k)$, $s_k = |\iota^{-1}(k)|$, an integer vector with $s_k \ge 0$ and $\sum_k s_k = K$, and there are $K!/\prod_k s_k!$ maps with a given occupancy. Hence $\Delta_K = \sum_s (\prod_k s_k!)^{-1} \bigotimes_i \Lambda_{\iota(i)}(V^2)$, the sum over all occupancies, with ι fixed for each occupancy s as in the proposition. Since $\Lambda_k(F)$ depends only on the Taylor coefficients of F of order at most 3 at $-k$, only occupancies with $s_k \le 4$, that is, the profiles, will contribute (Step 6).

Step 3. Let $\Lambda_k^{\mathrm{loc}} : \mathbb{Q}[X][w] \to \mathbb{Q}[X]$ be the map $w^j \mapsto c_{k,j}$, so that $\Lambda_k(F) = \Lambda_k^{\mathrm{loc}}(F(-k+w))$. Substituting $y_i = -\iota(i) + w_i$, the squared Vandermonde factors into the pairs inside one node, which give $(w_{i'} - w_i)^2$, and the cross pairs, which give the factors $(\iota(i') - \iota(i) + w_i - w_{i'})^2$ of (5.3). So $V^2 = \prod_k V(w^{(k)})^2 \cdot Q_s(w)$, where $w^{(k)}$ is the set of variables w_i with $\iota(i) = k$.

Step 4. Write $Q_s = \sum_\alpha q_\alpha w^\alpha$ with $q_\alpha \in \mathbb{Z}$. Permuting the variables inside one node permutes the cross pairs, and each factor of (5.3) is a square, so q_α is constant on the orbits of $\prod_k \mathrm{Sym}(w^{(k)})$. These orbits are labeled by the tuples $\lambda = (\lambda_k)$ of partitions, λ_k being the multiset of the exponents at $w^{(k)}$. So $Q_s = \sum_\lambda \mathrm{coef}_\lambda(s) \prod_k m_{\lambda_k}(w^{(k)})$, where m_μ is the monomial symmetric polynomial and $m_\emptyset = 1$.

Step 5. The map $\bigotimes_i \Lambda_{\iota(i)}^{\mathrm{loc}}$ is multiplicative on products of polynomials in disjoint groups of variables. Hence each λ contributes $\mathrm{coef}_\lambda(s) \prod_k (\Lambda_k^{\mathrm{loc}})^{\otimes s_k}\big(V(w^{(k)})^2 m_{\lambda_k}(w^{(k)})\big)$.

Step 6. For $s \geq 1$ and a partition μ with at most s parts, $V(w)m_\mu(w)$ is an antisymmetric polynomial in $w_1, \dots, w_s$ with integer coefficients, homogeneous of degree $\binom{s}{2} + |\mu|$. It is therefore an integer combination $\sum_\nu n_{\mu,\nu} \det[w_i^{\nu_b}]$ of alternants with $\nu_0 > \nu_1 > \cdots > \nu_{s-1} \geq 0$ and $\sum_b \nu_b = \binom{s}{2} + |\mu|$, because an antisymmetric polynomial is determined by its coefficients at the monomials with strictly decreasing exponents. By Lemma B.1 with $f_b = w^b$ and $g_{b'} = w^{\nu_{b'}}$,

$$(\Lambda_k^{\mathrm{loc}})^{\otimes s}\big(V^2 m_\mu\big) = s! \sum_\nu n_{\mu,\nu} \det\left[c_{k,b+\nu_{b'}}\right]_{b,b'<s} . \tag{B.3}$$

If some $\nu_{b'} \geq 4$, the corresponding column vanishes, as $c_{k,m} = 0$ for $m \geq 4$. So only $\nu \subseteq \{0,1,2,3\}$ contribute, and these are the row sets R of the minors $\det C_k[R]$, up to transposition and sign. For $s \geq 5$ there is no such ν, and in general $|\mu| = \sum \nu - \binom{s}{2} \leq (3s - \binom{s}{2}) - \binom{s}{2} = s(4-s)$. The factors $s_k!$ cancel those of Step 2, and the signs are absorbed in the integers $n_{\mu,\nu}$. □

Proof of Theorem 5.13. By Proposition 5.1 and the ultrametric inequality,

$$v(\Delta_K) \geq \min_{s,\lambda} \left[v(\mathrm{coef}_\lambda(s)) + \sum_k v(M_{k,\lambda_k}) \right] . \tag{B.4}$$

Local factors. Each $M_{k,\lambda}$ is an integer combination of minors $\det C_k[R]$ with $\sum R - \binom{s_k}{2} = |\lambda|$, so $v(M_{k,\lambda}) \geq -\mathsf{e}_p(k, s_k, |\lambda|)$.

Cross factors. Expand each factor of (5.3) as $\sum_{e=0}^{2} \binom{2}{e} d^{2-e}(w_i - w_{i'})^e$ with $d = \iota(i') - \iota(i) \neq 0$, and $(w_i - w_{i'})^e = \sum_{\alpha \leq e} \binom{e}{\alpha} (-1)^{e-\alpha} w_i^\alpha w_{i'}^{e-\alpha}$. A monomial of Q_s arises from a choice of (e, α) for every cross pair, with a coefficient that is an integer times $\prod d^{2-e}$, whose valuation is $2\sum_{\text{pairs}} v_p(d) - \sum_{\text{pairs}} e\, v_p(d)$. Of the e units of degree at a pair, α go to w_i and $e - \alpha$ to $w_{i'}$, and a unit that goes to a variable of the node k costs $v_p(d) \leq \eta_p(k)$. The node k receives $|\lambda_k|$ units in all, so

$$v(\mathrm{coef}_\lambda(s)) \geq 2 \sum_{k<m} s_k s_m v_p(k-m) - \sum_k |\lambda_k|\, \eta_p(k) . \tag{B.5}$$

Combination. With $\ell_k = |\lambda_k| \leq s_k(4 - s_k)$, every term of (5.2) has valuation at least $-\sum_k (\mathsf{e}_p(k, s_k, \ell_k) + \ell_k \eta_p(k)) + 2\sum_{k<m} s_k s_m v_p(k-m) \geq -\sum_k f_p(k, s_k) + 2\sum_{k<m} s_k s_m v_p(k-m)$. Finally, $v_p(x) = \#\{j \geq 1 :\ p^j \mid x\}$ gives

$$2 \sum_{k<m} s_k s_m v_p(k-m) = \sum_{j \geq 1} \sum_{\mathfrak{c} \bmod p^j} \sum_{\substack{k,m \in \mathfrak{c} \\ k \neq m}} s_k s_m = \sum_{j \geq 1} \sum_{\mathfrak{c} \bmod p^j} \left(S_{\mathfrak{c}}^2 - \sum_{k \in \mathfrak{c}} s_k^2 \right) , \tag{B.6}$$

a finite sum. Nothing in the argument uses that p is odd. □

For Theorem 5.7 we need four lemmas. Put $B(d) = \frac{1}{2}\log\bigl(1+d^2\bigr)$, so that $B(0) = 0$ and $0 \le B(d) - \log|d| \le \frac{1}{2d^2}$ for $d \ne 0$, and for a real vector $s = (s_k)_{k\in\mathcal{N}}$ let

$$E_B(s) = \sum_k s_k \log|h_k| + \sum_{k,m} s_k s_m B(k-m)\ . \tag{B.7}$$

The function B replaces $\log|k-m|$ because it is defined at $k=m$ and is conditionally negative definite (Lemma B.10).

Lemma B.8 (discrete energy)**.** $\log\|\Delta_K\|_1 \le \max_s E_B(s) + O(n\log n)$*, the maximum over the profiles* s *of Proposition 5.1, uniformly in* $K \le 3n+1$*.*

Proof. We apply $\|f+g\|_1 \le \|f\|_1 + \|g\|_1$ and $\|fg\|_1 \le \|f\|_1\|g\|_1$ to (5.2).

(1) There are at most 5^{3n+1} profiles, and for each node at most 12 partitions λ_k, since $|\lambda_k| \le 4$. So there are $e^{O(n)}$ pairs (s,λ).

(2) The cross factor $\mathrm{coef}_\lambda(s)$ is a coefficient of Q_s at a monomial of total degree $\sum_k|\lambda_k| \le 4(3n+1)$. By Cauchy's estimate on the polydisc $|w_i| = \frac{1}{10}$, and since $|d + w_i - w_{i'}| \le |d| + \frac{1}{5}$ there,

$$|\mathrm{coef}_\lambda(s)| \le 10^{4(3n+1)} \prod_{k\ne m} \bigl(|k-m| + \tfrac{1}{5}\bigr)^{s_k s_m}\ . \tag{B.9}$$

(3) Write $\widetilde{W}_k(z)/h_k = \exp\Bigl(-\sum_{j\ge1}\mathfrak{s}_j z^j/j\Bigr)$ with the power sums $\mathfrak{s}_j = \sum_d e_d d^{-j}$ over the nonzero roots and poles d of $\widetilde{W}_k$. All of them have $|d| \ge \frac{1}{3}$: the zeros $k-j-\frac{r}{3}$ lie in $\frac{1}{3}\mathbb{Z}\setminus\mathbb{Z}$, $|k-\frac{n}{2}| \ge \frac{1}{2}$, and $|k-m| \ge 1$. Moreover, the $3d$ are distinct integers for the zeros of order 3, and the d are distinct integers for the poles. Summing $|e_d|/|d|$ gives $|\mathfrak{s}_1| = O(\log n)$, and $|\mathfrak{s}_j| \le C3^j$ for $j \ge 2$ with an absolute constant C. The coefficients of $\exp\Bigl(\sum_j a_j z^j\Bigr)$ are polynomials in the a_j with nonnegative coefficients, so $|H_k[b]| \le |h_k|\cdot O(\log n)^b$ for $b \le 3$.

(4) $|\mathrm{HS}_3(k,M)| \le 3^M \sum_{u\ge1} u^{-M} \le 2\cdot 3^M$ for $M \ge 2$, so by (2.21) $\|c_{k,a}\|_1 \le C_1 \max_{b\le3}|H_k[b]|$ with an absolute constant C_1. The local factor M_{k,λ_k} is a combination of at most 6 minors with integer coefficients bounded by an absolute constant, since $s_k \le 4$ and $|\lambda_k| \le 4$, and each minor is a sum of at most 24 products of s_k local constants. Hence $\log\|M_{k,\lambda_k}\|_1 \le s_k\log|h_k| + O(s_k\log\log n) + O(1)$.

(5) For integers $|d| \ge 1$ we have $\log\bigl(|d|+\frac{1}{5}\bigr) = B(d) + \mathrm{err}(d)$ with $|\mathrm{err}(d)| \le \frac{1}{5|d|} + \frac{1}{2d^2} \le \frac{1}{|d|}$, so $\sum_{k\ne m} s_k s_m \mathrm{err}(k-m) \le \sum_k 4s_k \sum_{m\ne k}\frac{1}{|k-m|} = O(K\log n)$.

Collecting (1)–(5), every term of (5.2) has $\log\|\cdot\|_1 \le E_B(s) + O(n\log n)$, and there are $e^{O(n)}$ terms. □

Lemma B.10 (conditional negative definiteness)**.** *For real* x_i *and real* τ_i *with* $\sum_i\tau_i = 0$*,* $\sum_{i,j}\tau_i\tau_j B(x_i-x_j) \le 0$*. Consequently, for real vectors* s *and* $\tilde{s}$ *with* $\sum_k s_k = \sum_k \tilde{s}_k$*,*

$$E_B(s) \le E_B(\tilde{s}) + \langle\nabla, s-\tilde{s}\rangle\ , \qquad \nabla_k = \log|h_k| + 2\sum_m \tilde{s}_m B(k-m)\ . \tag{B.11}$$

Proof. Put $\xi_x = (\frac{1}{2}-ix)/(\frac{3}{2}+ix)$, so that $|\xi_x| < 1$. A direct computation gives $(\frac{3}{2}+ix)(\frac{3}{2}-iy) - (\frac{1}{2}-ix)(\frac{1}{2}+iy) = 2(1+i(x-y))$, hence $1+(x-y)^2 = \frac{1}{4}|1-\xi_x\bar{\xi}_y|^2|\frac{3}{2}+ix|^2|\frac{3}{2}+iy|^2$ and

$$B(x-y) = \log|1-\xi_x\bar{\xi}_y| + a(x) + a(y) - \log 2\ , \qquad a(x) = \log\bigl|\tfrac{3}{2}+ix\bigr|\ . \tag{B.12}$$

The terms $a(x_i)+a(x_j)-\log 2$ drop out against $\sum_i\tau_i = 0$, and with $\log|1-\zeta| = -\mathrm{Re}\sum_{m\ge1}\zeta^m/m$ for $|\zeta| < 1$,

$$\sum_{i,j}\tau_i\tau_j\log|1-\xi_{x_i}\bar{\xi}_{x_j}| = -\sum_{m\ge1}\frac{1}{m}\Bigl|\sum_i\tau_i\xi_{x_i}^m\Bigr|^2 \le 0\ . \tag{B.13}$$

For the consequence, expand the quadratic form E_B around $\tilde{s}$ with $\tau = s - \tilde{s}$: the linear term is $\langle \nabla, \tau \rangle$, and the quadratic term is $\sum_{k,m} \tau_k \tau_m B(k-m) \le 0$. □

Lemma B.14 (modulus of continuity)**.** *For $x < x'$ in $[-1,2]$ and $\delta = x' - x$,*

$$\int_{-1}^{2} \big|\log|x-y| - \log|x'-y|\big| \mathrm{d}y \le w(\delta) := 2\delta\left(1 + \log\frac{6+\delta}{\delta}\right). \tag{B.15}$$

Hence $|\mathcal{L}f(x) - \mathcal{L}f(x')| \le \|f\|_\infty \, w(|x-x'|)$.

Proof. Let $\bar{x} = \frac{x+x'}{2}$, so that $[-1,2] \subseteq [\bar{x}-3, \bar{x}+3]$. The integrand has a constant sign on each side of $\bar{x}$, and the reflection $y \mapsto 2\bar{x} - y$ exchanges x and x'. So the integral is at most $2G$, with $G = \int_{-3}^{0} (\log|y - \frac{\delta}{2}| - \log|y + \frac{\delta}{2}|)\,\mathrm{d}y = -2\mathcal{Q}_1(\frac{\delta}{2}) + \mathcal{Q}_1(3+\frac{\delta}{2}) - \mathcal{Q}_1(3-\frac{\delta}{2})$ and $\mathcal{Q}_1(y) = y\log|y| - y$. Here $-2\mathcal{Q}_1(\frac{\delta}{2}) = \delta(1 + \log\frac{2}{\delta})$, and $\mathcal{Q}_1(3+\frac{\delta}{2}) - \mathcal{Q}_1(3-\frac{\delta}{2}) = \int_{3-\delta/2}^{3+\delta/2} \log y \,\mathrm{d}y \le \delta \log\big(3 + \frac{\delta}{2}\big)$. □

Lemma B.16 (continuum limit)**.** *Let $1 \le \kappa \le 3 + \frac{1}{n}$, let σ be feasible for κ and piecewise constant with at most J pieces, and let $\tilde{s}_k = c_n \sigma(k/n)$ with c_n chosen so that $\sum_k \tilde{s}_k = K$; then $c_n = 1 + O_J(1/n)$. Uniformly in $k \in \mathcal{N}$:*

(a) $\log|h_k| = -6n\log n + n\,\mathcal{V}(k/n) + O(\log n)$;
(b) $2\sum_m \tilde{s}_m B(k-m) = 2\kappa n \log n + 2n\,\mathcal{L}\sigma(k/n) + O_J(\log n)$;
(c) $E_B(\tilde{s}) = (\kappa^2 - 6\kappa) n^2 \log n + n^2 \mathcal{E}[\sigma] + O_J(n \log n)$.

Proof. We use one elementary estimate. For $\theta \in [\frac{1}{3}, 1]$ and integers $0 \le M_1 \le M_2$,

$$\sum_{M_1 \le i < M_2} \log(i+\theta) = \int_{M_1}^{M_2} \log y \,\mathrm{d}y + O(\log(M_2+2)), \tag{B.17}$$

because $\int_{i-1+\theta}^{i+\theta} \log \le \log(i+\theta) \le \int_{i+\theta}^{i+1+\theta} \log$ for $i \ge 1$, the term $i = 0$ is $O(1)$, and shifting the limits of integration by at most 2 costs $O(\log(M_2+2))$, log being integrable at 0. Consequently, for real a with $|a| \le 3n$ at distance at least $\frac{1}{3}$ from $\mathbb{Z}$, or for $a \in \mathbb{Z}$ with the term $j = a$ omitted, and integers $A_1 < A_2$ with $|A_1|, |A_2| \le 3n$, we have $\sum_{A_1 \le j < A_2} \log|a-j| = \int_{A_1}^{A_2} \log|a-y|\,\mathrm{d}y + O(\log n)$; split the sum at a.

(a) By (2.18), $\log|h_k| = \log|\frac{n}{2} - k| + 3\sum_{r=1,2}\sum_{j<n} \log|k - \frac{r}{3} - j| - 4\sum_{m \ne k} \log|m-k|$. The first term is $O(\log n)$. By the elementary estimate, $\sum_{j<n} \log|k - \frac{r}{3} - j| = \int_0^n \log|k - \frac{r}{3} - y|\,\mathrm{d}y + O(\log n)$, and by Lemma B.14, after the substitution $y = ny'$, the shift $\frac{r}{3}$ changes the integral by $O(\log n)$. Rescaling, $\int_0^n \log|k-y|\,\mathrm{d}y = n\log n + n\,\mathcal{L}\mathbf{1}_{I_{\mathrm{in}}}(x)$ with $x = k/n$. Likewise $\sum_{m \ne k} \log|m-k| = 3n\log n + n\,\mathcal{L}\mathbf{1}_{[-1,2]}(x) + O(\log n)$. The coefficients of $n\log n$ add up to $3 \cdot 2 - 4 \cdot 3 = -6$, and the rest is $n(6\mathcal{L}\mathbf{1}_{I_{\mathrm{in}}} - 4\mathcal{L}\mathbf{1}_{[-1,2]})(x) = n\mathcal{V}(x)$.

(b) We have $\sum_{m \ne k} |B(k-m) - \log|k-m|| \le \frac{\pi^2}{6}$. On each piece $[\alpha, \beta)$ of σ, the elementary estimate gives $\sum_{m \in [\alpha n, \beta n),\, m \ne k} \log|k-m| = n(\beta-\alpha)\log n + n\int_\alpha^\beta \log|x-y|\,\mathrm{d}y + O(\log n)$. Multiply by the value of σ on the piece, which is at most 4, sum over the J pieces, and use $c_n = 1 + O(1/n)$ and $\int \sigma = \kappa$.

(c) By Lemma B.14, $\mathcal{V}$ and $\mathcal{L}\sigma$ have modulus of continuity $O(w(\delta))$, and $w(\frac{1}{n}) = O(\frac{\log n}{n})$. Hence $\sum_k \sigma(k/n)\phi(k/n) = n\langle \sigma, \phi \rangle + O_J(\log n)$ for $\phi \in \{\mathcal{V}, \mathcal{L}\sigma\}$: the $O(n)$ intervals $[\frac{k}{n}, \frac{k+1}{n})$ on which σ is constant contribute $O(w(\frac{1}{n}))$ each, and the at most J others $O(1)$ each. Sum (a) and (b) against $\tilde{s}_k$, and use $\sum_k \tilde{s}_k = \kappa n$. □

Proof of Theorem 5.7. By Lemmas B.8 and B.10, $\log\|\Delta_K\|_1 \le E_B(\tilde{s}) + \max_s \langle \nabla, s - \tilde{s} \rangle + O(n\log n)$. By Lemma B.16(a),(b), $\nabla_k = \mathrm{const}_n + n\,P_\sigma(k/n) + \delta_k$ with $\max_k |\delta_k| = O_J(\log n)$. The constant drops out because $\sum_k (s_k - \tilde{s}_k) = 0$, and $|\langle \delta, s - \tilde{s} \rangle| \le 2K \max_k |\delta_k| = O_J(n\log n)$. For a profile s let $\bar{s}(x) = s_k$ on $[\frac{k}{n}, \frac{k+1}{n})$ for $-n \le k < 2n$, and $\sigma' = \bar{s} + \frac{s_{2n}}{n|E|}\mathbf{1}_E$ with $E = \{\bar{s} \le 3\}$. Since

$|\{\bar s>3\}|\le\kappa/4<1$, we have $|E|\ge 2$, and σ' is feasible for $n\ge 3$. By Lemma B.14, $\sum_k s_kP_\sigma(k/n)=n\langle P_\sigma,\sigma'\rangle+O(\log n)$, and by the Riemann-sum estimate of Lemma B.16(c), $\sum_k\tilde s_kP_\sigma(k/n)=n\langle P_\sigma,\sigma\rangle+O_J(\log n)$. Hence

$$n\sum_k P_\sigma(k/n)(s_k-\tilde s_k)\le n^2\langle P_\sigma,\sigma'-\sigma\rangle+O_J(n\log n)\le n^2\,\mathrm{gap}(\sigma)+O_J(n\log n)\ . \tag{B.18}$$

Now insert Lemma B.16(c). All constants depend only on $\|\sigma\|_\infty\le 4$ and on J. □

Appendix C. Local bounds, the circle model and the prime sum

This appendix proves Lemmas 5.17, 5.18, 5.19, 5.28, 5.31 and 5.34 and Proposition 5.36.

Proof of Lemma 5.17. All integers whose valuations enter are nonzero and of absolute value less than $6n\le p^{\mathfrak l_p}$: $|k-m|\le 3n$; $3k-3j-r\in[-6n+1,6n-1]$, as $3j+r\in[1,3n-1]$; the u of (2.14) satisfy $|u|<3|k|\le 6n$; and $|n-2k|\le 3n$. Hence $\mu_k\le\mathfrak l_p$ (at $p=2$ the root $\frac{2k-n}{2}$ has $v_2=-1$), $\eta_p(k)\le\mathfrak l_p$, and $v(\mathrm{HS}_3(k,M))\ge -M\mathfrak l_p$. By Lemma 5.16 and (2.21), whose weights $i(i+1)$ and 2916 are integers, $v(c_{k,a})\ge -D_k-(6-a)\mathfrak l_p$ for $a\le 3$. A minor $\det C_k[R]$ is a signed sum of products $\prod_b c_{k,m_b}$ of s local constants with $\sum_b(6-m_b)\le 6s$, so $\mathsf e_p(k,s,\ell)\le sD_k+6s\mathfrak l_p$, and $\ell\eta_p(k)\le s(4-s)\mathfrak l_p\le 3s\mathfrak l_p$. □

Proof of Lemma 5.18. For $j\ge 1$ put $o_k^{(j)}=\#\{m\ne k:\ m\equiv k\ (p^j)\}$, $z_k^{(j)}=3\#\{(i,r):\ 0\le i<n,\ r\in\{1,2\},\ 3i+r\equiv 3k\ (p^j)\}$, and $g_k^{(j)}=1$ if $p^j\mid n-2k$ and 0 otherwise, for odd p, and $g_k^{(j)}=0$ for $p=2$. By (5.15) and $v_p(x)=\#\{j\ge 1:p^j\mid x\}$, $D_k=\sum_{j\ge1}(4o_k^{(j)}-z_k^{(j)}-g_k^{(j)})+[p=2]$. The first two quantities depend only on the class $\mathfrak c$ of k modulo $q=p^j$, as 3 is invertible modulo q; write $N_\mathfrak c$ for the number of nodes in $\mathfrak c$ and $z_\mathfrak c$ for $z_k^{(j)}$. By Lemma 5.17, $T_p\le 9K\mathfrak l_p+[p=2]K+\max_s\sum_{j\ge1}\Xi_j(s)$ with

$$\Xi_j(s)=\sum_{\mathfrak c\bmod q}\left[S_\mathfrak c(4(N_\mathfrak c-1)-z_\mathfrak c)-\sum_{k\in\mathfrak c}s_kg_k^{(j)}-S_\mathfrak c^2+\sum_{k\in\mathfrak c}s_k^2\right]. \tag{C.1}$$

If $q>3n$, every class has $N_\mathfrak c\le 1$, and $\Xi_j(s)\le 0$. If $q\le 3n$, we use $\sum_{k\in\mathfrak c}s_k^2\le 4S_\mathfrak c$ and $g\ge 0$ to get $\Xi_j(s)\le\sum_\mathfrak c(S_\mathfrak cA_\mathfrak c-S_\mathfrak c^2)$ with $A_\mathfrak c=4N_\mathfrak c-z_\mathfrak c$. The $3n+1$ nodes are consecutive integers, so $N_\mathfrak c\in\{\lfloor\frac{3n+1}{q}\rfloor,\lceil\frac{3n+1}{q}\rceil\}$, and since $i\mapsto 3i+r$ is a bijection of $\mathbb Z/q$, each of the two counts in $z_\mathfrak c/3$ is the number of $i\in[0,n)$ in one residue class, hence lies in $\{\lfloor\frac nq\rfloor,\lceil\frac nq\rceil\}$. So all $A_\mathfrak c$ lie in an interval of length $4+6=10$, which contains their mean $\bar A=(4(3n+1)-6n)/q=(6n+4)/q$. For real ξ and $S_\mathfrak c$ we have $S_\mathfrak cA_\mathfrak c-S_\mathfrak c^2\le\xi S_\mathfrak c+\frac14(A_\mathfrak c-\xi)^2$; with $\xi=\bar A-2K/q$ and $\sum_\mathfrak c(A_\mathfrak c-\bar A)=0$,

$$\sum_\mathfrak c(S_\mathfrak cA_\mathfrak c-S_\mathfrak c^2)\le K\bar A-\frac{K^2}{q}+\frac14\sum_\mathfrak c(A_\mathfrak c-\bar A)^2\le\frac{K(6n+4-K)}{q}+25q\ . \tag{C.2}$$

Summing over j with $p^j\le 3n$, $\sum_j p^{-j}\le\frac{1}{p-1}$ and $\sum_{p^j\le3n}25p^j\le 25\cdot 3n\frac{p}{p-1}\le 150n$. □

Proof of Lemma 5.19. Here p is single-level, so all valuations are 0 or 1 and only the classes modulo p matter. By Lemma 5.16 with $\mu_k\le 1$ and $v(\mathrm{HS}_3(k,M))\ge -M$, we have $v(\beta_{k,m})\ge\min_{1\le i\le4-m}(-D_k-(4-i-m)-(i+2))=-D_k-6+m$ and $v(\alpha_{k,m})\ge -D_k-1+m$. Take $\check v_m(k)=-D_k-6+m$ in Lemma 5.28. For admissible (R,π), $\sum_b(D_k+6-R_b-\pi(b))+(\sum R-\binom s2)[o_k>0]\le sD_k+6s-2\binom s2$, because $\sum_b\pi(b)=\binom s2$ and $\sum R\ge\binom s2$. Every class has $N_\mathfrak c\ge\lfloor\frac{3n+1}{p}\rfloor\ge 7$ nodes, and with $z_\mathfrak c\le 6\lceil\frac np\rceil$ and $g_k\le 1$ we get $D_k=4(N_\mathfrak c-1)-z_\mathfrak c-g_k\ge 4(\frac{3n}{p}-2)-6(\frac np+1)-1=\frac{6n}{p}-15\ge 0$. Hence $sD_k+6s-s(s-1)\ge 0$, the clipping in (5.29) is inactive, and $f^+(k,s)+s^2\le s(D'_\mathfrak c+7)$ with $D'_\mathfrak c=4(N_\mathfrak c-1)-z_\mathfrak c$. So $V_\mathfrak c(S)\le S(D'_\mathfrak c+7)-S^2$. The mean of $D'_\mathfrak c+7$ over the p classes is

$\frac{6n+4-4p}{p}+7$, and the values lie in an interval of length 10, as in the proof of Lemma 5.18. The same weak-duality step gives $T_p \le T_p^+ \le K(\frac{6n+4-4p}{p}+7) - \frac{K^2}{p} + 25p = \frac{K(6n+4-K)}{p} + 3K + 25p$. □

Proof of Lemma 5.28. A minor is $\det C_k[R] = \sum_\pi \pm \prod_b c_{k,R_b+\pi(b)}$, and the terms with some $R_b + \pi(b) \ge 4$ vanish. So $-v(\det C_k[R]) \le \max_\pi \sum_b -\check v_{R_b+\pi(b)}(k)$. At a single-level prime $\eta_p(k) = [o_k > 0]$, so $f_p(k,s) \le f^+(k,s)$; the clipping at 0 only increases the right side. The tree (5.14) has one level, and grouping its maximum by classes gives

$$T_p \le \max_s \sum_{\mathfrak{c} \bmod p} \left[\sum_{k\in\mathfrak{c}} \left(f^+(k,s_k) + s_k^2 \right) - S_\mathfrak{c}^2 \right] \le \max \left\{ \sum_\mathfrak{c} V_\mathfrak{c}(S_\mathfrak{c}) : \ \sum_\mathfrak{c} S_\mathfrak{c} = K \right\} = T_p^+ \ . \tag{C.3}$$

Finally $\sum_\mathfrak{c} V_\mathfrak{c}(S_\mathfrak{c}) = \xi \sum_\mathfrak{c} S_\mathfrak{c} + \sum_\mathfrak{c} (V_\mathfrak{c}(S_\mathfrak{c}) - \xi S_\mathfrak{c}) \le \xi K + \sum_\mathfrak{c} \psi_\mathfrak{c}(\xi)$. □

Proof of Lemma 5.31. The data of a node y of the class of x change only when y crosses -1 or 2, where a node enters or leaves the node range and the number of partners changes; 0, where the range of the virtual poles changes sign; a point of $\frac{\varrho}{3}(\mathbb{Z} \setminus 3\mathbb{Z})$, where a virtual pole or the upper end of the zero range appears ($t\varrho = 3y$); or a point of $1 + \frac{\varrho}{3}(\mathbb{Z}\setminus 3\mathbb{Z})$, the lower end of the zero range ($t\varrho = 3y-3$). Modulo ϱ these are the seven points (5.32), each of the form $\alpha + \beta\varrho$ with $\alpha \in \{-1,0,1,2\}$ and $\beta \in \{0, \frac13, \frac23\}$. Two of them coincide modulo ϱ if and only if $\alpha_1 - \alpha_2 = (N' - \beta_1 + \beta_2)\varrho$ for an integer N'. If $\alpha_1 = \alpha_2$, this is impossible for distinct points, since $\beta_1 - \beta_2 \notin \mathbb{Z}$. Otherwise $\varrho = 3j/N$ with $j = |\alpha_1 - \alpha_2| \in \{1,2,3\}$ and $N = |3N' - 3\beta_1 + 3\beta_2|$, a positive integer. So on a cell the seven points are distinct, and their cyclic order is constant. If x is not one of the seven points modulo ϱ, no node of x is at a critical point, and so the type of (x,ϱ) is locally constant on the set of such pairs. For each arc, the set of pairs (x,ϱ) with ϱ in the cell and x in the arc is connected, so the type is constant on it. The arc lengths are differences of consecutive points, each represented with an integer w that is fixed on the cell, hence affine in ϱ.

Now let $\varrho > 3$. A class contains at most one node, since $[-1,2]$ has length $3 < \varrho$, so $o = 0$. For $y > 0$, $t \in \mathrm{VP}$ means $0 < t\varrho < 3y \le 6$, so $t = 1$ and $y > \frac{\varrho}{3} > 1$; then $3y - 3 \le 3 < \varrho$, so the virtual pole is covered, and it is the only special point of the node, because $2\varrho > 6 \ge 3y$ excludes $t = 2$ from $\mathcal{Z}$ and $y > 1 > 1 - \frac{\varrho}{3}$ excludes $t = -1$. For $y < 0$ there is no virtual pole, since $-\varrho < -3 \le 3y$. So at every node either $\mathrm{VP} = \mathcal{Z} = \{1\}$ with $D = -3$ and $\mu_{k,1} = 0$, or $\mathrm{VP} = \emptyset$ and $\mathcal{Z} \subseteq \{-1\}$, and in all cases the three terms of (5.25) are nonnegative. Then $f^+ = 0$ at every node, $V_\mathfrak{c}(S) = 0$ for every class, and $T_p^+ = 0$. For a prime $p > 3n$ the discrete data agree with these continuum data, because $|2k - n| \le 3n < p$ gives $g_k = 0$. □

Proof of Lemma 5.34. For $n \ge n_0$, $p \ge \epsilon n$ implies $p \ge 5$ and $p^2 > 6n$, so Lemma 5.28 applies with $\check v = \hat v$. The map $k_0 \mapsto k_0/n$ identifies $\mathbb{Z}/p$ with the grid $\frac1n\mathbb{Z}/\varrho\mathbb{Z}$ of spacing $\frac1n$ on the circle $\mathbb{R}/\varrho\mathbb{Z}$, $\varrho = p/n$, and the node data of Definition 5.20 at $k = k_0 + mp$ are the continuum data at $y = x_0 + m\varrho$, except for g_k: the conditions $pt < 3k$, $3k - 3n < pt$, $-n \le k \le 2n$ and $k \gtrless 0$ are the continuum conditions multiplied by n. Call a class *exceptional* if it contains a node with $g_k = 1$ or one of the nodes $-n$, 0, $2n$; there are at most four exceptional classes, because the nodes with $g_k = 1$, those with $2k \equiv n \pmod p$, form a single class. A node k of a non-exceptional class is not at a critical point, because $3k = pt$ and $3k - 3n = pt$ with $3 \nmid t$ are impossible, as $3 \nmid p$. So the type of the class is the type of the arc that contains its point, which is constant on the cell; at an endpoint $\varrho \in \{a,b\}$ the arcs of positive length carry the types of the cell, by the local constancy of the type. An open arc of length ℓ contains between $n\ell - 1$ and $n\ell + 1$ points of the grid, and there are at most seven arcs. So the number of non-exceptional classes of type τ is $n \operatorname{meas}_\tau^{[a,b]}(\varrho) + O(1)$. Finally $V_\mathfrak{c}(0) = 0$, so $\psi_\mathfrak{c} \ge 0$, and a class has at most $3/\epsilon + 2$ nodes, so $0 \le \psi_\mathfrak{c}(\xi) \le C_\epsilon(1 + |\xi|)$ for all classes, exceptional or not. Insert all this into Lemma 5.28 with $K = \kappa n$. □

Proof of Proposition 5.36. We use three standard facts about primes. The first is the prime number theorem [25, 26] in the form $\vartheta(x) = \sum_{p\le x} \log p = x + o(x)$, and its consequence $\sum_{p\le x} p\log p = \frac{x^2}{2} + o(x^2)$, by partial summation. The second is Mertens' theorem with the value of its constant, $\sum_{p\le x} \frac{\log p}{p} = \log x + E + o(1)$ with $E = -\gamma - \sum_p \sum_{j\ge 2} \frac{\log p}{p^j}$, see e.g. [27, (2.5), (2.8)]. Since $\sum_{j\ge 2} p^{-j} = \frac{1}{p(p-1)}$ and $\frac{\log p}{p-1} = \frac{\log p}{p} + \frac{\log p}{p(p-1)}$, it gives

$$\sum_{p\le x} \frac{\log p}{p-1} = \log x - \gamma + o(1)\ . \tag{C.4}$$

The third is the consequence $\sum_{p\le x,\, p\ne 3} \frac{\log p}{p-1} = \log x - \gamma - \frac{\log 3}{2} + o(1)$. We split the primes $p \ne 3$ into four ranges, with $Y = n^{0.9}$; any exponent in $(\frac12, 1)$ would do.

(i) $p \le Y$. By Lemma 5.18 and (C.4), the leading terms give $K(6n+4-K)(\log Y - \gamma - \frac{\log 3}{2}) + o(n^2)$, and the other terms contribute at most $(9K\mathfrak{l}_2 + 154n)\vartheta(Y) = O(n^{1.9}\log n)$.

(ii) $Y < p \le \epsilon n$. For n large these primes satisfy $p^2 > 6n$ and $p \le 0.4n$, so Lemma 5.19 applies. Since $\sum_{Y<p\le \epsilon n} \frac{\log p}{p} = \log \frac{\epsilon n}{Y} + o(1)$ by Mertens' theorem, and by the prime number theorem, the sum over these primes is at most $K(6n+4-K)\log\frac{\epsilon n}{Y} + 3K\vartheta(\epsilon n) + 25\sum_{p\le\epsilon n} p\log p + o(n^2) = K(6n+4-K)\log\frac{\epsilon n}{Y} + (3\kappa\epsilon + 12.5\epsilon^2)n^2 + o(n^2)$.

(iii) $\epsilon n < p \le 3n$. By Lemma 5.34, $T_p \le n\,\mathcal{G}_{[a,b]}(\xi_\nu, p/n, K/n) + C$ for $p/n \in [a_{\nu-1}, a_\nu]$. For fixed ξ_ν the function $\mathcal{G}_{[a,b]}$ is affine in ϱ and in κ, so its maximum over $[a_{\nu-1}, a_\nu]\times[\kappa_1,\kappa_2]$ is attained at a corner, and $T_p \le n\bar{\mathcal{G}}_\nu + C$. By the prime number theorem, $\sum \max(T_p, 0)\log p \le \sum_\nu (n\max(\bar{\mathcal{G}}_\nu, 0) + C)(\vartheta(a_\nu n) - \vartheta(a_{\nu-1}n)) = n^2 \sum_\nu \max(\bar{\mathcal{G}}_\nu, 0)(a_\nu - a_{\nu-1}) + o(n^2)$.

(iv) $p > 3n$. Then $T_p \le T_p^+ = 0$ by Lemma 5.31.

Adding (i)–(iv), $\log Y + \log\frac{\epsilon n}{Y} = \log n - \log\frac{1}{\epsilon}$, and $K(6n+4-K) = \kappa(6-\kappa)n^2 + O(n)$. The function $\kappa(6-\kappa)$ is increasing on $[2.8, 3]$, so the negative term $-\kappa(6-\kappa)(\gamma + \frac{\log 3}{2} + \log\frac{1}{\epsilon})n^2$ is largest at $\kappa = \kappa_1$, and $3\kappa\epsilon$ is largest at $\kappa = \kappa_2$.

The prime number theorem and Mertens' theorem enter only in the passage from sums over primes to their main terms, and the error of this passage is contained in the term $o(n^2)$. For a fixed small n the individual primes matter and this term may be large; the proposition makes no statement about such n, and the proof of Theorem 1.1 uses it only for n large. □

Appendix D. Proof of Lemma 6.17

We go through the nodes k by type and bound the weighted valuation (6.16) of every contribution, using the bounds of Lemma 6.7 for the local constants and the bounds for the Taylor coefficients of the E_x given after (6.14). We write $m = a + b \le 3$. Note that $\theta_x \ge \frac12$ for every condition x.

(1) *U nodes.* All three parts of the local constants have $v_p \ge 0$, and the Taylor coefficients of E_x and $E_{x'}$ are integral, so $\varpi \ge \theta_x + \theta_{x'} \ge 1$.

(2) *A node R of type Ra or Rc.* If $R_x \ne R$, all the Taylor coefficients $(E_x)_a(R)$, $a \le 3$, vanish, and there is no contribution. So let $R_x = R_{x'} = R$. Then only $a = j_x$ and $b = j_{x'}$ contribute. The part β^{HS} has $v_p \ge m - 7$ by (6.8), so $\varpi \ge (j_x + j_{x'} - 7) + (\frac72 - j_x) + (\frac72 - j_{x'}) = 0$. The part β^{rest} has $v_p \ge m - 4$, so $\varpi \ge 3$, and α has $v_p \ge m - 2$, so $\varpi \ge 5$.

(3) *A node R of type Rb.* Here $\theta_x = 2 - j_x$ for the conditions at R, and β^{HS} has $v_p \ge m - 4$. The orders $0, 1$ are prescribed at R and the orders $2, 3$ are not, and since $m \le 3$, at most one of a, b is at least 2.

(3i) $R_x = R_{x'} = R$, $a = j_x$, $b = j_{x'}$: $\varpi \ge (a+b-4) + (2-j_x) + (2-j_{x'}) = 0$.

(3ii) $R_x = R_{x'} = R$, $a \in \{2,3\}$, $b = j_{x'}$: $\varpi \ge (a+b-4) + (2-j_x) + (2-b) = a - j_x \ge 1$; likewise with the roles of a and b exchanged.

(3iii) $R_x = R \neq R_{x'}$: then $(E_{x'})_b(R) = 0$ for $b \leq 1$, so $b \in \{2,3\}$, hence $a \leq 1$ and $a = j_x$. So $\varpi \geq (j_x + b - 4) + (2 - j_x) + \theta_{x'} = b - 2 + \theta_{x'} \geq \frac{1}{2}$; likewise with x and x' exchanged.

(3iv) $R_x \neq R$ and $R_{x'} \neq R$: then $a, b \geq 2$, so $m \geq 4$, and there is no contribution.

The parts β^{rest} and α have $v_p \geq m-1$ and $v_p \geq m+1$, which exceed the bound $m-4$ for β^{HS} by 3 and 5, so their contributions have $\varpi \geq 3$ and $\varpi \geq 5$.

(3′) *A node R of type Rb_g.* The same computation with the bound $m-3$ for β^{HS} and $\theta_x = \frac{3}{2} - j_x$ gives $\varpi \geq 0$ in (3i), $\varpi \geq a - j_x \geq 1$ in (3ii), and $\varpi \geq (j_x + b - 3) + (\frac{3}{2} - j_x) + \theta_{x'} = b - \frac{3}{2} + \theta_{x'} \geq 1$ in (3iii). The parts β^{rest} ($v_p \geq m$) and α ($v_p \geq m+2$) are again better by 3 and 5.

(4) *A node $\bar{R}$ of type La, the partner of a node R of type Ra.* Here β^{HS} has $v_p \geq m-7$. We use the decomposition (6.15). If $R_x = R_{x'} = R$, the product with $\xi_{x,a}\xi_{x',b}$, which is nonzero only for $a \leq j_x$ and $b \leq j_{x'}$, has $\varpi \geq (m-7) + (j_x - a) + (j_{x'} - b) + (7 - j_x - j_{x'}) = 0$. A product that contains $\xi'_{x,a}$ instead has a factor of valuation at least $4 - a$ instead of $j_x - a$, which raises ϖ by at least 1, and likewise for $\xi'_{x',b}$. If $R_x = R \neq R_{x'}$, then $v_p((E_{x'})_b(\bar{R})) \geq 4 - b$ and $\varpi \geq (a + b - 7) + (j_x - a) + (4 - b) + (\frac{7}{2} - j_x) + \theta_{x'} = \frac{1}{2} + \theta_{x'} \geq 1$. If $R_x \neq R$ and $R_{x'} \neq R$, then $\varpi \geq (m-7) + (4-a) + (4-b) + \theta_x + \theta_{x'} \geq 2$. The parts β^{rest} ($v_p \geq m-4$) and α ($v_p \geq m-2$) are better by 3 and 5 in every case.

(4′) *A node $\bar{R}$ of type Lc, the partner of a node R of type Rc.* There is no virtual pole, so only β^{rest} ($v_p \geq m-4$) and α ($v_p \geq m-2$) occur. These are the bounds of (4) improved by 3 and 5, and the Taylor coefficients satisfy the same bounds as in (4), so $\varpi \geq 3$ and $\varpi \geq 5$.

(5) *A node $\bar{R}$ of type Lb or Lb_g, the partner of a node R of type Rb or Rb_g.* There is no virtual pole. At Lb, β^{rest} has $v_p \geq m-1$ and α has $v_p \geq m+1$; at Lb_g the bounds are m and $m+2$. Put $w_x(a) = a + v_p((E_x)_a(\bar{R}))$. By the bounds after (6.14), if $R_x = R$, then $w_x(a) \geq j_x$ for $a \leq j_x$ and $w_x(a) \geq \max(a, 2) \geq 2$ for $a > j_x$, so $w_x(a) + \theta_x \geq 2$ if R is of type Rb ($\theta_x = 2 - j_x$) and $w_x(a) + \theta_x \geq \frac{3}{2}$ if R is of type Rb_g ($\theta_x = \frac{3}{2} - j_x$). If $R_x \neq R$, then $w_x(a) \geq \max(a, 2) \geq 2$ and $w_x(a) + \theta_x \geq \frac{5}{2}$. Hence the part β^{rest} has $\varpi \geq -1 + (w_x(a) + \theta_x) + (w_{x'}(b) + \theta_{x'}) \geq -1 + 2 + 2 = 3$ at Lb and $\varpi \geq \frac{3}{2} + \frac{3}{2} = 3$ at Lb_g, and the part α has $\varpi \geq 5$ at both.

At a condition node R' different from R_x and $R_{x'}$ there is no contribution, by (2), (3iv) and (3′); the partners of such nodes are covered by (4), (4′) and (5). Collecting the cases, every contribution to Γ^β has $\varpi \geq 0$ and every contribution to Γ^α has $\varpi \geq 1$. The contributions to $\Gamma^\beta_{xx'}$ with $R_x \neq R_{x'}$ are those of (1), (3iii), (4) with $R_x = R \neq R_{x'}$ or with neither condition at R, (4′) and (5), all with $\varpi \geq \frac{1}{2}$. The contributions with $\varpi = 0$ are those of (2), (3i) and (3′) for β^{HS}, and the products $\beta^{\mathrm{HS}}_{\bar{R},a+b}\xi_{x,a}\xi_{x',b}$ of (4). □

Appendix E. The computations

This appendix describes how the upper bounds of Propositions 5.11 and 5.40 are obtained. In both cases a computer evaluates a finite expression that the lemmas of the main text show to be an upper bound for the constant in question. Real numbers are handled either as exact rationals or in ball arithmetic, in which every number is represented by a ball that contains it and every operation returns a ball that contains the exact result [18]. Only the upper ends of the final balls are used.

The bound for $F_i^\dagger$. Put $t = \kappa/m_0$, $A_0 = \langle \sigma_0, \mathcal{V} \rangle$ and $B_0 = \langle \sigma_0, \mathcal{L}\sigma_0 \rangle$. Since $\mathcal{E}$ is quadratic, $\mathcal{E}[\sigma_\kappa] = tA_0 + t^2 B_0$, $P_{\sigma_\kappa} = \mathcal{V} + 2t\mathcal{L}\sigma_0 =: P_t$ and $\langle P_t, \sigma_\kappa \rangle = tA_0 + 2t^2 B_0$, so that

$$\mathcal{E}[\sigma_\kappa] + \mathrm{gap}(\sigma_\kappa) = M_\kappa(P_t) - t^2 B_0 \,, \qquad M_\kappa(P) = \sup\{\langle P, \sigma' \rangle : \ \sigma' \text{ feasible for } \kappa\} \,. \tag{E.1}$$

With $\mathcal{Q}_1(w) = w \log|w| - w$ and $\mathcal{Q}_2(w) = \frac{w^2}{2} \log|w| - \frac{3}{4} w^2$, which satisfy $\mathcal{Q}_2' = \mathcal{Q}_1$ and $\mathcal{Q}_1' = \log|\cdot|$, we have $\int_\alpha^\beta \log|x - y| \, \mathrm{d}y = \mathcal{Q}_1(\beta - x) - \mathcal{Q}_1(\alpha - x)$ and

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} \log|x - y| \, \mathrm{d}y \, \mathrm{d}x = \mathcal{Q}_2(b_2 - a_1) - \mathcal{Q}_2(b_2 - b_1) - \mathcal{Q}_2(a_2 - a_1) + \mathcal{Q}_2(a_2 - b_1) \,. \tag{E.2}$$

So A_0, B_0 and the values of P_t are finite sums of values of $\mathcal{Q}_1$ and $\mathcal{Q}_2$. One finds $B_0 < 0$ (about -3.78), so $-t^2 B_0$ is increasing in t.

To bound M_κ, split $[-1,2]$ into N_{cell} intervals Q_m of length $\delta = 3/N_{\mathrm{cell}}$, and let $\pi_m \geq \sup\{P_t(x) : x \in Q_m,\ t \in [t_a, t_b]\}$, where $t_{a,b} = \kappa_{a,b}/m_0$ for a subinterval $[\kappa_a, \kappa_b]$ of I_i. For feasible σ' put $\bar\sigma_m = \int_{Q_m} \sigma' \in [0, 4\delta]$. For every real c_*,

$$\langle P_t, \sigma' \rangle \leq \sum_m \pi_m \bar\sigma_m = c_* \kappa + \sum_m (\pi_m - c_*)\bar\sigma_m \leq c_* \kappa + 4\delta \sum_m (\pi_m - c_*)^+ \ . \tag{E.3}$$

If $c_* \geq \min_m \pi_m$, it follows that $M_\kappa(P_t) \leq M^+ := c_*\kappa_b + (\kappa_b - \kappa_a)\max(0, -\min_m \pi_m) + 4\delta \sum_m (\pi_m - c_*)^+$ for all $\kappa \in [\kappa_a, \kappa_b]$. We take for c_* the threshold of the filling with density 4 at κ_b, that is, the value π_m such that the intervals Q_m with larger values have total length about $\kappa_b/4$; any other $c_* \geq \min_m \pi_m$ would also give a valid bound. The function $P_t(x)$ is affine in t, so it suffices to bound it at $t \in \{t_a, t_b\}$, and by Lemma B.14, $\sup_{Q_m} P_t \leq P_t(x_m) + b_\infty\, w(\delta/2)$, where x_m is the midpoint of Q_m and $b_\infty = 4 + 2t_b \max \sigma_0$ bounds $\|2 \cdot \mathbf{1}_{I_{\mathrm{in}}} - 4 \cdot \mathbf{1}_{I_{\mathrm{out}}} + 2t\sigma_0\|_\infty$, the density whose potential is P_t. This is the procedure of [13]. We use $N_{\mathrm{cell}} = 36000$ and cut each I_i into 20 subintervals $[\kappa_a, \kappa_b]$. All values of $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are computed as balls at a precision of 96 bits, π_m is the upper end of the enclosure, and the bound for $F_i^\dagger$ is the largest, over the 20 subintervals, of the upper ends of the balls for $(M^+ - t_b^2 B_0)/\log 2$.

The bound for c_i. The bound is obtained by evaluating (5.39) for one explicit partition and explicit rationals ξ_ν, and all steps up to the last one are done in exact rational arithmetic. The points of $\mathcal{B}$ in $(\frac{1}{20}, 3)$, together with $\frac{1}{20}$ and 3, divide $[\frac{1}{20}, 3]$ into cells. For each cell we fix an interior point ϱ_*, order the seven points (5.32) on $\mathbb{R}/\varrho_*\mathbb{Z}$, and record for each arc its length, as an affine function of ϱ, and its type, computed at the midpoint of the arc; by Lemma 5.31 the type does not depend on these choices. The type determines $\hat v$ at each node by (5.25), then f^+ by (5.29), the function $V_{\mathfrak{c}}$ by a dynamic program over the nodes of the class, and $\psi_\tau(\xi) = \max_S (V_{\mathfrak{c}}(S) - \xi S)$ as the maximum over the vertices of the upper concave hull of $V_{\mathfrak{c}}$. Each cell is cut into 32 subcells of equal length; these form the partition $a_0 < \cdots < a_M$. On each subcell, for each interval I_i, a rational ξ_ν of denominator at most 10^6 is chosen by a numerical minimization of the corner maximum (5.37); the choice affects only the quality of the bound, not its validity. Then $\bar{\mathcal{G}}_\nu$ is computed exactly, and the sum $\sum_\nu \max(\bar{\mathcal{G}}_\nu, 0)(a_\nu - a_{\nu-1})$ is an exact rational number. For the five intervals $I_1, \ldots, I_5$ these sums are 57.5996087, 57.8686577, 58.1240232, 58.3660091 and 58.5946334, rounded to seven decimals; we quote them so that the reader can evaluate (5.39) from them. Finally, (5.39) is evaluated in ball arithmetic with Euler's constant and the logarithms $\log 3$, $\log 20$ and $\log 2$, and the upper ends of the balls give the bounds of Proposition 5.40.

Table 1. The values $G(\kappa_{1,i}) = (10.5\,\kappa_{1,i} - \kappa_{1,i}^2/4)\log_2 3$ are computed in ball arithmetic and rounded down to two decimals, the bounds of Propositions 5.11 and 5.40 are rounded up to two decimals, and the last column is computed from these printed values. The coefficients (6.6) and the determinants of the blocks in Section 6, including (6.21), are exact rational computations that are small enough to be checked by hand.

Appendix F. A second proof, on the modular curve $X_0(3)$

This appendix gives a second proof of Theorem 1.1, by a different method. It shares with the main proof only the statement, the normalization (2.4) of $\zeta_3(5)$ and standard facts; it uses neither the Volkenborn functional nor any estimate of Sections 3–6 and Appendices A–E. Instead, it rests on published theorems of Coleman, of Coleman and Mazur, and of Buzzard on overconvergent modular forms (Theorems F.20 and F.23), whose statements and hypotheses we quote but whose proofs we have not checked. The method is the determinant argument of Yi for $\zeta_7(3)$ on $X_0(7)$ [15]: a determinant of Taylor coefficients of power series built from modular forms, a bound for its denominators at the primes $\ell \neq 3$ through the Eisenstein series of weight $\ell - 1$, and an energy

bound at the real place. We write out for level 3 every step that we take from [15], so that this proof, too, should not depend on the correctness of [15]. New relative to [15] are the Sym^4 form of his congruence (Proposition F.31) and the layer theorem (Theorem F.40). The notation of this appendix is independent of the main text, except for $\zeta_3(5)$, L_3, the trivial character $\mathbf{1}$, the Teichmüller character ω and $\langle a\rangle$ of Section 2.1, $\mathbb{Q}$, $\mathbb{Z}$, v_p, the coefficient notation $[x^a]F$ and $t^+ = \max(t,0)$. In $O(\cdot)$ and $o(\cdot)$, $d\to\infty$ while η_5, ϵ and the measure μ of Section F.5 stay fixed; no other parameter depends on d.

Notation. Let τ lie in the upper half plane $\mathbb{H}$, $q = e^{2\pi i\tau}$ and $\theta = q\,\mathrm{d}/\mathrm{d}q = (2\pi i)^{-1}\mathrm{d}/\mathrm{d}\tau$, and for $k\in\mathbb{Z}$ put $\sigma_k(n) = \sum_{a|n} a^k$, with $\sigma_k(n/3) = 0$ if $3\nmid n$, and $\sigma_k^{(3)}(n) = \sum_{a|n,\,3\nmid a} a^k$. Let

$$x(\tau) = \left(\frac{\eta(3\tau)}{\eta(\tau)}\right)^{12} = q\prod_{n\ge1}\frac{(1-q^{3n})^{12}}{(1-q^n)^{12}} \in q + q^2\mathbb{Z}[[q]]\ , \tag{F.1}$$

where η is Dedekind's function, let $q(x)\in x+x^2\mathbb{Z}[[x]]$ be the compositional inverse of $x(q)$, let $D = x\,\mathrm{d}/\mathrm{d}x$, and

$$B = \theta\log x = 1 + 12\sum_{n\ge1}\sigma_1^{(3)}(n)q^n\ , \qquad G_5(q) = \sum_{3\nmid a}\frac{q^a}{a^5(1-q^a)} = \sum_{n\ge1}\sigma_{-5}^{(3)}(n)q^n\ . \tag{F.2}$$

We write B also for $B(q(x))\in 1+x\mathbb{Z}[[x]]$, so that $\theta = BD$ on series in x, and put $u = DB/B\in x\mathbb{Z}[[x]]$, $P = 1+27x$ and $x_e = -1/27$, the value of x at the elliptic point (Lemma F.5). For a constant c let

$$H_c = B^2(G_5(q(x)) + c)\ , \qquad g_0 = 1\ , \qquad g_i = D^{i-1}H_c \quad (1\le i\le 5)\ . \tag{F.3}$$

We call $g_0,\dots,g_5$ the germs. Finally put $\eta_5 = \zeta_3(5)/2\in\mathbb{Q}_3$. The series G_5 is an Eichler integral: $\theta^5 G_5$ is a modular form of weight 6 (proof of Lemma F.7). Moreover $G_5+\eta_5$ is the q-expansion of an overconvergent 3-adic modular form of weight -4 (F.21), which the factor B^2, of weight 4, turns into a function of x. Informally, an overconvergent p-adic modular form of weight k is a rigid analytic section of the k-th power of the sheaf of invariant differentials over a region of the p-adic modular curve that contains the ordinary component of the cusp ∞ and a neighbourhood of it reaching into the supersingular locus [28, 29, 30]; the precise statements that we use are Theorems F.20 and F.23. This is why, for $c=\eta_5$, the germs converge on the 3-adic disc $|x|_3 < 3^6$ (Proposition F.24).

The plan. Suppose that $\zeta_3(5)$, and hence η_5, is rational. For every $d\ge1$ we construct a nonzero rational number $\mathcal{D}_d = \mathcal{D}_d(\eta_5)$, the determinant of a $5d\times 5d$ matrix of Taylor coefficients of the germs $x^k g_i$ with $1\le i\le 5$ and $0\le k<d$ (Section F.2). By the product formula, $\log|\mathcal{D}_d| = v_3(\mathcal{D}_d)\log 3 + \sum_{\ell\ne3} v_\ell(\mathcal{D}_d)\log\ell$. We shall prove three estimates.

(i) For every $0<\epsilon<6$, $v_3(\mathcal{D}_d)\ge 15(6-\epsilon)d^2 - O(d)$ (Proposition F.26). It holds because the germs converge on the 3-adic disc $|x|_3<3^6$.

(ii) $\sum_{\ell\ne3} v_\ell(\mathcal{D}_d)\log\ell \ge -\frac{13289}{168}d^2 - o(d^2)$ (Proposition F.42). It comes from Proposition F.31, which writes the germs modulo powers of ℓ through rational functions with few poles.

(iii) $\log|\mathcal{D}_d| \le U\,d^2 + o(d^2)$ with an explicit constant $U\le 16.6040183$ (Proposition F.52 and (F.58)). It comes from Jensen's formula on circles in the complex q-disc and an energy bound for Gram determinants.

Second proof of Theorem 1.1. Suppose that $\eta_5\in\mathbb{Q}$. By the product formula and (i)–(iii), with $\epsilon = \frac{1}{10}$, so that $15(6-\epsilon) = \frac{177}{2}$,

$$\left(\frac{177}{2}\log 3 - \frac{13289}{168}\right)d^2 - o(d^2) \le v_3(\mathcal{D}_d)\log 3 + \sum_{\ell\ne3} v_\ell(\mathcal{D}_d)\log\ell = \log|\mathcal{D}_d| \le U\,d^2 + o(d^2)\ . \tag{F.4}$$

Since $\frac{177}{2}\log 3 - \frac{13289}{168} - U > 1.52$, this is impossible for large d. □

The rest of this appendix proves (i)–(iii). Section F.1 collects the modular identities, Section F.2 constructs $\mathcal{D}_d$, and Sections F.3, F.4 and F.5 prove (i), (ii) and (iii).

F.1. **Modular identities on $X_0(3)$.** This section shows that H_c satisfies the inhomogeneous linear differential equation $L_5 H_c = R_5$ (Lemma F.10). Here L_5 is an explicit operator of order 5 whose solutions are the functions $B^2\tau^a$, $0 \le a \le 4$, and R_5 is an explicit rational function. Sections F.2 and F.4 rest on this equation.

Lemma F.5. *(a) The group $\Gamma_0(3)$ has genus 0, two cusps ∞ and 0, and one elliptic point, of order 3, the orbit of $\tau_e = \frac{1}{2} + \frac{i}{2\sqrt{3}}$. The function x is $\Gamma_0(3)$-invariant, holomorphic and nonvanishing on $\mathbb{H}$, has a simple zero at the cusp ∞ and a simple pole at the cusp 0, and identifies $X_0(3)$ with $\mathbb{P}^1$; see e.g.* [3, Section 3]. *(b) $x(-1/(3\tau)) = 3^{-6}/x(\tau)$. (c) $j = (1+27x)(1+243x)^3/x$, and $x(\tau_e) = x_e$.*

Proof. (a) The point τ_e is fixed by $\left(\begin{smallmatrix} 1 & -1 \\ 3 & -2 \end{smallmatrix}\right) \in \Gamma_0(3)$, whose fixed points solve $3\tau^2 - 3\tau + 1 = 0$; this matrix has trace -1 and hence order 3 in $\mathrm{PSL}_2(\mathbb{Z})$. The function x is the Hauptmodul $(\eta(3\tau)^{24}/\eta(\tau)^{24})^{1/2}$ of $\Gamma_0(3)$ used in [3, Section 3]. It is holomorphic and nonvanishing on $\mathbb{H}$ because η is, its simple zero at the cusp ∞ can be read off from (F.1), and its simple pole at the cusp 0 follows from (b). (b) From $\eta(-1/\tau) = (\tau/i)^{1/2}\eta(\tau)$, see e.g. [31], we get

$$x(-1/(3\tau)) = \left(\frac{\eta(-1/\tau)}{\eta(-1/(3\tau))}\right)^{12} = \left(\frac{(\tau/i)^{1/2}\eta(\tau)}{(3\tau/i)^{1/2}\eta(3\tau)}\right)^{12} = \frac{3^{-6}}{x(\tau)}\ .$$

(c) The function j is $\Gamma_0(3)$-invariant and holomorphic on $\mathbb{H}$, hence rational in x with poles only at the cusps. At ∞, $j = q^{-1} + O(1)$ and $x = q + O(q^2)$, a simple pole in x. At the cusp 0, $j(-1/(3\tau)) = j(3\tau) = q^{-3} + O(1)$, while $x(-1/(3\tau)) = 3^{-6}q^{-1} + O(1)$, a pole of order 3 in x. So xj is a polynomial in x of degree at most 4, determined by its first five Taylor coefficients at $x = 0$, which one computes from the q-expansions. The fiber of $x \mapsto j$ over $j = 0$ consists of $x = -1/243$, with ramification index 3, and $x = -1/27$, with index 1. Every point of $\mathbb{H}$ over $j = 0$ is $\mathrm{SL}_2(\mathbb{Z})$-elliptic of order 3, so the unramified one is the $\Gamma_0(3)$-elliptic point, and $x(\tau_e) = -1/27$. □

Lemma F.6. *The series B is a holomorphic modular form of weight 2 on $\Gamma_0(3)$ with $B(-1/(3\tau)) = -3\tau^2 B(\tau)$. It does not vanish at the two cusps, and its zeros in $\mathbb{H}$ are exactly the orbit of τ_e, each of order 2 in $\tau - \tau_e$.*

Proof. Since $B = (2\pi i)^{-1}x'(\tau)/x(\tau)$ is the logarithmic derivative of an invariant function, it has weight 2, and differentiating $\log x(-1/(3\tau)) = -\log x(\tau) - 6\log 3$ gives $B(-1/(3\tau))\cdot(3\tau^2)^{-1} = -B(\tau)$. We have $B = 1 + O(q)$ at ∞, and $B(-1/(3\tau))/(-3\tau^2) = B(\tau) \to 1$ at the cusp 0. On $\mathbb{H}$, x is a local coordinate except at the elliptic points, where $x - x_e$ vanishes to order 3 in $\tau - \tau_e$; so x' vanishes exactly on the orbit of τ_e, to order 2, and $x \neq 0$ on $\mathbb{H}$. □

Lemma F.7. $V := 2Du + u^2 = \dfrac{24x}{P^2}$ *and* $R_5 := B^{-3}\theta^5 G_5 = \dfrac{x(1+39x)}{P^2}$.

Proof. V is invariant. For $\gamma = \left(\begin{smallmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{smallmatrix}\right) \in \Gamma_0(3)$ we have $B(\gamma\tau) = (c_\gamma\tau + d_\gamma)^2 B(\tau)$, the operator D commutes with $\tau \mapsto \gamma\tau$, and $D\tau = (2\pi i B)^{-1}$. Hence $u(\gamma\tau) = u + \delta$ with $\delta = 2c_\gamma/(2\pi i B(c_\gamma\tau + d_\gamma))$, and $D\delta = -u\delta - \delta^2/2$. So $V(\gamma\tau) = 2Du + 2D\delta + u^2 + 2u\delta + \delta^2 = V(\tau)$. For the Fricke involution $\tau \mapsto -1/(3\tau)$, the relation $x(-1/(3\tau)) = 3^{-6}/x(\tau)$ turns D into $-D$, and Lemma F.6 gives $u(-1/(3\tau)) = -u - \delta'$ with $\delta' = 2/(2\pi i\tau B)$ and $D\delta' = -u\delta' - \delta'^2/2$; the same computation shows that V is invariant under it.

The poles of V. So V is a rational function of x, holomorphic wherever $B \neq 0$. Near τ_e, $u = (2\pi i)^{-1}B'/B^2$ has a pole of order 3 in τ and Du one of order 6, so V has order at least -6 in $\tau - \tau_e$, that is, at least -2 in $x - x_e$. At $x = 0$ we have $V \in \mathbb{Z}[[x]]$ and $V(0) = 0$, since $u = 12x + O(x^2)$, and by the Fricke symmetry also $V(\infty) = 0$. Hence P^2V is a polynomial of

degree at most 2 vanishing at 0, its x^2-coefficient is $729\,V(\infty)=0$, and $P^2V=24x$ because $V=24x+O(x^2)$.

R_5. Grouping the divisors, $\theta^5G_5=\sum_n(\sigma_5(n)-\sigma_5(n/3))q^n=(E_6(3\tau)-E_6(\tau))/504$ with the Eisenstein series $E_6=1-504\sum_n\sigma_5(n)q^n$, a holomorphic form of weight 6 on $\Gamma_0(3)$. So R_5 is invariant and rational in x, with a pole of order at most 2 at x_e, since B^3 vanishes to order 6 in $\tau-\tau_e$, and $R_5=x+O(x^2)$. By $E_6(-1/\tau)=\tau^6E_6(\tau)$ and Lemma F.6, $R_5(-1/(3\tau))=(729E_6(3\tau)-E_6(\tau))/(13608\,B(\tau)^3)$, which tends to $728/13608=13/243$ at ∞; so $R_5=13/243$ at $x=\infty$. Hence $P^2R_5=x+39x^2$, as $729\cdot13/243=39$. □

Since θ acts simply on q-series, we express $D^i(B^2Y)$, for a holomorphic function Y of τ and with $D=B^{-1}\theta$, through the functions $B^{2-m}\theta^mY$, $0\le m\le i$. The coefficients are polynomials in an indeterminate u, to be evaluated at $u=DB/B$. Let $\mathfrak{R}=\mathbb{Q}(x)[u]$, with u an indeterminate, and with the derivation $\tilde D$ that extends D by $\tilde Du=(V-u^2)/2$, the Riccati form of Lemma F.7, and let $\partial=\partial/\partial u$. Define $N_{i,m}\in\mathfrak{R}$ by $N_{0,0}=1$ and

$$N_{i+1,m}=(2-m)\,u\,N_{i,m}+\tilde DN_{i,m}+N_{i,m-1}\ ,\qquad N_{i,m}=0\text{ unless }0\le m\le i\ . \tag{F.8}$$

Lemma F.9. *(a) For every holomorphic function Y of τ, $D^i(B^2Y)=B^2\sum_{m=0}^i N_{i,m}(u)\,B^{-m}\theta^mY$. (b) If V is kept as a symbol on which $\tilde D$ acts as D, then $N_{i,m}\in\mathbb{Z}[\frac12][u,V,DV,D^2V,\dots]$ is isobaric of weight $i-m$, where u has weight 1 and D^jV weight $2+j$, and $N_{i,i}=1$. (c) $\partial N_{i,m}=\mathfrak{c}_m\,N_{i,m+1}$ with $\mathfrak{c}_m=(m+1)(4-m)/2$, that is, $(\mathfrak{c}_0,\dots,\mathfrak{c}_4)=(2,3,3,2,0)$.*

Proof. (a) By induction on i, using $D(B^{2-m}\theta^mY\,N)=B^{2-m}\theta^mY\,((2-m)uN+DN)+B^{1-m}\theta^{m+1}Y\,N$, which follows from $D=B^{-1}\theta$ and $DB=uB$; evaluation at $u=DB/B$ is a homomorphism of differential rings by the Riccati equation. (b) Each step of (F.8) raises the weight by one, since $\tilde Du$ has weight 2. (c) On $\mathfrak{R}$ we have $\partial\tilde D=\tilde D\partial-u\partial$. If (c) holds at level i, then $\partial N_{i+1,m}=(2-m)N_{i,m}+\mathfrak{c}_m((1-m)uN_{i,m+1}+\tilde DN_{i,m+1})+\mathfrak{c}_{m-1}N_{i,m}$ and $\mathfrak{c}_mN_{i+1,m+1}=\mathfrak{c}_m((1-m)uN_{i,m+1}+\tilde DN_{i,m+1}+N_{i,m})$, and the two agree because $\mathfrak{c}_m=\mathfrak{c}_{m-1}+2-m$ and $\mathfrak{c}_{-1}=0$. The case $i=0$ is trivial. □

For example, $N_{1,0}=2u$, $N_{2,0}=3u^2+V$, $N_{2,1}=3u$, $N_{3,0}=3u^3+5Vu+DV$, $N_{3,1}=\frac12(9u^2+5V)$, $N_{3,2}=3u$, $N_{4,0}=\frac12(3u^4+24Vu^2+14\,DV\,u+5V^2+2D^2V)$, $N_{4,1}=\frac12(6u^3+24Vu+7DV)$, $N_{4,2}=3u^2+4V$ and $N_{4,3}=2u$.

Lemma F.10. *Let $L_5=D^5+\mathfrak{a}_3D^3+\mathfrak{a}_2D^2+\mathfrak{a}_1D+\mathfrak{a}_0$ with*

$$\mathfrak{a}_3=-\frac{120x}{P^2}\ ,\qquad \mathfrak{a}_2=\frac{180x(27x-1)}{P^3}\ ,\qquad \mathfrak{a}_1=-\frac{36x(2187x^2-388x+3)}{P^4}\ ,$$
$$\mathfrak{a}_0=\frac{24x(27x-1)(729x^2-366x+1)}{P^5}\ . \tag{F.11}$$

Then $L_5(B^2Y)=B^{-3}\theta^5Y$ for every holomorphic function Y of τ, in particular for every q-series. Consequently $L_5H_c=R_5$ for every c, and $L_5(B^2\tau^a)=0$ for $0\le a\le4$.

Proof. By Lemma F.9(a), $L_5(B^2Y)=B^2\sum_m(\sum_j\mathfrak{a}_jN_{j,m}(u))B^{-m}\theta^mY$ with $\mathfrak{a}_5=1$ and $\mathfrak{a}_4=0$. With $V=24x/P^2$, a direct computation of the inner sums, which are polynomials in u over $\mathbb{Q}(x)$, gives 0 for $m\le4$ and 1 for $m=5$. Then $L_5H_c=B^{-3}\theta^5(G_5+c)=R_5$, and $\theta^5\tau^a=0$ for $a\le4$. □

F.2. **The determinant.** We show that the six germs are independent over $\mathbb{C}(x)$ for every c, that a nonzero combination of them with polynomial coefficients of degree less than d vanishes at $x=0$ to order at most $6d+69$, and we construct $\mathcal{D}_d$. This is Yi's argument, written for level 3 and Sym^4.

Lemma F.12. *There is no $v\in\mathbb{C}(x)$ with $L_5v=R_5$.*

Proof. At a point $x_0 \notin \{0, x_e, \infty\}$, a pole of v of order k_0 gives a pole of $L_5 v$ of order $k_0 + 5$, while R_5 is regular there. At 0 the indicial polynomial of L_5 is ρ^5, and at x_e its roots are $\frac{4}{3}, \frac{5}{3}, 2, \frac{7}{3}, \frac{8}{3}$, the exponents of the local solutions $B^2\tau^a$. So a pole of order $k_0 \geq 1$ at 0 or at x_e gives a pole of $L_5 v$ of order k_0, respectively $k_0 + 5 \geq 6$, while R_5 is regular at 0 and has a pole of order at most 2 at x_e. Hence v is a polynomial. All $\mathfrak{a}_j$ are $O(1/x)$ at ∞ and $D^5 x^m = m^5 x^m$, so if $\deg v = m > 0$, the leading term of $L_5 v$ is m^5 times that of v, while R_5 is bounded at ∞; so v is constant. Then $L_5 v = \mathfrak{a}_0 v$, which is impossible, because $P^5 \mathfrak{a}_0$ is a polynomial of degree 4 while $P^5 R_5 = x(1 + 39x)P^3$ has degree 5. □

A *differential module* over $(\mathbb{C}(x), D)$ is a finite-dimensional $\mathbb{C}(x)$-vector space $\mathcal{M}$ with an additive map $D : \mathcal{M} \to \mathcal{M}$ such that $D(fm) = (Df)m + f\,Dm$ for $f \in \mathbb{C}(x)$ and $m \in \mathcal{M}$; a submodule is a D-stable subspace, and $\mathcal{M}$ is irreducible if 0 and $\mathcal{M}$ are its only submodules.

Lemma F.13. *The differential module $\mathcal{M}_5 = \mathbb{C}(x)[D]/\mathbb{C}(x)[D]L_5$ is irreducible.*

Proof. Near a point $x_0 \notin \{0, x_e, \infty\}$ with a preimage τ_0, the solutions of $L_5 f = 0$ are the span of $B^2\tau^a$, $0 \leq a \leq 4$, by Lemma F.10. A loop in $\mathbb{P}^1 \setminus \{0, x_e, \infty\}$ lifts to a path from τ_0 to $\gamma\tau_0$, and every $\gamma \in \Gamma_0(3)$ arises, since x makes $\mathbb{H}$ minus the elliptic orbit a covering of $\mathbb{P}^1 \setminus \{0, x_e, \infty\}$ with deck group $\Gamma_0(3)/\{\pm 1\}$. Continuation sends $B^2\tau^a$ to $B^2(\gamma\tau)(\gamma\tau)^a = B^2(\tau)(a_\gamma\tau + b_\gamma)^a(c_\gamma\tau + d_\gamma)^{4-a}$, so the monodromy contains Sym^4 of $\Gamma_0(3)$ acting on binary quartics in $(X, Y) = (\tau, 1)$. The elements $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 0 \\ 3 & 1 \end{smallmatrix}\right)$ are unipotent, with logarithms $Y\partial_X$ and $3X\partial_Y$. A subspace invariant under a unipotent element is invariant under its logarithm, a polynomial in the element minus the identity, hence under the commutator $3(Y\partial_Y - X\partial_X)$, whose eigenvalues $12, 6, 0, -6, -12$ on the monomials $X^a Y^{4-a}$ are distinct. So a nonzero invariant subspace contains some $X^a Y^{4-a}$, and then, by the two logarithms, all of them. If $\mathcal{M}_5$ had a proper nonzero submodule, the quotient would be cyclic, generated by the image of 1, with annihilator a monic operator of order between 1 and 4 dividing L_5 on the right, and its local solutions would form a monodromy-invariant subspace of that dimension, a contradiction. □

Proposition F.14. *Let $\mathcal{W}$ be the differential module with basis $e_0, \dots, e_5$, $De_0 = 0$, $De_i = e_{i+1}$ for $1 \leq i \leq 4$, and $De_5 = R_5 e_0 - \mathfrak{a}_0 e_1 - \mathfrak{a}_1 e_2 - \mathfrak{a}_2 e_3 - \mathfrak{a}_3 e_4$. Its only proper nonzero differential submodule is $\mathbb{C}(x)e_0$. For every $c \in \mathbb{C}$, the map $e_0 \mapsto 1$, $e_i \mapsto D^{i-1}H_c$ is an injective morphism of differential modules into $\mathbb{C}((x))$; in particular $1, H_c, \dots, D^4 H_c$ are linearly independent over $\mathbb{C}(x)$.*

Proof. We have $\mathcal{W}/\mathbb{C}(x)e_0 \cong \mathcal{M}_5$ via $\bar{e}_1 \mapsto 1$, since $L_5 \bar{e}_1 = 0$. Let S be a proper nonzero submodule. If its image in $\mathcal{M}_5$ is zero, then $S = \mathbb{C}(x)e_0$. Otherwise the image is $\mathcal{M}_5$ by Lemma F.13. Since $e_0 \in S$ would give $S = \mathcal{W}$, we have $e_0 \notin S$, so $S \cap \mathbb{C}(x)e_0 = 0$, and S is a complement of $\mathbb{C}(x)e_0$ containing $s_1 = e_1 - ve_0$ for some $v \in \mathbb{C}(x)$. Then $D^k s_1 = e_{k+1} - (D^k v)e_0$ for $k \leq 4$, and $D^5 s_1 + \sum_j \mathfrak{a}_j D^j s_1 = (R_5 - L_5 v)e_0 \in S \cap \mathbb{C}(x)e_0 = 0$, so $L_5 v = R_5$, which Lemma F.12 excludes. The map is a morphism because $D(D^4 H_c) = R_5 - \sum_j \mathfrak{a}_j D^j H_c$ by Lemma F.10, and its kernel is a submodule not containing e_0, hence zero. □

Proposition F.15 (zero estimate)**.** *Let $\mathcal{F} = \sum_{j=0}^{5} a_j g_j \neq 0$ with polynomials a_j of degree less than d, and $c \in \mathbb{C}$. Then $\mathrm{ord}_0 \mathcal{F} \leq 6d + 69$.*

Proof. If $a_1 = \cdots = a_5 = 0$, then $\mathrm{ord}_0 \mathcal{F} \leq d - 1$. Otherwise, identifying $\mathcal{W}$ with its image (Proposition F.14), the $\mathbb{C}(x)$-span of $\mathcal{F}, D\mathcal{F}, D^2\mathcal{F}, \dots$ is a differential submodule not contained in $\mathbb{C}(x)e_0$, hence equal to $\mathcal{W}$. So $\mathcal{F}, D\mathcal{F}, \dots, D^5\mathcal{F}$ are linearly independent over $\mathbb{C}(x)$, since a dependence at step $k < 6$ would make the span D-stable of dimension k. Write $De = e\,\mathcal{C}$ for the frame $e = (e_0, \dots, e_5)$; the entries of $P^5\mathcal{C}$ are 0, P^5, $P^5 R_5$ and $-P^5\mathfrak{a}_j$, polynomials of degree at most 5. Put $P^{5k}D^k\mathcal{F} = e\,\mathbf{a}^{(k)}$ with $\mathbf{a}^{(0)} = (a_0, \dots, a_5)^{\mathsf{T}}$. Then

$$\mathbf{a}^{(k+1)} = P^5 D\mathbf{a}^{(k)} - 5k\,P^4(DP)\,\mathbf{a}^{(k)} + (P^5\mathcal{C})\,\mathbf{a}^{(k)}\ , \tag{F.16}$$

so $\deg \mathbf{a}^{(k)} \le d-1+5k$, and $\mathrm{Wr} = \det\big(\mathbf{a}^{(0)},\dots,\mathbf{a}^{(5)}\big)$ is a nonzero polynomial of degree at most $\sum_{k=0}^{5}(d-1+5k) = 6d+69$. Adding to the row of e_0 the other rows multiplied by the germs g_j does not change Wr, and it turns that row into $(P^{5k}D^k\mathcal{F})_{0\le k\le 5}$, whose entries have order at least $\operatorname{ord}_0\mathcal{F}$ at 0, as D does not lower the order; the other rows are polynomials. Expanding along that row, $\operatorname{ord}_0\mathcal{F} \le \operatorname{ord}_0 \mathrm{Wr} \le 6d+69$. □

The determinant. For $d \ge 1$ the map $\Phi_d : \mathbb{C}[x]^6_{<d} \to \mathbb{C}[[x]]$, $a \mapsto \sum_j a_j g_j$, is injective by Proposition F.14. The orders $\operatorname{ord}_0\Phi_d(a)$ take exactly $6d$ distinct values $\lambda_0 < \cdots < \lambda_{6d-1}$, the *jumps*, since each quotient $\{\operatorname{ord}_0\Phi_d \ge n\}/\{\operatorname{ord}_0\Phi_d \ge n+1\}$ has dimension at most 1; all jumps are at most $6d+69$ by Proposition F.15. The block of a_0, the coefficient of $g_0 = 1$, realizes the orders $0,\dots,d-1$, so $\lambda_j = j$ for $j < d$, and since the jumps are distinct,

$$j \le \lambda_j \le (6d+69) - (6d-1-j) = j+70\ . \tag{F.17}$$

Put $I_d = \{\lambda_d,\dots,\lambda_{6d-1}\} \subset [d, N_d)$ with $N_d = 6d+70$, and let M be the $5d\times 5d$ matrix

$$M_{\lambda,(i,k)} = [x^{\lambda-k}]\,g_i \qquad (\lambda\in I_d,\ 1\le i\le 5,\ 0\le k<d)\ , \qquad \mathcal{D}_d(c) = \det M\ . \tag{F.18}$$

The sign of $\mathcal{D}_d(c)$ depends on the order of rows and columns and plays no role. Let $\tilde M$ be the $6d\times 6d$ matrix with entries $[x^\lambda](x^k g_i)$, rows $\lambda\in\{\lambda_0,\dots,\lambda_{6d-1}\}$ and columns (i,k), $0\le i\le 5$, $0\le k<d$. It is nonsingular, because in a basis of the source adapted to the order filtration it becomes triangular with nonzero diagonal. Its columns with $i=0$ are the unit vectors of the rows $\lambda_j = j < d$, so $\det\tilde M = \pm\det M$ and $\mathcal{D}_d(c)\ne 0$. For $c\in\mathbb{Q}$ the jumps are defined over $\mathbb{Q}$ and do not change under $\mathbb{Q}\subset\mathbb{C}$ or $\mathbb{Q}\subset\mathbb{Q}_3$, so $\mathcal{D}_d(c)$ is one nonzero rational number, which we estimate at every place.

F.3. **The 3-adic estimate.** The 3-adic estimate rests on Proposition F.24: for $c=\eta_5$ the germs converge on the open disc $|x|_3 < 3^6$. Our rigid analytic spaces are defined over a finite extension of the completion of $\mathbb{Q}_3^{\mathrm{ur}}$, as in [30, Section 3]; since $H_{\eta_5}\in\mathbb{Q}_3[[x]]$, its radius of convergence does not depend on this choice.

We write $X_0(3)^{\mathrm{an}}$ for the rigid analytic space attached to $X_0(3)$ [32]; through x, its points over $\mathbb{C}_3$ are the points of $\mathbb{P}^1(\mathbb{C}_3)$. A point is *supersingular* if the elliptic curve that it parametrizes has potentially good supersingular reduction, which here means $|j|_3 < 1$, and *ordinary* otherwise; this depends only on its j-invariant.

Lemma F.19. *In the coordinate x, the ordinary locus of $X_0(3)^{\mathrm{an}}$ is $\{|x|_3\le 1\}\sqcup\{|x|_3\ge 3^6\}$, the two components containing the cusps ∞ ($x=0$) and 0 ($x=\infty$), and the supersingular locus is the annulus $1<|x|_3<3^6$. Hence $W_0(3)$, the complement of the ordinary component containing the cusp 0, is the open disc $\{|x|_3<3^6\}$. The elliptic point x_e, with $|x_e|_3 = 27$, lies in the supersingular annulus.*

Proof. In characteristic 3 the only supersingular j-invariant is 0. Indeed, an elliptic curve over $\bar{\mathbb{F}}_3$ has a model $Y^2 = h(X) = X^3 + a_2X^2 + a_4X + a_6$, for which $\Delta = a_2^2a_4^2 - a_2^3a_6 - a_4^3$ and $j = a_2^6/\Delta$. A point $Q=(X_Q,Y_Q)$ has order 3 if and only if $Y_Q\ne 0$ and the third intersection $-2Q$ of the curve with the tangent at Q is Q. The tangent has slope $m = h'(X_Q)/(2Y_Q)$ and $-2Q$ has X-coordinate $m^2 - a_2 - 2X_Q$, so this means $h'(X_Q)^2 = 4a_2h(X_Q)$, that is, $a_2X_Q^3 + a_2a_6 - a_4^2 = 0$. If $a_2\ne 0$, this equation has a root X_Q, at which $h(X_Q)\ne 0$ because the curve is smooth, so the curve has a point of order 3, is ordinary, and has $j\ne 0$. If $a_2 = 0$, then $j=0$ and $\Delta = -a_4^3\ne 0$, so the equation has no root and the curve has no point of order 3, that is, it is supersingular. So a point is supersingular if and only if $|j|_3<1$, and whether it is ordinary depends only on j. Put $v = v_3(x)\in\mathbb{R}$. By Lemma F.5(c) and the Newton polygons of $1+27x$ and $1+243x$, $v_3(j) = -v$ for $v>-3$, $v_3(j)=3$ for $-5<v<-3$, $v_3(j) = 18+3v$ for $v<-5$, and $v_3(j)\ge 3$ for $v=-3,-5$. So $v_3(j)\le 0$ exactly for $v\ge 0$ or $v\le -6$. The two ordinary pieces are closed discs, one around $x=0$ and one around $x=\infty$, so they are connected, and they contain the two cusps. □

We now quote the two published inputs, in our notation. For $p>2$ let ω be the Teichmüller character, $\langle a\rangle = a/\omega(a)$ for $a\in\mathbb{Z}_p^\times$ as in Section 2.1. For a continuous character $\kappa\neq\mathbf{1}$ of $\mathbb{Z}_p^\times$ put $\sigma_\kappa^*(n)=\sum_{a|n,\,p\nmid a}\kappa(a)a^{-1}$ and $G_\kappa^*(q)=\zeta^*(\kappa)/2+\sum_{n\geq1}\sigma_\kappa^*(n)q^n$, where $\zeta^*(\kappa)$ is Coleman's p-adic zeta function of κ; if $\kappa(a)=\langle a\rangle^s\chi(a)$ with $s\in\mathbb{C}_p$, $|s|<p^{(p-2)/(p-1)}$, and χ of finite order, then $\zeta^*(\kappa)=L_p(1-s,\chi)$ [29, Section B1].

Theorem F.20 (Coleman [29, Section B1]; Coleman and Mazur [33, Corollary 2.2.6])**.** *Let $p>2$. If $\kappa(a)=\langle a\rangle^k\chi(a)$, where k is an integer and χ is a character of finite order on $\mathbb{Z}_p^\times$ with $\chi(-1)=1$, then $G_\kappa^*(q)$ is the q-expansion of an overconvergent modular form of weight k on $\Gamma_1(\mathrm{lcm}(p,f_\chi))$ with character $\chi\omega^{-k}$, where f_χ is the conductor of χ.*

Coleman states this with a sketch of proof, and Coleman and Mazur give a complete proof for the characters in the connected component $\mathcal{B}$ of the identity of weight space [33, Section 1.4 and Corollary 2.2.6]. Weight space is the rigid analytic space whose $\mathbb{C}_p$-points are the continuous characters $\mathbb{Z}_p^\times\to\mathbb{C}_p^\times$, and the characters in $\mathcal{B}$ are, by restriction, identified with the continuous characters of $1+p\mathbb{Z}_p$; these are the characters trivial on $\mu(\mathbb{Z}_p)$. In their normalization, the Eisenstein series E_κ has constant term 1, so that $\zeta^*(\kappa)E_\kappa=2G_\kappa^*$; the passage to G_κ^* uses $\zeta^*(\kappa)\neq0$, which holds for these characters by [29, Section B1]. At $p=3$ the Teichmüller character has order 2, so for $\kappa(a)=a^{-4}=\langle a\rangle^{-4}$ we have $k=s=-4$ and $\chi=\mathbf{1}$, with $|s|_3=1<3^{1/2}$; the level is $\Gamma_1(3)$ with character $\omega^4=\mathbf{1}$, that is, $\Gamma_0(3)$. Then $\sigma_\kappa^*=\sigma_{-5}^{(3)}$ and $\zeta^*(\kappa)=L_3(5,\mathbf{1})=\zeta_3(5)$, so

$$G_{-4}^*:=G_{a^{-4}}^*=\eta_5+\sum_{n\geq1}\sigma_{-5}^{(3)}(n)q^n=\eta_5+G_5(q) \tag{F.21}$$

is an overconvergent form of weight -4 and tame level 1 (the tame level is the part of the level prime to p). The character $a\mapsto a^{-4}$ is trivial on $\mu(\mathbb{Z}_3)=\{\pm1\}$, so it lies in $\mathcal{B}$ and the complete proof of [33] applies.

Lemma F.22. *Atkin's operator U_3, which acts on q-expansions by $\sum_n a_nq^n\mapsto\sum_n a_{3n}q^n$, fixes G_{-4}^*.*

Proof. We have $\sigma_{-5}^{(3)}(3n)=\sigma_{-5}^{(3)}(n)$ because $3n$ and n have the same divisors prime to 3; the constant term is unchanged. □

Theorem F.23 (Buzzard [30, Theorem 5.2(2)])**.** *Let p be a prime, $N\geq5$ an integer prime to p, and k an integer. Let $X_1(N;p)$ be the modular curve of level $\Gamma_1(N)\cap\Gamma_0(p)$, and let $W_0(p)\subset X_1(N;p)^{\mathrm{an}}$ be the set of points whose reduction lies in the component of the special fiber that contains the reduction of the cusp ∞, that is, the complement of the component of the ordinary locus which contains the cusp 0. If f is an overconvergent modular form of level N and weight k which is an eigenvector of U_p with nonzero eigenvalue, then f extends to a section of $(\underline{\omega}^{\mathrm{an}})^{\otimes k}$ on $W_0(p)$.*

Here $\underline{\omega}$ is the sheaf of invariant differentials of the universal elliptic curve, the superscript an denotes the associated rigid analytic space, and U_p acts on q-expansions by $\sum_n a_nq^n\mapsto\sum_n a_{pn}q^n$. Buzzard's setting includes $p=3$ [30, Section 3], and for small tame level he suggests adding an auxiliary level structure, working at higher level and then taking invariants [30, Section 3]. We carry out this reduction for $p=3$ with the tame level $N=5$ and a trace, and we turn the resulting section of $\underline{\omega}^{\otimes-4}$ into a function by multiplying it with the form B^2 of weight 4.

Proposition F.24. *The series $H_{\eta_5}=B^2(G_5(q(x))+\eta_5)\in\mathbb{Q}_3[[x]]$ and all $D^iH_{\eta_5}$ converge on $|x|_3<3^6$. Hence for every $\epsilon>0$ there is Θ_ϵ with*

$$v_3([x^n]\,g_i)\geq(6-\epsilon)n-\Theta_\epsilon\qquad(n\geq0,\ 1\leq i\leq5,\ c=\eta_5)\ . \tag{F.25}$$

Proof. Away from the cusps, the points of $X_1(5;3)$ correspond to triples (E,ι,C) of an elliptic curve E, a point ι of exact order 5 on E and a subgroup $C\subset E[3]$ of order 3, and those of $X_0(3)$ to pairs

(E, C). Let $\pi_5 : X_1(5;3) \to X_0(3)$ be the morphism $(E, \iota, C) \mapsto (E, C)$ that forgets ι, a finite flat morphism of smooth projective curves of some degree $\delta \geq 1$. Whether a point lies in $W_0(3)$ depends only on (E, C), namely on whether E has supersingular reduction or ordinary reduction with C the canonical subgroup [28, Ch. 3], [30, Sections 3–4]; if E has ordinary reduction, its canonical subgroup is the subgroup of $E[3]$ of order 3 whose points reduce to the origin, the 3-torsion of the formal group of E. On the ordinary locus of $X_0(3)$ the condition that C is the canonical subgroup is open and closed; it holds at the cusp ∞, where $C = \mu_3$ on the Tate curve $\mathbb{G}_m/q^{\mathbb{Z}}$, the elliptic curve over $\mathbb{Z}((q))$ that describes a neighbourhood of the cusp ∞, and fails at the cusp 0. Since the two ordinary components of Lemma F.19 are connected, the disc $W_0(3) = \{|x|_3 < 3^6\}$ is exactly the set of points (E, C) with E supersingular or C canonical. So $W_5 := \pi_5^{-1}(W_0(3))$ is Buzzard's $W_0(3) \subset X_1(5;3)^{\mathrm{an}}$, and $\pi_5 : W_5 \to W_0(3)$ is finite flat of degree δ, with a trace Tr such that $\mathrm{Tr} \circ \pi_5^*$ is multiplication by δ.

By Theorem F.20, G^*_{-4} is overconvergent of weight -4 and tame level 1, which Coleman defines through two auxiliary tame levels [29]: with these taken to be 5 and 7, G^*_{-4} is given by sections $\mathcal{S}$ and $\mathcal{S}'$ of $\underline{\omega}^{\otimes -4}$ over affinoids $X_1(15)(w)$ and $X_1(21)(w)$, for some $0 < w < \frac{3}{4}$, with q-expansion $G^*_{-4}(q)$ and with the same pull-back to $X_1(105)(w)$. For $N' \in \{5, 7, 35\}$, Coleman's $X_1(N')(w)$ is the image in $X_1(N')$ of the affinoid of a modular curve of level divisible by N' and prime to 3, on which a lift $\mathcal{A}$ of the Hasse invariant to that curve satisfies $v_3(\mathcal{A}) \leq w$, and $X_1(3N')(w)$ consists of the pairs (E, α), $\alpha : \mu_{3N'} \hookrightarrow E$, such that $\alpha(\mu_3)$ is the canonical subgroup of E and $(E, \alpha|_{\mu_{N'}})$ lies in $X_1(N')(w)$ [29]. Since the Hasse invariant has a simple zero at each supersingular point, $\mathcal{A}$ can serve as the parameter on the supersingular discs of that curve, as Buzzard notes for E_{p-1} [30], and since $w < \frac{3}{4}$ the condition $v_3(\mathcal{A}) \leq w$ depends only on E, as does the corresponding condition for Buzzard's parameters on $X_1(N')$ [30]. So $X_1(N')(w)$ is Buzzard's strict neighborhood $X_1(N')_{\geq 3^{-w}}$ of the ordinary locus, and $X_1(105)(w)$ is the full preimage of $X_1(15)(w)$ and of $X_1(21)(w)$.

The value of $\mathcal{S}$ at (E, α) lies in $H^0(E, \Omega^1)^{\otimes -4}$. It does not depend on $\alpha|_{\mu_5}$, since it is the value of $\mathcal{S}'$ at $(E, \alpha|_{\mu_3}\alpha')$ for every $\alpha' : \mu_7 \hookrightarrow E$, and it does not change under $\alpha \mapsto -\alpha$, since -1 on E is an isomorphism from (E, α) to $(E, -\alpha)$ and acts trivially on $H^0(E, \Omega^1)^{\otimes -4}$. As $\alpha|_{\mu_3}$ is determined up to sign by $\alpha(\mu_3)$, the value depends only on E. The map $(E, \alpha) \mapsto (E, \alpha(\zeta_5), \alpha(\mu_3))$, with ζ_5 a fixed primitive fifth root of unity, from $X_1(15)(w)$ to its image is the quotient by the diamond operator $\langle 11 \rangle : (E, \alpha) \mapsto (E, \alpha \circ [11])$, which is -1 on μ_3 and the identity on μ_5. Since $\langle 11 \rangle$ changes only the level structure, $\underline{\omega}$ on $X_1(15)$ is the pull-back of $\underline{\omega}$ on $X_1(5;3)$, with $\langle 11 \rangle$ acting trivially on the fibres, so a section of $\underline{\omega}^{\otimes -4}$ invariant under $\langle 11 \rangle$ descends. The section $\mathcal{S}$ is invariant, since its value depends only on E, and gives a section $\pi_5^* G^*_{-4}$ of $\underline{\omega}^{\otimes -4}$ over this image, which is $X_1(5)_{\geq 3^{-w}}$, embedded in $X_1(5;3)$ through the canonical subgroup [30]. Hence $\pi_5^* G^*_{-4}$ is an overconvergent form of weight -4 and level 5 in the sense of [30], with q-expansion $G^*_{-4}(q)$, and, by Lemma F.22 and the q-expansion principle (a form on a connected region containing the cusp ∞ is determined by its q-expansion; see [28] for the algebraic statement and the proof of [30, Theorem 5.2] for its use on such regions), a U_3-eigenvector with eigenvalue 1.

Theorem F.23, with $p = 3$, $N = 5$ and $k = -4$, extends it to a section ϕ of $\underline{\omega}^{\otimes -4}$ on W_5. The form B^2 is a classical form of weight 4 on $\Gamma_0(3)$ with integral q-expansion (F.2) (Lemma F.6), so $\pi_5^*(B^2)$ is a global section of $\underline{\omega}^{\otimes 4}$, and $\tilde{\mathcal{H}} := \pi_5^*(B^2)\,\phi$ is a rigid analytic function on W_5. Put $\mathcal{H} = \delta^{-1}\mathrm{Tr}(\tilde{\mathcal{H}})$, a rigid analytic function on the open disc $W_0(3) = \{|x|_3 < 3^6\}$ (Lemma F.19), hence a power series in x converging on that disc. On a neighbourhood of the cusp ∞ on which G^*_{-4} is defined, $\phi = \pi_5^* G^*_{-4}$, so there $\tilde{\mathcal{H}} = \pi_5^*(B^2 G^*_{-4})$ and $\mathcal{H} = B^2 G^*_{-4}$. Near the cusp ∞, by the compatibility of the algebraic and rigid Tate curves [32, Section 3.1], the function $B^2 G^*_{-4}$ is $B^2(q)(G_5(q) + \eta_5)$ in the Tate parameter q, with $x = x(q)$, that is, $H_{\eta_5}(x)$. Hence $\mathcal{H} = H_{\eta_5}$ near $x = 0$, and by the uniqueness of Taylor expansions H_{η_5} is the expansion of $\mathcal{H}$ on the whole disc. The elliptic point x_e inside the disc causes no difficulty, since $\mathcal{H}$ is a function on the coarse curve.

Finally D preserves convergence on an open disc, and (F.25) follows from convergence on the closed disc of radius $3^{6-\epsilon}$. □

Calegari uses the same description of the ordinary locus of $X_0(3)$ in his proof that $\zeta_3(3)$ is irrational [3, Section 3].

Proposition F.26. *Suppose that $\eta_5 \in \mathbb{Q}$. For every $0 < \epsilon < 6$, $v_3(\mathcal{D}_d(\eta_5)) \geq 15(6-\epsilon)d^2 - 5\Theta_\epsilon d$, with Θ_ϵ from* (F.25).

Proof. The entries of M are Taylor coefficients of the 3-adic series of Proposition F.24, of index $\lambda - k \geq 1$ for $\lambda \in I_d$ and $k < d$. By (F.25), every term of the Leibniz expansion of $\det M$ has valuation at least $(6-\epsilon)(\sum_{\lambda\in I_d}\lambda - \sum_{(i,k)} k) - 5d\Theta_\epsilon$. Since I_d consists of $5d$ distinct integers $\geq d$, $\sum_{\lambda\in I_d}\lambda \geq \sum_{n=d}^{6d-1} n = (35d^2-5d)/2$, and $\sum_{(i,k)} k = 5d(d-1)/2$; the difference is at least $15d^2$. □

F.4. **The primes $\ell \neq 3$.** We now prove $\sum_{\ell\neq 3}\nu_\ell \log \ell \leq \frac{13289}{168}d^2 + o(d^2)$ with $\nu_\ell = -v_\ell(\mathcal{D}_d)$. The denominators of the germs at ℓ come only from the terms of G_5 with $\ell \mid a$. Modulo ℓ, the series $u = DB/B$ agrees with the logarithmic derivative r of a rational function of degree about $\ell/3$, built from the Eisenstein series of weight $\ell - 1$ (Lemma F.27). Hence, modulo ℓ^j, the combinations $\ell^5\sum_i a_i D^{i-1}H_c$ are governed by rational functions $F_0(a), \dots, F_{j-1}(a)$ with few poles (Proposition F.31 and (F.35)), and counting their dimensions bounds the number h_j of elementary divisors of $\ell^5 M$ of valuation below j (Proposition F.36 and Theorem F.40). For the remaining primes a crude common denominator suffices. Fix a prime $\ell \geq 5$, put $s = \lfloor(\ell-1)/3\rfloor$, and let $E_{\ell-1} = 1 - \frac{2(\ell-1)}{\beta_{\ell-1}}\sum_n \sigma_{\ell-2}(n)q^n$ be the Eisenstein series of weight $\ell-1$, where $\beta_{\ell-1}$ is the Bernoulli number.

Lemma F.27. *One has $E_{\ell-1}(q(x))/B^{(\ell-1)/2} = Q_\ell(x)/P^s$ with $Q_\ell \in \mathbb{Z}_{(\ell)}[x]$, $Q_\ell(0) = 1$, $\deg Q_\ell = s$, and leading coefficient $27^s(-3)^{(\ell-1)/2}$, an ℓ-adic unit. The rational function*

$$r := -\frac{2}{\ell-1} D\log(Q_\ell/P^s) \tag{F.28}$$

has only simple poles, at the roots of Q_ℓ and at x_e, is bounded at ∞, and lies in $\mathbb{Z}_{(\ell)}[[x]]$, and $\xi := (u-r)/\ell \in \mathbb{Z}_{(\ell)}[[x]]$.

Proof. By the von Staudt–Clausen theorem $v_\ell(\beta_{\ell-1}) = -1$, so $E_{\ell-1} \in 1 + \ell q\mathbb{Z}_{(\ell)}[[q]]$ and $E_{\ell-1}(q(x)) \in 1 + \ell x\mathbb{Z}_{(\ell)}[[x]]$. The quotient $E_{\ell-1}/B^{(\ell-1)/2}$ is $\Gamma_0(3)$-invariant of weight 0, with poles only at the zeros of B, of order at most $\ell-1$ in $\tau - \tau_e$ (Lemma F.6), that is, at most s in $x - x_e$. It equals 1 at $x = 0$, and at $x = \infty$, by $E_{\ell-1}(-1/(3\tau)) = (3\tau)^{\ell-1}E_{\ell-1}(3\tau)$ and Lemma F.6, it tends to $3^{\ell-1}(-3)^{-(\ell-1)/2} = (-3)^{(\ell-1)/2}$. So $Q_\ell = P^s E_{\ell-1}/B^{(\ell-1)/2}$ is a polynomial of degree s with leading coefficient $27^s(-3)^{(\ell-1)/2}$, and its Taylor series lies in $\mathbb{Z}_{(\ell)}[[x]]$. Finally $D\log(Q_\ell/P^s) = D\log E_{\ell-1}(q(x)) - \frac{\ell-1}{2}u$ gives $u - r = \frac{2}{\ell-1}D\log E_{\ell-1}(q(x)) \in \ell\mathbb{Z}_{(\ell)}[[x]]$; the statements about the poles hold for the logarithmic derivative of any rational function. □

Lemma F.29. *The reduction $\bar{Q}_\ell \in \mathbb{F}_\ell[x]$ has degree s, is squarefree, and is prime to $\bar{P}$. Hence Q_ℓ has s distinct roots in $\bar{\mathbb{Q}}_\ell$, all ℓ-integral and different from 0 and x_e.*

Proof. Reducing $E_{\ell-1} = B^{(\ell-1)/2}Q_\ell/P^s$ modulo ℓ gives $\bar{B}^{(\ell-1)/2} = \bar{P}^s/\bar{Q}_\ell$, hence, as $\frac{2}{\ell-1} \equiv -2$,

$$\bar{u} = \frac{2}{\ell-1}D\log\frac{\bar{P}^s}{\bar{Q}_\ell} = D\log\frac{\bar{Q}_\ell^2}{\bar{P}^{2s}}\ , \qquad 2D\bar{u} + \bar{u}^2 = \bar{V} = \frac{24x}{\bar{P}^2}\ . \tag{F.30}$$

Let α be a root of $\bar{Q}_\ell$ of multiplicity μ, $1 \leq \mu \leq s < \ell$, and $t = x - \alpha$; $\alpha \neq 0$ since $Q_\ell(0) = 1$. If $\bar{P}(\alpha) \neq 0$, then $\bar{u} = 2\mu\alpha/t + O(1)$, and the coefficient of t^{-2} in $2D\bar{u} + \bar{u}^2$ is $4\mu(\mu-1)\alpha^2$, which vanishes because $\bar{V}$ is regular at α; so $\mu = 1$. If $\alpha = x_e$, then $\bar{u} = \rho x_e/t + O(1)$ with $\rho = 2\mu - 2s$, and comparison with $\bar{V} = -\frac{8}{9}x_e^2 t^{-2} + O(t^{-1})$ gives $\rho^2 - 2\rho = -\frac{8}{9}$, that is, $(3\rho-2)(3\rho-4) = 0$ in

$\mathbb{F}_\ell$. Since $\mu = s + \rho/2$, and $3s \equiv -1$ or $3s \equiv -2 \pmod{\ell}$ according as $\ell = 3s+1$ or $\ell = 3s+2$, this gives $\mu \equiv 0$ or $\mu \equiv 2s+1 \pmod{\ell}$ in both cases, and neither lies in $[1, s]$. The degree is that of Lemma F.27, the leading coefficient being a unit, and the roots of Q_ℓ are integral and reduce to the distinct roots of $\bar{Q}_\ell$, none equal to $\bar{x}_e$. □

Now fix $\ell \geq 5$ and $c \in \mathbb{Z}_{(\ell)}$. Write $G_5 = G_5^{[\ell]} + \ell^{-5} G_5(q^\ell)$, where $G_5^{[\ell]}$ omits the terms with $\ell \mid a$ and is ℓ-integral, and put $T_m = (\theta^m G_5)(q(x)^\ell) \in \mathbb{Q}[[x]]$. The coefficient of Y^n in $\theta^m G_5(Y)$ is $n^m \sigma_{-5}^{(3)}(n)$, which is ℓ-integral for $n < \ell$, and $q(x)^\ell \in x^\ell \mathbb{Z}[[x]]$, so T_m is ℓ-integral below x^{ℓ^2}. Moreover $DT_m = \ell B^{-1} T_{m+1}$, since $D \log q(x) = B^{-1}$.

Proposition F.31 (the Sym^4 form of Yi's congruence)**.** *Put*

$$C_{m,j} = \frac{1}{j!} \prod_{t=m}^{m+j-1} \mathfrak{e}_t = \binom{m+j}{m} \frac{(4-m)!}{(4-m-j)!\, 2^j} \in \mathbb{Z}\big[\tfrac{1}{2}\big] \ , \tag{F.32}$$

and $b_n = \sum_{m+j=n} C_{m,j}\, \xi^j\, B^{2-m}\, T_m$ for $0 \leq n \leq 4$. Then, for $0 \leq i \leq 4$, one has the exact identity of formal series

$$\ell^5 D^i H_c = \sum_{n=0}^{i} \ell^n\, N_{i,n}(r)\, b_n + \ell^5 \mathcal{E}_i \ , \qquad \mathcal{E}_i = D^i\Big(B^2 (G_5^{[\ell]} + c)\Big) \in \mathbb{Z}_{(\ell)}[[x]] \ , \tag{F.33}$$

and every b_n is ℓ-integral below x^{ℓ^2}.

Proof. We have $\ell^5 H_c = B^2 T_0 + \ell^5 B^2 (G_5^{[\ell]} + c)$. Lemma F.9(a) with $\theta^m T_0 = \ell^m T_m$ gives $D^i(B^2 T_0) = \sum_m \ell^m B^{2-m} T_m N_{i,m}(u)$. Insert $u = r + \ell\xi$ and expand the polynomial $N_{i,m}$ in u by Taylor's formula; by Lemma F.9(c), $\partial^j N_{i,m}/j! = C_{m,j} N_{i,m+j}$, and collecting the powers ℓ^{m+j} gives (F.33). The closed form of $C_{m,j}$ is a rearrangement of the product. The integrality follows from $\xi \in \mathbb{Z}_{(\ell)}[[x]]$ (Lemma F.27), $B^{\pm 1} \in \mathbb{Z}[[x]]$, $C_{m,j} \in \mathbb{Z}[\frac{1}{2}]$, and the integrality of T_m below x^{ℓ^2}. □

At level 7, Yi proves the analogous congruence modulo ℓ^3 for $H = B(G + c)$, with a form G of weight -2, and the three germs H, DH, D^2H [15, (27)]; Proposition F.31 is an exact identity of the same kind for weight -4 and five germs. From now on let $\ell \geq 5$ with $\ell^2 > N_d$, and $c \in \mathbb{Z}_{(\ell)}$; we call these primes the *large primes*. Let $A = \ell^5 M$, which acts on $a = (a_1, \dots, a_5) \in \mathbb{Z}_{(\ell)}[x]^5_{<d} \cong \mathbb{Z}_{(\ell)}^{5d}$ by sending a to the coefficients of $\ell^5 \sum_i a_i D^{i-1} H_c$ at the rows I_d; A is ℓ-integral by (F.33), since $I_d \subset [d, N_d)$ and $N_d < \ell^2$. Let $\mathfrak{e}_1, \dots, \mathfrak{e}_{5d}$ be the ℓ-adic valuations of its elementary divisors over $\mathbb{Z}_{(\ell)}$, and $h_j = \#\{i : \mathfrak{e}_i < j\}$. Since $\det A = \ell^{25d} \mathcal{D}_d$,

$$\nu_\ell = -v_\ell(\mathcal{D}_d) = 25d - \sum_i \mathfrak{e}_i \leq \sum_i \max(0, 5 - \mathfrak{e}_i) = \sum_{j=1}^{5} h_j \ . \tag{F.34}$$

For $a \in \mathbb{Q}[x]^5_{<d}$ put $F_n(a) = \sum_{i=1}^5 a_i\, N_{i-1,n}(r)$, $0 \leq n \leq 4$. By (F.33),

$$\ell^5 \sum_{i=1}^{5} a_i D^{i-1} H_c = \sum_{n=0}^{4} \ell^n b_n F_n(a) + \ell^5 \sum_{i=1}^{5} a_i \mathcal{E}_{i-1} \ , \tag{F.35}$$

so modulo ℓ^j only $F_0(a), \dots, F_{j-1}(a)$ matter. Since $N_{i,i} = 1$, $F_n(a) = a_{n+1} + \sum_{i>n+1} a_i N_{i-1,n}(r)$.

Proposition F.36. *For $1 \leq j \leq 5$,*

$$h_j \leq \min\left(5d,\ \#\{\lambda \in I_d : \lambda \geq \ell\},\ \sum_{n<j} \dim F_n(K_n)\right) \ , \tag{F.37}$$

where $K_n = \{a \in \mathbb{Q}[x]^5_{<d} :\ F_0(a) = \cdots = F_{n-1}(a) = 0\}$.

Proof. Let $S_j = \{a \in \mathbb{Z}_{(\ell)}^{5d} : Aa \equiv 0 \bmod \ell^j\}$ and $\bar{S}_j$ its image in $\mathbb{F}_\ell^{5d}$. In a Smith basis, $a \in S_j$ if and only if its i-th coordinate lies in $\ell^{\max(0,j-\mathfrak{e}_i)}\mathbb{Z}_{(\ell)}$, so $\dim \bar{S}_j = \#\{i : \mathfrak{e}_i \geq j\}$ and $h_j = 5d - \dim \bar{S}_j$.

The kernel bound. K_j is a $\mathbb{Q}$-subspace of dimension $5d - \operatorname{rank}_{\mathbb{Q}}(F_0, \dots, F_{j-1})$, and $K_j \cap \mathbb{Z}_{(\ell)}^{5d}$ is saturated, so it reduces injectively to a subspace of $\mathbb{F}_\ell^{5d}$ of that dimension. For a in it, (F.35) reduces to $\ell^5 \sum_i a_i D^{i-1} H_c = \sum_{n \geq j} \ell^n b_n F_n(a) + \ell^5 \sum_i a_i \mathcal{E}_{i-1}$, and $F_n(a) \in \mathbb{Z}_{(\ell)}[[x]]$ because r, V and the derivatives of V are ℓ-integral series and the $N_{i,m}$ have coefficients in $\mathbb{Z}[\frac{1}{2}]$. With b_n integral below x^{ℓ^2} this gives $a \in S_j$. So $h_j \leq \operatorname{rank}_{\mathbb{Q}}(F_0, \dots, F_{j-1}) = \sum_{n<j}(\dim K_n - \dim K_{n+1}) = \sum_{n<j} \dim F_n(K_n)$.

The row bound. For $\lambda < \ell$ the coefficients $[x^n]H_c$, $n \leq \lambda$, are ℓ-integral, since they involve $\sigma_{-5}^{(3)}(m)$ only for $m < \ell$; so the rows $\lambda < \ell$ of A vanish modulo ℓ^5. Modulo ℓ^j ($j \leq 5$), A therefore has at most $\#\{\lambda \in I_d : \lambda \geq \ell\}$ nonzero rows, hence at most that many elementary divisors that are nonzero modulo ℓ^j, and h_j is the number of these. □

Lemma F.38 (weight count)**.** *$F_n(\mathbb{Q}[x]^5_{<d}) \subseteq (PQ_\ell)^{-(4-n)}\mathbb{Q}[x]_{\leq d-1+(4-n)(s+1)}$, a space of dimension $d + (4-n)(s+1)$.*

Proof. By Lemma F.9(b), $N_{i-1,n}(r)$ is a sum of monomials in r and the $D^j V$ of weight $i - 1 - n$. The function r has simple poles at the roots of PQ_ℓ (Lemma F.27), $D^j V$ has a pole of order $2 + j$ at x_e only, and all are bounded at ∞. So $N_{i-1,n}(r)(PQ_\ell)^{i-1-n}$ is a polynomial of degree at most $(i-1-n)(s+1)$, and $a_i N_{i-1,n}(r)(PQ_\ell)^{4-n}$ has degree at most $d - 1 + (4-n)(s+1)$. □

With Proposition F.36, Lemma F.38 already gives a bound for the denominators with which the argument closes (Remark F.46). The following local lemma improves the bounds for $n \geq 1$; it more than doubles the final margin.

Lemma F.39 (local lemma)**.** *Let α be a root of Q_ℓ, $t = x - \alpha$, and $a \in K_j$ with $0 \leq j \leq 4$. Then $\operatorname{ord}_\alpha F_j(a) \geq 2j - 4$.*

Proof. By Lemma F.29, $\alpha \neq 0, x_e$, so near α the function V and its derivatives are regular, and, α being a simple root, $r = w + \tilde{r}$ with $w = \rho_\alpha/t$, $\rho_\alpha = -2\alpha/(\ell - 1) \neq 0$, and $\tilde{r}$ regular. Taylor's formula and Lemma F.9(c) give $F_m(a) = \sum_{n \geq m} C_{m,n-m} w^{n-m} \tilde{F}_n$ with $\tilde{F}_n = \sum_i a_i N_{i-1,n}(\tilde{r})$ regular at α. With $f_m = F_m/((4-m)!\,2^m)$ and $\tilde{f}_n = \tilde{F}_n/((4-n)!\,2^n)$, the closed form of $C_{m,j}$ turns this into $f_m = \sum_n \binom{n}{m} w^{n-m} \tilde{f}_n$, that is, $f(Z) = \tilde{f}(Z + w)$ for the quartics $f(Z) = \sum_m f_m Z^m$ and $\tilde{f}(Y) = \sum_n \tilde{f}_n Y^n$. Let $\hat{f}(y) = t^4 \tilde{f}(y/t) = \sum_n \tilde{f}_n t^{4-n} y^n \in \mathbb{C}[[t]][y]$; then $f_m = t^{m-4}\, \partial_y^m \hat{f}(\rho_\alpha)/m!$. The hypothesis $f_0 = \cdots = f_{j-1} = 0$ says that $(y - \rho_\alpha)^j$ divides $\hat{f}$ in $\mathbb{C}((t))[y]$, hence in $\mathbb{C}[[t]][y]$, by division by a monic polynomial with constant coefficients. Write $\hat{f} = \sum_{b \geq 0} t^b p_b(y)$. The coefficient of $t^b y^n$ is the coefficient of t^{b-4+n} in $\tilde{f}_n$, so y^{4-b} divides p_b for $b \leq 4$, and $(y - \rho_\alpha)^j$ divides every p_b. For $b < j$, $\deg p_b \leq 4 < (4-b) + j$ forces $p_b = 0$. Hence $\partial_y^j \hat{f}(\rho_\alpha) = O(t^j)$, and $f_j = O(t^{2j-4})$. □

Theorem F.40 (layer theorem)**.** *For the large primes,*

$$\dim F_n(K_n) \leq \mathcal{L}_n := \begin{cases} d + (4-n) + (4-2n)s\ , & n = 0, 1, 2\ , \\ \max(0,\ d + 1 - 2s)\ , & n = 3\ , \\ \max(0,\ d - 4s)\ , & n = 4\ , \end{cases} \tag{F.41}$$

and therefore $h_j \leq \min(5d,\ \#\{\lambda \in I_d : \lambda \geq \ell\},\ \mathcal{L}_0 + \cdots + \mathcal{L}_{j-1})$.

Proof. For $a \in K_n$, the function $F_n(a)$ has at x_e a pole of order at most $4 - n$ (Lemma F.38), at each of the s distinct roots of Q_ℓ (Lemma F.29) a pole of order at most $\max(0, 4 - 2n)$ and, for $n \geq 3$, a zero of order at least $2n - 4$ (Lemma F.39), no other finite pole, and it is $O(x^{d-1})$ at ∞. For $n \leq 2$, $P^{4-n} Q_\ell^{4-2n} F_n(a)$ is a polynomial of degree at most $d - 1 + (4-n) + (4-2n)s$. For $n = 3$, $PF_3(a)$ is a polynomial of degree at most d divisible by Q_ℓ^2. For $n = 4$, $F_4(a) = a_5$ has

degree less than d and is divisible by Q_ℓ^4. Counting dimensions and applying Proposition F.36 proves the claim. □

Proposition F.42. *Suppose that $\eta_5 \in \mathbb{Q}$. Then $\sum_{\ell\neq 3} \nu_\ell \log \ell \le \left(\int_0^6 \Psi(z)\,\mathrm{d}z\right) d^2 + o(d^2)$, where*

$$\Psi(z) = \sum_{j=1}^{5} \min\left(\mathrm{rc}(z),\ \sum_{n<j} \Lambda_n(z)\right), \qquad \mathrm{rc}(z) = \min\left(5, (6-z)^+\right), \tag{F.43}$$

with the normalized row count rc *and $(\Lambda_0, \dots, \Lambda_4) = (1+\frac{4}{3}z,\ 1+\frac{2}{3}z,\ 1,\ (1-\frac{2}{3}z)^+,\ (1-\frac{4}{3}z)^+)$, the normalized $\mathcal{L}_n/d$ of* (F.41) *with $z = \ell/d$. Explicitly,*

$$\Psi(z) = \begin{cases} 15 + \frac{20}{3}z\ , & 0 \le z \le \frac{3}{4}\ , \\ 16 + \frac{16}{3}z\ , & \frac{3}{4} \le z \le 1\ , \\ 21 + \frac{1}{3}z\ , & 1 \le z \le \frac{4}{3}\ , \\ 25 - \frac{8}{3}z\ , & \frac{4}{3} \le z \le \frac{15}{7}\ , \\ 30 - 5z\ , & \frac{15}{7} \le z \le 6\ , \end{cases} \qquad \int_0^6 \Psi = \frac{13289}{168}\ . \tag{F.44}$$

Proof. Let $b_c \ge 1$ be the denominator of η_5.

A common denominator. The coefficient $[x^n]G_5(q(x))$ involves $\sigma_{-5}^{(3)}(m)$ only for $m \le n$, and $q(x)$ and B have integral coefficients, so $b_c\,\mathrm{lcm}(1,\dots,n)^5[x^n]H_{\eta_5} \in \mathbb{Z}$; and D multiplies coefficients by integers. Hence every entry of M has a denominator dividing $b_c\,\mathrm{lcm}(1,\dots,N_d)^5$, and for every prime ℓ

$$\nu_\ell \le 5d(v_\ell(b_c) + 5\lfloor \log N_d / \log \ell \rfloor)\ . \tag{F.45}$$

In particular $\nu_\ell \le 0$ for $\ell > N_d$ with $\ell \nmid b_c$.

The small primes. The primes $\ell \neq 3$ with $\ell = 2$, $\ell \mid b_c$, or $5 \le \ell \le \sqrt{N_d}$ contribute by (F.45) at most $5d\big(\log b_c + 5\log N_d\,(2 + \log_2 b_c + \sqrt{N_d})\big) = O(d^{3/2}\log d)$, since b_c has at most $\log_2 b_c$ prime factors.

The large primes. For the others, (F.34) and Theorem F.40 give

$$\nu_\ell \le \sum_{j=1}^{5} \min\left(5d,\ \#\{\lambda \in I_d : \lambda \ge \ell\},\ \mathcal{L}_0 + \cdots + \mathcal{L}_{j-1}\right)\ .$$

We have $\#\{\lambda \in I_d : \lambda \ge \ell\} \le (6d-\ell)^+ + 70$ by (F.17); and $|s - \ell/3| \le 1$ gives $\mathcal{L}_n \le d\Lambda_n(\ell/d) + 4$ for each n. So in each of the five minima the second argument exceeds $d(6-\ell/d)^+$ by at most 70 and the third exceeds $d\sum_{n<j}\Lambda_n(\ell/d)$ by at most 20, and $\nu_\ell \le d\,\Psi(\ell/d) + 350$.

The summation. With $\vartheta(X) = \sum_{\ell\le X}\log\ell = X + o(X)$ by the prime number theorem, and since Ψ is continuous, piecewise linear and zero on $[6,\infty)$, partial summation gives $\sum_{\ell<N_d} d\,\Psi(\ell/d)\log\ell = d^2\int_0^6\Psi + o(d^2)$ and $\sum_{\ell<N_d} 350\log\ell = O(d)$. The formula (F.44) and the integral follow from (F.43) by exact piecewise linear arithmetic. □

Remark F.46. The local lemma and the layer theorem are not needed for the conclusion. If Theorem F.40 is replaced by the elementary bound $\dim F_n(K_n) \le d + (4-n)(s+1)$ of Lemma F.38, the same computation applies with $\Lambda_n = 1 + \frac{4-n}{3}z$, the integral becomes $\frac{33569}{420}$, and $\frac{177}{2}\log 3 - \frac{33569}{420} - U$ is still positive, about 0.70.

F.5. **The archimedean estimate.** We prove $\limsup_d d^{-2}\log|\mathcal{D}_d| \le U(\mu)$ for every finite convex combination μ of circle measures, and we bound $U(\mu)$ for one such μ. The leading Taylor coefficients of combinations of the germs are bounded by Jensen's formula on circles $|q| = r_i$ in the complex q-disc (Lemma F.48). The polynomial coefficients are measured in a weighted L^2 norm, whose

Gram determinant is controlled by the logarithmic energy of the image measure (Lemma F.51). Throughout, $c \in \mathbb{Q}$ is fixed and embedded in $\mathbb{C}$.

Pullbacks and the potential. The product (F.1) and the series (F.2) converge locally uniformly on $|q| < 1$, and $x(q)/q$ is holomorphic and nonvanishing there with value 1 at 0. So $H_c(x(q)) = B(q)^2(G_5(q)+c)$ is holomorphic on $|q| < 1$, and since $D = B^{-1}\theta$, each $g_i(x(q))$ is meromorphic with poles only at the zeros of $B(q)$. Here and below, $g_i(x(q))$ denotes the function $(B^{-1}\theta)^{i-1}(B^2(G_5+c))$ of q, and similarly for combinations of the germs. Near $q = 0$ it is the composition of the series g_i with $x(q)$, but the series g_i in x need not converge on the image curves used below. Choose radii $r_i = e^{-2\pi y_i} < 1$ and weights $\varpi_i > 0$ with $\sum_i \varpi_i = 1$, and fix ρ_0 with $\max_i r_i < \rho_0 < 1$. The functions $g_i(x(q))$ have finitely many poles in $|q| \le \rho_0$, and a polynomial $\Pi(q)$ with $\Pi(0) = 1$, independent of d, clears them for all six germs. Let μ_i be the pushforward under $x(q)$ of the normalized arc length on $|q| = r_i$, let $\mu = \sum_i \varpi_i \mu_i$, and let $\mathcal{P}_\mu(z) = \int \log|z-w| \,\mathrm{d}\mu(w)$.

Lemma F.47. *$\mathcal{P}_\mu$ is continuous on $\mathbb{C}$, and μ has finite logarithmic energy $I = \iint \log|z-w| \,\mathrm{d}\mu(z)\,\mathrm{d}\mu(w)$ (we use this sign convention, the opposite of that of* [34]*).*

Proof. Near a point q_0 of one of the circles, $x(q) = x(q_0) + \mathfrak{u}(q)^{m_0}$ with a local biholomorphic coordinate $\mathfrak{u}$ and $m_0 \ge 1$. For $z \in \mathbb{C}$ put $z' = z - x(q_0)$. If $|\mathfrak{u}^{m_0} - z'| \le \varrho$, then one of the m_0 factors $\mathfrak{u} - \mathfrak{u}_k$, $\mathfrak{u}_k^{m_0} = z'$, has modulus at most ϱ^{1/m_0}; on a short arc the inverse coordinate is Lipschitz, so the arc length of the preimage of the disc $\mathbb{D}(z,\varrho)$ is at most $C\varrho^{1/m_0}$, uniformly in z. A finite cover of the circles gives $\mu(\mathbb{D}(z,\varrho)) \le C\varrho^{1/m_1}$ for $0 < \varrho \le 1$, with a fixed C and m_1 the largest of the m_0. Integrating in ϱ, the contribution of $|z-w| < e^{-T}$ to $\mathcal{P}_\mu(z)$ tends to 0 uniformly in z as $T \to \infty$, so the continuous truncated potentials converge uniformly to $\mathcal{P}_\mu$. In particular $\mathcal{P}_\mu$ is bounded on the support of μ, and $I = \int \mathcal{P}_\mu \,\mathrm{d}\mu$ is finite. □

Put $W_i = \int \mathcal{P}_\mu \,\mathrm{d}\mu_i$. Fix a closed disc Ω containing the image curves in its interior, and $h > 0$ such that the discs of radius h around all points of these curves lie in Ω. Give the polynomials of degree less than d the norm $\|a\|_d^2 = \int_\Omega |a|^2 e^{-2d\mathcal{P}_\mu}\mathrm{d}A$, and the source $\mathbb{C}[x]^6_{<d}$ of Φ_d the orthogonal sum norm. Let $m_\mu(h)$ be a modulus of continuity of $\mathcal{P}_\mu$ on Ω.

Lemma F.48. *If a source vector of norm one has the image $\mathcal{F} = \sum_j a_j g_j = \mathfrak{c}_n x^n + O(x^{n+1})$, then $\log|\mathfrak{c}_n| \le dW_i + 2\pi n y_i + d\,m_\mu(h) + C_h$ for every i, with C_h independent of d, n and the vector.*

Proof. Let z be a point of the image curves. For $|\zeta - z| \le h$ we have $e^{-2d\mathcal{P}_\mu(\zeta)} \ge e^{-2d(\mathcal{P}_\mu(z)+m_\mu(h))}$, so the submean inequality for $|a_j|^2$ on the disc $\mathbb{D}(z,h) \subset \Omega$ gives

$$|a_j(z)| \le (\pi h^2)^{-1/2} e^{d(\mathcal{P}_\mu(z)+m_\mu(h))}\|a_j\|_d \,.$$

On $|q| = r_i$, $|\Pi(q)\mathcal{F}(x(q))| \le \sum_j |a_j(x(q))|\,|\Pi g_j(x(q))|$, and the Cauchy–Schwarz inequality over the six blocks, with fixed bounds for the Πg_j on $|q| \le \max_i r_i$, gives $\log|\Pi\mathcal{F}(x(q))| \le d\mathcal{P}_\mu(x(q)) + d\,m_\mu(h) + C_h$. The function $\psi = \Pi(q)\mathcal{F}(x(q))/q^n$ is holomorphic on $|q| \le r_i$, and $\psi(0) = \mathfrak{c}_n$ because near $q = 0$ the function $\mathcal{F}(x(q))$ is the composition of series and $x(q) = q + O(q^2)$. Jensen's inequality $\log|\psi(0)| \le \int \log|\psi| \,\mathrm{d}\vartheta/2\pi$ on $|q| = r_i$ gives the claim, since $\int \mathcal{P}_\mu(x(q))\,\mathrm{d}\vartheta/2\pi = W_i$ and $\log r_i = -2\pi y_i$. □

Let $\mathcal{G}_d$ be the Gram matrix of $1, z, \dots, z^{d-1}$ for $\|\cdot\|_d$. Expanding the Vandermonde determinant $\det\bigl[z_k^b\bigr]_{k,b}$ and its complex conjugate and integrating term by term, which is Andréief's identity, gives

$$\det \mathcal{G}_d = \frac{1}{d!}\int_{\Omega^d} \prod_{i<j} |z_i - z_j|^2 e^{-2d\sum_i \mathcal{P}_\mu(z_i)} \prod_i \mathrm{d}A(z_i) \,. \tag{F.49}$$

Lemma F.50 (energy inequality)**.** *For every Borel probability measure ν on Ω, $I(\nu) - 2\int \mathcal{P}_\mu \,\mathrm{d}\nu \le -I$, where $I(\nu) = \iint \log|z-w| \,\mathrm{d}\nu\,\mathrm{d}\nu \in [-\infty, \infty)$.*

Proof. This is [34, Lemma I.1.8], in our sign convention; we include a proof. If $I(\nu) = -\infty$ there is nothing to prove, $\mathcal{P}_\mu$ being bounded on Ω. Otherwise ν has no atoms, and $\log|z-w|$ is absolutely integrable against $\nu\otimes\nu$, $\nu\otimes\mu$ and $\mu\otimes\mu$, because it is bounded above on $\Omega-\Omega$ and the three integrals are $I(\nu)$, $\int\mathcal{P}_\mu\mathrm{d}\nu$ and I. Put $\sigma=\nu-\mu$, of total mass 0; the claim is $I(\sigma)\le 0$. Let $\sigma_\varepsilon=\sigma*\chi_\varepsilon$ with a smooth radial probability density χ_ε supported in $|z|\le\varepsilon$, and let $\mathcal{P}_{\sigma_\varepsilon}=\log|\cdot|*\sigma_\varepsilon$ be its potential. By the zero total mass, $\mathcal{P}_{\sigma_\varepsilon}=O(|z|^{-1})$ and $\nabla\mathcal{P}_{\sigma_\varepsilon}=O(|z|^{-2})$ at infinity, and $\Delta\mathcal{P}_{\sigma_\varepsilon}=2\pi\sigma_\varepsilon$, so integration by parts gives $I(\sigma_\varepsilon)=\int\mathcal{P}_{\sigma_\varepsilon}\,\mathrm{d}\sigma_\varepsilon=-\frac{1}{2\pi}\int_{\mathbb{C}}|\nabla\mathcal{P}_{\sigma_\varepsilon}|^2\mathrm{d}A\le 0$. Now $I(\sigma_\varepsilon)=\iint k_\varepsilon(z-w)\,\mathrm{d}\sigma\,\mathrm{d}\sigma$ with $k_\varepsilon=\log|\cdot|*\chi_\varepsilon*\chi_\varepsilon$. By the submean property of the subharmonic function $\log|\cdot|$ and the radiality of χ_ε, $\log|\cdot|\le k_\varepsilon$; moreover k_ε is bounded above on compact sets, and $k_\varepsilon\to\log|\cdot|$ off 0. Dominated convergence in the three pairings separately gives $I(\sigma)=\lim_{\varepsilon\to0}I(\sigma_\varepsilon)\le 0$. □

Lemma F.51. $\limsup_{d\to\infty}d^{-2}\log\det\mathcal{G}_d\le -I$.

Proof. Let $k_T=\max(\log|z-w|,-T)$, continuous and bounded on $\Omega\times\Omega$. For $z_1,\dots,z_d\in\Omega$ with empirical measure $\nu_d=d^{-1}\sum_i\delta_{z_i}$ we have $2\sum_{i<j}\log|z_i-z_j|\le d^2\iint k_T\,\mathrm{d}\nu_d\,\mathrm{d}\nu_d+dT$, since the diagonal contributes $-dT$. Choose configurations $z^{(d)}$ at which the integrand in (F.49) is within a factor 2 of its supremum. Then $\log\det\mathcal{G}_d\le d^2\big(\iint k_T\mathrm{d}\nu_d\mathrm{d}\nu_d-2\int\mathcal{P}_\mu\mathrm{d}\nu_d\big)+dT+O(d)$ for every T, with ν_d independent of T. Along a subsequence realizing the lim sup, $\nu_d\to\nu$ weakly, and for fixed T the bracket converges to $\iint k_T\mathrm{d}\nu\mathrm{d}\nu-2\int\mathcal{P}_\mu\mathrm{d}\nu$. Letting $T\to\infty$, by monotone convergence, gives at most $I(\nu)-2\int\mathcal{P}_\mu\mathrm{d}\nu$, which is at most $-I$ by Lemma F.50. □

Proposition F.52. $\limsup_{d\to\infty}d^{-2}\log|\mathcal{D}_d(c)|\le U(\mu):=\int_0^6\min_i(W_i+2\pi y_i k)\,\mathrm{d}k-3I$.

Proof. Choose an orthonormal basis $v_0,\dots,v_{6d-1}$ of the source adapted to the order filtration of Section F.2: v_j is a unit vector in $\{\mathrm{ord}_0\Phi_d\ge\lambda_j\}$ orthogonal to $\{\mathrm{ord}_0\Phi_d>\lambda_j\}$, so that $\Phi_d(v_j)=\mathfrak{c}_j x^{\lambda_j}+O(x^{\lambda_j+1})$ with $\mathfrak{c}_j\ne0$. The matrix $([x^{\lambda_{j'}}]\Phi_d(v_j))_{j',j}$ is triangular with diagonal $(\mathfrak{c}_j)$, and by Lemma F.48, $\log|\mathfrak{c}_j|\le\min_i(dW_i+2\pi y_i\lambda_j)+d\,m_\mu(h)+C_h$. The matrix of the monomials on the same rows has determinant $\pm\mathcal{D}_d(c)$, and the change of basis from the monomials to (v_j) has absolute determinant $(\det\mathcal{G}_d)^{-3}$, there being six orthogonal blocks. Hence

$$\log|\mathcal{D}_d(c)|\le 3\log\det\mathcal{G}_d+\sum_{j=0}^{6d-1}\min_i\,(dW_i+2\pi y_i\lambda_j)+6d^2m_\mu(h)+O_h(d)\ . \tag{F.53}$$

Replacing λ_j by j costs $O(d)$ by (F.17), and the sum is then a Riemann sum equal to $d^2\int_0^6\min_i(W_i+2\pi y_i k)\,\mathrm{d}k+O(d)$. Take $\limsup_d$ with Lemma F.51, and then let $h\to0$. □

To evaluate $U(\mu)$ we need the mutual energies $\mathcal{I}(y,y')=\iint\log|z-w|\,\mathrm{d}\mu_y(z)\,\mathrm{d}\mu_{y'}(w)$ of the circle measures with radii $e^{-2\pi y}$ and $e^{-2\pi y'}$.

Lemma F.54. *With Euler's function φ,*

$$\mathcal{I}(y,y')=-2\pi\min(y,y')+\sum_{\substack{j\ge1\\ 9j^2yy'<1}}\frac{4\pi\varphi(3j)}{9j^2}\Big(\arccos\Big(3j\sqrt{yy'}\Big)-3j\sqrt{yy'}\sqrt{1-9j^2yy'}\Big)\ . \tag{F.55}$$

Proof. Fix $\tau=u_0+iy$ outside the elliptic orbit and $w=x(e^{2\pi i\tau})$. Since x identifies $X_0(3)$ with $\mathbb{P}^1$, the zeros of $x(q)-w$ in $|q|<1$ are the points $e^{2\pi i\gamma\tau}$ with $\gamma\in\Gamma_\infty\backslash\Gamma_0(3)$, where Γ_∞ is the stabilizer of ∞ in $\Gamma_0(3)$; they are all simple. The cosets are the identity and the cosets with a primitive bottom row $(3j,m)$, $j\ge1$, and $\mathrm{Im}\,\gamma\tau=y/((3ju_0+m)^2+9j^2y^2)$. Jensen's formula on $|q|=e^{-2\pi y'}$ gives $\int\log|x(q)-w|\,\mathrm{d}\vartheta/2\pi=\log|w|+\sum_\gamma 2\pi(\mathrm{Im}\,\gamma\tau-y')^+$. We average over $u_0\in[0,1)$. The mean of $\log|w|$ is $-2\pi y$, because $\log|x(q)/q|$ is harmonic with value 0 at $q=0$, and together with the identity coset this gives $-2\pi\min(y,y')$. For fixed j, the substitution $v=3ju_0+m$, and the fact

that almost every $v \in \mathbb{R}$ lies in exactly $\varphi(3j)$ of the intervals $[m, m+3j]$ with $\gcd(3j,m)=1$, give the contribution

$$\frac{\varphi(3j)}{3j} \int_{\mathbb{R}} \left(\frac{2\pi y}{v^2 + 9j^2y^2} - 2\pi y' \right)^+ \mathrm{d}v \ . \tag{F.56}$$

The integrand is supported on $|v| \le v_* = \sqrt{y/y' - 9j^2y^2}$, which is empty unless $9j^2yy' < 1$. With $a = 3j\sqrt{yy'}$, $\int_{-v_*}^{v_*} \frac{2\pi y\,\mathrm{d}v}{v^2+9j^2y^2} = \frac{4\pi}{3j} \arctan \frac{\sqrt{1-a^2}}{a} = \frac{4\pi}{3j} \arccos a$ and $4\pi y' v_* = \frac{4\pi}{3j} a\sqrt{1-a^2}$. Only finitely many j contribute, and the elliptic orbit has measure zero. □

The bound for U. We take the 34 circles and weights of Table 3, with rational $y_i > 0$ and $\sum_i \varpi_i = 1$. Then $W_i = \sum_j \varpi_j \mathcal{I}(y_i, y_j)$ and $I = \sum_i \varpi_i W_i$. Assign to the circle i the segment $[\alpha_i, \beta_i]$ of Table 3. These segments tile $[0,6]$, and on each of them $\min_{i'}(W_{i'} + 2\pi y_{i'}k) \le W_i + 2\pi y_i k$, so

$$U(\mu) \le J - 3I \ , \qquad J = \sum_i \big((\beta_i - \alpha_i) W_i + \pi y_i (\beta_i^2 - \alpha_i^2) \big) \ . \tag{F.57}$$

We evaluate I and J from (F.55) in ball arithmetic at a precision of 400 bits [18], deciding every condition $9j^2 y_i y_j < 1$ in exact rational arithmetic. This gives $I = 2.3300529587\ldots$, $J = 23.5941770943\ldots$, and

$$U := J - 3I \le 16.6040183 \ ; \tag{F.58}$$

we quote I and J so that a reader who evaluates (F.55) on Table 3 can compare. With (F.58), $\frac{177}{2}\log 3 - \frac{13289}{168} - U > 1.52$, as used in the proof at the beginning of this appendix.

i	y_i	ϖ_i	segment $[\alpha_i, \beta_i]$
1	4811/100000	15863/166667	[1357267/250000, 6]
2	2707/50000	34665/333334	[4804939/1000000, 1357267/250000]
3	6093/100000	46900/500001	[2121097/500000, 4804939/1000000]
4	857/12500	82975/1000002	[468043/125000, 2121097/500000]
5	1929/25000	12424/166667	[824241/250000, 468043/125000]
6	8683/100000	66269/1000002	[2899411/1000000, 824241/250000]
7	9771/100000	28582/500001	[127821/50000, 2899411/1000000]
8	2749/25000	51181/1000002	[2249323/1000000, 127821/50000]
9	6187/50000	46379/1000002	[1971069/1000000, 2249323/1000000]
10	557/4000	20644/500001	[215419/125000, 1971069/1000000]
11	1567/10000	17935/500001	[301631/200000, 215419/125000]
12	8817/50000	10775/333334	[657103/500000, 301631/200000]
13	4961/25000	4847/166667	[1139719/1000000, 657103/500000]
14	22331/100000	4253/166667	[197323/200000, 1139719/1000000]
15	2513/10000	21353/1000002	[85849/100000, 197323/200000]
16	707/2500	8519/500001	[756249/1000000, 85849/100000]
17	1273/4000	15553/1000002	[26517/40000, 756249/1000000]
18	35813/100000	2381/166667	[577227/1000000, 26517/40000]
19	20151/50000	13103/1000002	[498597/1000000, 577227/1000000]
20	45353/100000	1989/166667	[427003/1000000, 498597/1000000]
21	25519/50000	3529/333334	[14539/40000, 427003/1000000]
22	28717/50000	4502/500001	[154727/500000, 14539/40000]
23	64633/100000	1247/166667	[3307/12500, 154727/500000]
24	36367/50000	6887/1000002	[111617/500000, 3307/12500]
25	1637/2000	6319/1000002	[185321/1000000, 111617/500000]
26	92109/100000	2872/500001	[75431/500000, 185321/1000000]
27	103653/100000	1729/333334	[59869/500000, 75431/500000]
28	23329/20000	4669/1000002	[91727/1000000, 59869/500000]
29	4102/3125	4141/1000002	[66877/1000000, 91727/1000000]
30	36929/25000	3583/1000002	[9077/200000, 66877/1000000]
31	166231/100000	501/166667	[13673/500000, 9077/200000]
32	37413/20000	1189/500001	[13079/1000000, 13673/500000]
33	210511/100000	267/166667	[3471/1000000, 13079/1000000]
34	7403/3125	289/500001	[0, 3471/1000000]

TABLE 3. The 34-circle measure of (F.58). Circle i has radius $r_i = e^{-2\pi y_i}$ in the q-disc, weight ϖ_i, and the assigned segment $[\alpha_i, \beta_i] \subset [0, 6]$ in (F.57); the segments tile $[0, 6]$, and $\sum_i \varpi_i = 1$.

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